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Research Article

$Z_2 \times Z_3$ Equivariant Bifurcation in Coupled Two Neural Network Rings

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We study a Hopfield-type network that consists of a pair of one-way rings each with three neurons and two-way coupling between the rings. The rings have symmetric group $\Gamma = Z_3 \times Z_2$, which means the global symmetry Z_2 and internal symmetry Z_3. We discuss the spatiotemporal patterns of bifurcating periodic oscillations by using the symmetric bifurcation theory of delay differential equations combined with representation theory of Lie groups. The existence of multiple branches of bifurcating periodic solution is obtained. We also found that the spatiotemporal patterns of bifurcating periodic oscillations alternate according to the change of the propagation time delay in the coupling; that is, different ranges of delays correspond to different patterns of neural network oscillators. The oscillations of corresponding neurons in the two loops can be in phase, antiphase, $T/3, 2T/3, 4T/3, 5T/6$, or $7T/6$ periods out of phase depending on the delay. Some numerical simulations support our analysis results.

1. Introduction

The theory of spatiotemporal pattern formation in systems of coupled nonlinear oscillators with symmetry has grown extensively in recent years. Its impact has been felt in a wide variety of fields of applied science. Coupled networks of nonlinear dynamical systems have become important models for studying the behavior of large complex systems. These models allow us to investigate fundamental features of physical systems, biological systems, and so on. The central question is to understand how specific properties of the individual behavior and the coupling architecture can give rise to the emergence of new collective phenomena [1–5]. Couple can lead to oscillators' synchronization, chaos, symmetric bifurcation, and so on [6].

Networks with a ring topology, where locally coupled oscillators or oscillatory populations form a closed loop of signal transmission, appear to be relevant for many practical situations. These systems sometimes show symmetric properties. In general, symmetric systems typically exhibit more complicated bifurcations than nonsymmetric systems, and as well they may increase the dimension of the space and the number of variables involved. Some bifurcations can have a smaller codimension in a class of systems with specified symmetries. Other bifurcations, on the contrary, may not occur in the presence of certain symmetries [7, 8].

Time delays have been incorporated into coupled models by many authors, since in real systems the signal inevitably propagates from one oscillator to the next over a finite distance and with a finite speed; a time delay can not be negligible. From the mathematical point of view, the presence of delays makes the problem harder to handle. In fact, the state vector characterizing a nonlinear delayed system evolves in an infinite dimensional functional space. Networks with interacting loops and time delays are common in physiological systems. For example, there are many interacting loops and feedback systems in the model of brain's motor circuitry [9, 10].

In this paper, we focus on the simplest Hopfield network with delays. This model consists of two coupling unidirectional rings, each with three oscillators. See Figure 1.

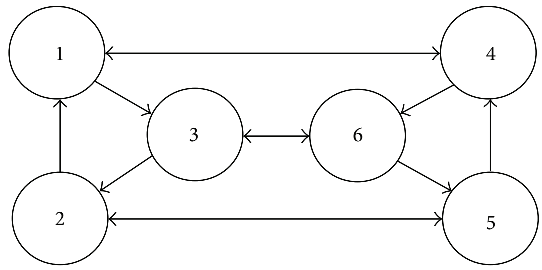

Figure 1: The architecture of the model (1).

The case leads to the following system of delay differential equations:

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + b\tanh(x_2(t)) + c\tanh(x_4(t-\tau)),\\ \dot{x}_2(t) &= -x_2(t) + b\tanh(x_3(t)) + c\tanh(x_5(t-\tau)),\\ \dot{x}_3(t) &= -x_3(t) + b\tanh(x_1(t)) + c\tanh(x_6(t-\tau)),\\ \dot{x}_4(t) &= -x_4(t) + b\tanh(x_5(t)) + c\tanh(x_1(t-\tau)),\\ \dot{x}_5(t) &= -x_5(t) + b\tanh(x_6(t)) + c\tanh(x_2(t-\tau)),\\ \dot{x}_6(t) &= -x_6(t) + b\tanh(x_4(t)) + c\tanh(x_3(t-\tau)),\end{aligned} \quad (1)$$

where $\tau \geq 0$ is the time delay. Let $X = (x_1, x_2, x_3, x_4, x_5, x_6)$ represent the state variables. For $\Gamma = Z_3 \times Z_2$, where Z_3 and Z_2 are the cycle group, the action on X follows $\rho \in Z_3, \kappa \in Z_2$:

$$\begin{aligned}\rho(x_1, x_2, x_3, x_4, x_5, x_6) &= (x_2, x_3, x_1, x_5, x_6, x_4),\\ \kappa(x_1, x_2, x_3, x_4, x_5, x_6) &= (x_4, x_5, x_6, x_1, x_2, x_3).\end{aligned} \quad (2)$$

We will determine the effects of symmetric coupling between parallel copies of a network structure in the presence of delays. In the following, we focus on the symmetric properties of (1). Let $C([-\tau, 0], R^6)$ denote the Banach space of continuous mapping from $[-\tau, 0]$ to R^6 equipped with the supremum norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$ for $\varphi \in C([-\tau, 0], R^6)$. Let $\sigma \in R, A \geq 0, X : [\sigma - \tau, \sigma + A] \to R^6$, $t \in [\sigma, \sigma + A]$ be defined by $X_t(\theta) = X(t + \theta)$ for $-\tau \leq \theta \leq 0$. Define the mapping $f : C([-\tau, 0], R^6) \to R^6$ by

$$[f(\varphi)] = \begin{pmatrix} -\varphi_1(0) + \tanh(\varphi_2(0)) + \tanh(\varphi_4(-\tau))\\ -\varphi_2(0) + \tanh(\varphi_3(0)) + \tanh(\varphi_5(-\tau))\\ -\varphi_3(0) + \tanh(\varphi_1(0)) + \tanh(\varphi_6(-\tau))\\ -\varphi_4(0) + \tanh(\varphi_5(0)) + \tanh(\varphi_1(-\tau))\\ -\varphi_5(0) + \tanh(\varphi_6(0)) + \tanh(\varphi_2(-\tau))\\ -\varphi_6(0) + \tanh(\varphi_4(0)) + \tanh(\varphi_3(-\tau)) \end{pmatrix}, \quad (3)$$

where $\varphi \in C([-\tau, 0], R^6)$.

It is clear that (1) has symmetric group $\Gamma = Z_3 \times Z_2$, which means the global symmetry Z_2 and internal symmetry Z_3.

In the next section we focus on the linear stability analysis of the trivial equilibrium. This then leads us to a discussion of the bifurcations of the trivial equilibrium. In Section 3, we present a characterization of all possible periodic solutions, their twisted isotropy subgroups, and corresponding fixed-point subspaces. We obtain some important results about spontaneous bifurcations of multiple branches of periodic solutions and their spatiotemporal patterns, which describe the oscillatory mode of each neuron. Finally, some numerical simulations are carried out to support the analysis results.

2. Elementary Analysis

It is clear that (0,0,0,0,0,0) is an equilibrium point of (1). The linearization of (1) at the origin leads to

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + bx_2(t) + cx_4(t-\tau),\\ \dot{x}_2(t) &= -x_2(t) + bx_3(t) + cx_5(t-\tau),\\ \dot{x}_3(t) &= -x_3(t) + bx_1(t) + cx_6(t-\tau),\\ \dot{x}_4(t) &= -x_4(t) + bx_5(t) + cx_1(t-\tau),\\ \dot{x}_5(t) &= -x_5(t) + bx_6(t) + cx_2(t-\tau),\\ \dot{x}_6(t) &= -x_6(t) + bx_4(t) + cx_3(t-\tau).\end{aligned} \quad (4)$$

The associated characteristic equation of (4) takes the form

$$\det(\Delta(\lambda, \tau)) = 0, \quad (5)$$

where

$$\Delta(\lambda, \tau) = \lambda I_6 - \mathrm{Circ}(A_0, A_1),$$
$$A_0 = \begin{pmatrix} 0 & b & 0\\ 0 & 0 & b\\ b & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} c & 0 & 0\\ 0 & c & 0\\ 0 & 0 & c \end{pmatrix} e^{-\lambda\tau}. \quad (6)$$

Rewrite (4) as

$$\dot{U}(t) = LU(t) \quad (7)$$

with

$$L\varphi = \begin{pmatrix} -\varphi_1(0) + b\varphi_2(0) + c\varphi_4(-\tau)\\ -\varphi_2(0) + b\varphi_3(0) + c\varphi_5(-\tau)\\ -\varphi_3(0) + b\varphi_1(0) + c\varphi_6(-\tau)\\ -\varphi_4(0) + b\varphi_5(0) + c\varphi_1(-\tau)\\ -\varphi_5(0) + b\varphi_6(0) + c\varphi_2(-\tau)\\ -\varphi_6(0) + b\varphi_4(0) + c\varphi_3(-\tau) \end{pmatrix}^T. \quad (8)$$

The infinitesimal generator of the C_0-semigroup generated by linear system (4) is $\mathscr{A}(\tau)$ with

$$\begin{aligned}\mathscr{A}(\tau)\phi &= \dot{\phi}, \quad \phi \in \mathrm{Dom}(\mathscr{A}(\tau)),\\ \mathrm{Dom}(\mathscr{A}(\tau)) &= \left\{\phi \in C, \dot{\phi} \in C, \dot{\phi}(0) = L(\tau)\phi\right\}.\end{aligned} \quad (9)$$

Regarding τ as the parameter, we determine when the infinitesimal generator $A(\tau)$ of the C^0-semigroup generated by linear system (7) has a pair of pure imaginary eigenvalues.

Using Lemma 2.1 in [11], the characteristic equation then factors as

$$\begin{aligned}\Delta(\lambda) &= \left[\lambda + 1 - b - ce^{-\lambda\tau}\right]\left[\lambda + 1 - be^{(2\pi/3)i} - ce^{-\lambda\tau}\right] \\ &\quad\times\left[\lambda + 1 - be^{(4\pi/3)i} - ce^{-\lambda\tau}\right] \\ &\quad\times\left[\lambda + 1 - b + ce^{-\lambda\tau}\right]\left[\lambda + 1 - be^{(2\pi/3)i} + ce^{-\lambda\tau}\right] \\ &\quad\times\left[\lambda + 1 - be^{(4\pi/3)i} + ce^{-\lambda\tau}\right] \\ &= \Delta_1\Delta_2\Delta_3\Delta_4\Delta_5\Delta_6 = 0.\end{aligned} \tag{10}$$

It is not difficult to verify that $a + bi$ is a root of $\Delta_2 = 0$ or $\Delta_5 = 0$ if and only if $a - bi$ is a root of $\Delta_4 = 0$ or $\Delta_6 = 0$.

In order to study the distribution of zeros of (10), it is sufficient to investigate $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_4 = 0$, and $\Delta_5 = 0$. We make the following assumption:

(H_1): $|b - 1| < c$;

(H_2): $|1 + (b/2)| < c$.

If the assumptions (H_1), (H_2) hold, then the roots of $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_4 = 0$, and $\Delta_5 = 0$ have negative real parts when $\tau = 0$. In the sequel, we consider the distribution of zeros of $\Delta = 0$.

Case 1 ($\Delta_1 = 0$). Let $i\omega$ ($\omega > 0$) be a zero of Δ_1; then the critical frequency is identified as

$$\omega_1 = \sqrt{c^2 - (1 - b)^2}, \tag{11}$$

and the critical delay is

$$\tau_k^1 = \begin{cases} \dfrac{1}{\omega_1}\left[2k\pi + 2\pi - \arccos\left(\dfrac{1-b}{c}\right)\right], & 1 - b > 0; \\ \dfrac{1}{\omega_1}\left[2k\pi + \pi + \arccos\left(\dfrac{1-b}{c}\right)\right], & 1 - b < 0; \end{cases} \quad k = 0, 1, 2, \ldots. \tag{12}$$

Moreover, we differentiate the equality $\Delta_1 = 0$ with respect to τ to get

$$\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k^1,\omega=\omega_1} = \frac{1-b}{(1-b)^2 + (\omega_1)^2} \begin{cases} > 0, & 1 - b > 0; \\ < 0, & 1 - b < 0; \end{cases} \quad k = 0, 1, 2, \ldots. \tag{13}$$

Next, we consider the generalized eigenspace corresponding to pure imaginary eigenvalues of $\mathscr{A}(\tau)$.

Let assumptions (H_1) and (H_2) hold such that (10) has roots $\pm i\omega_1$ when $\tau = \tau_1^k$. Using Theorem 2.1 in [11], we have the generalized eigenspace $U_{\pm i\omega_1}$ consisting of eigenvectors of $\mathscr{A}(\tau_1^k)$ corresponding to $\pm i\omega_1$ is

$$U_{\pm i\omega_1} = \left\{\sum_{r=1}^{2} x_r\varsigma_r,\ x_r \in R\right\}, \tag{14}$$

where

$$\begin{aligned}\varsigma_1(\theta) &= \cos(\omega_1\theta)\operatorname{Re}\{V_1\} - \sin(\omega_1\theta)\operatorname{Im}\{V_1\}, \\ \varsigma_2(\theta) &= \sin(\omega_1\theta)\operatorname{Re}\{V_1\} + \cos(\omega_1\theta)\operatorname{Im}\{V_1\}, \\ V_1 &= (1,1,1,1,1,1)^T, \quad \theta \in [-1, 0].\end{aligned} \tag{15}$$

Case 2 ($\Delta_2 = 0$). Letting $i\omega$ ($\omega \neq 0$) be a zero of Δ_2, then

$$\begin{aligned}\omega_2^+ &= \frac{\sqrt{3}}{2}b + \sqrt{c^2 - 1 + \frac{b^2}{2}}; \\ \omega_2^- &= \frac{\sqrt{3}}{2}b - \sqrt{c^2 - 1 + \frac{b^2}{2}},\end{aligned} \tag{16}$$

$$\tau_k^{2+} = \begin{cases} \dfrac{1}{\omega_2^+}\left[2k\pi + 2\pi - \arccos\left(\dfrac{1+(b/2)}{c}\right)\right], & 1 + \dfrac{b}{2} > 0; \\ \dfrac{1}{\omega_2^+}\left[2k\pi + \pi + \arccos\left(\dfrac{1+(b/2)}{c}\right)\right], & 1 + \dfrac{b}{2} < 0; \end{cases} \quad k = 0, 1, 2, \ldots,$$

$$\tau_k^{2-} = \begin{cases} \dfrac{1}{\omega_2^-}\left[2k\pi + \arccos\left(\dfrac{1+(b/2)}{-c}\right)\right], & 1 + \dfrac{b}{2} > 0; \\ \dfrac{1}{\omega_2^-}\left[2k\pi + \pi - \arccos\left(\dfrac{1+(b/2)}{-c}\right)\right], & 1 + \dfrac{b}{2} < 0; \end{cases} \quad k = 0, 1, 2, \ldots. \tag{17}$$

For further analysis, we found that the transversality conditions are met:

$$\begin{aligned}\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k^{2+},\omega=\omega_2^+} &= \frac{\omega_2^+ - (\sqrt{3}/2)b}{\omega_2^+\left((1+(b/2))^2 + \left(\omega_2^+ - (\sqrt{3}/2)b\right)^2\right)} > 0, \quad k = 0, 1, 2, \ldots, \\ \operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k^{2-},\omega=\omega_2^-} &= \frac{\omega_2^- - (\sqrt{3}/2)b}{\omega_2^-\left((1+(b/2))^2 + \left(\omega_2^- - (\sqrt{3}/2)b\right)^2\right)} < 0, \quad k = 0, 1, 2, \ldots.\end{aligned} \tag{18}$$

The generalized eigenspace $U_{\pm i\omega_2}$ consisting of eigenvectors of $\mathscr{A}(\tau_2^{k\pm})$ corresponding to $\pm i\omega_2^{\pm}$ is

$$U_{\pm i\omega_2^{\pm}} = \left\{\sum_{r=1}^{4} x_r \varepsilon_r,\ x_r \in R\right\}, \tag{19}$$

where

$$\begin{gathered}\varepsilon_1(\theta) = \cos(\omega_2^{\pm}\theta)\operatorname{Re}\{V_2\} - \sin(\omega_2^{\pm}\theta)\operatorname{Im}\{V_2\},\\ \varepsilon_2(\theta) = \sin(\omega_2^{\pm}\theta)\operatorname{Re}\{V_2\} + \cos(\omega_2^{\pm}\theta)\operatorname{Im}\{V_2\},\\ \varepsilon_3(\theta) = \cos(\omega_2^{\pm}\theta)\operatorname{Re}\{V_3\} - \sin(\omega_3^{\pm}\theta)\operatorname{Im}\{V_2\},\\ \varepsilon_4(\theta) = \sin(\omega_2^{\pm}\theta)\operatorname{Re}\{V_3\} + \cos(\omega_3^{\pm}\theta)\operatorname{Im}\{V_3\},\\ V_2 = \left(1, e^{-2\pi/3}e^{-4\pi/3}, 1, e^{-2\pi/3}, e^{-4\pi/3}\right)^T,\\ V_3 = \left(1, e^{-4\pi/3}, e^{-8\pi/3}, 1, e^{-4\pi/3}, e^{-8\pi/3}\right)^T,\\ \theta \in [-1, 0].\end{gathered} \tag{20}$$

In a similar manner it can be shown that, for the fourth factor, $\Delta_4 = 0$, and fifth factor, $\Delta_5 = 0$, we have the following.

Case 3 ($\Delta_4 = 0$). In this case,

$$\begin{gathered}\omega_4 = \sqrt{c^2 - (1-b)^2};\\ \tau_k^4 = \begin{cases}\dfrac{1}{\omega_4}\left[2k\pi + \pi - \arccos\left(\dfrac{c}{1-b}\right)\right], & 1-b > 0;\\ \dfrac{1}{\omega_4}\left[2k\pi + \arccos\left(\dfrac{-c}{1-b}\right)\right], & 1-b < 0;\end{cases}\\ k = 0, 1, 2, \ldots,\end{gathered} \tag{21}$$

and the transversality conditions are also met:

$$\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k^4,\omega=\omega_4} = \frac{1-b}{(1-b)^2 + (\omega_4)^2}\begin{cases}>0, & 1-b>0;\\ <0, & 1-b<0;\end{cases}\quad k = 0, 1, 2, \ldots. \tag{22}$$

The generalized eigenspace $U_{\pm i\omega_4}$ consisting of eigenvectors of $\mathscr{A}(\tau_k^4)$ corresponding to $\pm i\omega_4$ is

$$U_{\pm i\omega_4} = \left\{\sum_{r=1}^{2} x_r \varsigma_r,\ x_r \in R\right\}, \tag{23}$$

where

$$\begin{gathered}\varsigma_1(\theta) = \cos(\omega_4\theta)\operatorname{Re}\{V_4\} - \sin(\omega_4\theta)\operatorname{Im}\{V_4\},\\ \varsigma_2(\theta) = \sin(\omega_4\theta)\operatorname{Re}\{V_4\} + \cos(\omega_4\theta)\operatorname{Im}\{V_4\},\\ V_4 = (1,1,1,1,1,1)^T, \quad \theta \in [-1, 0].\end{gathered} \tag{24}$$

Case 4 ($\Delta_5 = 0$). Using the same method of case two, we have

$$\begin{gathered}\omega_5^+ = \frac{\sqrt{3}}{2}b + \sqrt{c^2 - \left(1 + \frac{b}{2}\right)^2},\\ \omega_5^- = \frac{\sqrt{3}}{2}b - \sqrt{c^2 - \left(1 + \frac{b}{2}\right)^2},\\ \tau_k^{5+} = \begin{cases}\dfrac{1}{\omega_5^+}\left[2k\pi + \pi - \arccos\left(\dfrac{1+(b/2)}{-c}\right)\right], & 1 + \dfrac{b}{2} > 0;\\ \dfrac{1}{\omega_5^+}\left[2k\pi + \arccos\left(\dfrac{1+(b/2)}{-c}\right)\right], & 1 + \dfrac{b}{2} < 0;\end{cases}\\ k = 0, 1, 2, \ldots,\\ \tau_k^{5-} = \begin{cases}\dfrac{1}{\omega_5^-}\left[2k\pi + 2\pi - \arccos\left(\dfrac{1+(b/2)}{-c}\right)\right], & 1 + \dfrac{b}{2} > 0;\\ \dfrac{1}{\omega_5^-}\left[2k\pi + \pi + \arccos\left(\dfrac{1+(b/2)}{-c}\right)\right], & 1 + \dfrac{b}{2} < 0;\end{cases}\\ k = 0, 1, 2, \ldots,\\ \operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k^{5+},\omega=\omega_5^+} = \frac{\omega_5^+ - (\sqrt{3}/2)b}{\omega_5^+\left((1+(b/2))^2 + \left(\omega_5^+ - (\sqrt{3}/2)b\right)^2\right)} > 0,\\ k = 0, 1, 2, \ldots,\\ \operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k^{5-},\omega=\omega_5^-} = \frac{\omega_5^- - (\sqrt{3}/2)b}{\omega_5^-\left((1+(b/2))^2 + \left(\omega_5^- - (\sqrt{3}/2)b\right)^2\right)} < 0,\\ k = 0, 1, 2, \ldots.\\ U_{\pm i\omega_5^{\pm}} = \left\{\sum_{r=1}^{4} x_r \epsilon_r, x_r \in R\right\},\end{gathered} \tag{25}$$

where

$$\begin{gathered}\epsilon_1(\theta) = \cos(\omega_5^{\pm}\theta)\operatorname{Re}\{V_5\} - \sin(\omega_5^{\pm}\theta)\operatorname{Im}\{V_5\},\\ \epsilon_2(\theta) = \sin(\omega_5^{\pm}\theta)\operatorname{Re}\{V_5\} + \cos(\omega_5^{\pm}\theta)\operatorname{Im}\{V_5\},\\ \epsilon_3(\theta) = \cos(\omega_5^{\pm}\theta)\operatorname{Re}\{V_6\} - \sin(\omega_5^{\pm}\theta)\operatorname{Im}\{V_6\},\end{gathered}$$

TABLE 1: The twisted isotropy subgroups for Γ-equivariant system (1).

τ	Group action	Twisted isotropy subgroups	Fixed-point subspaces
τ_k^1	$\rho z = z;\ \kappa z = z$	$\sum(\rho,\kappa)$	$\varsigma_1(\theta), \varsigma_2(\theta)$
$\tau_k^{2\pm}$	$\rho z = e^{i2\pi/3}z;\ \kappa z = z;$ $\rho z = e^{i4\pi/3}z;\ \kappa z = z;$	$\sum\left(\rho e^{i2\pi/3},\kappa\right);$ $\sum\left(\rho e^{i4\pi/3},\kappa\right);$	$\varepsilon_1(\theta), \varepsilon_2(\theta), \varepsilon_3(\theta), \varepsilon_4(\theta)$.
$\tau_k^{4\pm}$	$\rho z = z;\ \kappa z = -z$	$\sum(\rho,-\kappa)$	$\varsigma_3(\theta), \varsigma_4(\theta)$
$\tau_k^{5\pm}$	$\rho z = e^{i2\pi/3}z;\ \kappa z = -z;$ $\rho z = e^{i4\pi/3}z;\ \kappa z = -z;$	$\sum\left(\rho e^{i2\pi/3},-\kappa\right);$ $\sum\left(\rho e^{i4\pi/3},-\kappa\right);$	$\epsilon_1(\theta), \epsilon_2(\theta), \epsilon_3(\theta), \epsilon_4(\theta)$

TABLE 2: Bifurcating periodic solutions.

Twisted isotropy subgroups $\sum$	Periodic solutions
$\sum(\rho,\kappa)$	$(x(t), x(t), x(t), x(t), x(t), x(t))$
$\sum\left(\rho e^{i2\pi/3},\kappa\right)$	$\left(x(t), x\left(t+\frac{T}{3}\right), x\left(t+\frac{2T}{3}\right), x(t), x\left(t+\frac{T}{3}\right), x\left(t\frac{2T}{3}\right)\right)$
$\sum\left(\rho e^{i4\pi/3},\kappa\right)$	$\left(x(t), x\left(t+\frac{2T}{3}\right), x\left(t+\frac{4T}{3}\right), x(t), x\left(t+\frac{2T}{3}\right), x\left(t\frac{4T}{3}\right)\right)$
$\sum(\rho,-\kappa)$	$(x(t), x(t), x(t), -x(t), -x(t), -x(t))$
$\sum\left(\rho e^{i2\pi/3},-\kappa\right)$	$\left(x(t), x\left(t+\frac{T}{3}\right), x\left(t+\frac{2T}{3}\right), x\left(t+\frac{T}{2}\right), x\left(t+\frac{5T}{6}\right), x\left(t\frac{7T}{6}\right)\right)$
$\sum\left(\rho e^{i4\pi/3},-\kappa\right)$	$\left(x(t), x\left(t+\frac{2T}{3}\right), x\left(t+\frac{T}{3}\right), x\left(t+\frac{T}{2}\right), x\left(t+\frac{7T}{6}\right), x\left(t+\frac{5T}{6}\right)\right)$

$$\begin{aligned}\epsilon_4(\theta) &= \sin\left(\omega_5^{\pm}\theta\right)\operatorname{Re}\{V_6\} + \cos\left(\omega_5^{\pm}\theta\right)\operatorname{Im}\{V_6\},\\ V_5 &= \left(1, e^{-2\pi/3}, e^{-4\pi/3}, -1, -e^{-2\pi/3}, -e^{-4\pi/3}\right)^T,\\ V_6 &= \left(1, e^{-4\pi/3}, e^{-8\pi/3}, -1, -e^{-4\pi/3}, -e^{-8\pi/3}\right)^T,\\ &\theta\in[-1,0].\end{aligned} \tag{26}$$

3. Multiple Hopf Bifurcations

In order to study the Hopf bifurcation of the origin, we consider the action of $\Gamma\times S^1$, where $\Gamma = Z_2\times Z_3$ and S^1 is the temporal. The action of the group S^1 is defined as follows:

$$\begin{aligned}&\theta\left(x_1, x_2, x_3, x_4, x_5, x_6\right)\\ &\quad = \left(e^{i\theta}x_1, e^{i\theta}x_2, e^{i\theta}x_3, e^{i\theta}x_4, e^{i\theta}x_5, e^{i\theta}x_6\right),\end{aligned} \tag{27}$$

where $\theta\in S^1$. It is clear that

$$\Gamma = Z_3\times Z_2 = \left\{1, \rho, \rho^2, \kappa, \kappa\rho, \kappa\rho^2\right\}. \tag{28}$$

For fixed k, j, let $T = 2\pi/\omega_{\pm}$. Denote by P_T the Banach space of all continuous T-periodic solutions. Then $\Gamma\times S^1$ acts on P_T by

$$(\gamma,\theta)\,x(t) = \gamma x(t+\theta), \quad (\gamma,\theta)\in\Gamma\times S^1,\ x\in P_T. \tag{29}$$

Denote by SP_T the subspace of P_T consisting of all T-periodic solutions of (4) with $\tau = \tau_k^{j\pm}$ $(j = 1,2,4,5)$. Then, for each subgroup $\Sigma\le\Gamma\times S^1$,

$$\operatorname{Fix}\left(\Sigma, SP_T\right) = \left\{x\in SP_T; (r,\theta)\,x = x\ \forall(\gamma,\theta)\in\Sigma\right\} \tag{30}$$

is a subspace.

In the following, by discussing the isotropy subgroup and fixed-point subspaces, we will give the possible bifurcating solutions. From Section 2, we have obtained the generalized eigenspace corresponding to pure imaginary eigenvalues of $\mathscr{A}$ $(\tau_k^{j\pm},\ j = 1,2,4,5;\ k = 0,1,\ldots)$. Hence, we know their corresponding isotropy subgroup; see Table 1.

The equivariant bifurcation theorem asserts the existence of branches of small amplitude periodic solutions to system (1), whose spatiotemporal symmetries can be completely characterized by isotropy subgroup.

In case one, $\Delta_1 = 0$ implies that the purely imaginary eigenvalues associated with Hopf bifurcation are simple. It follows that the action of $Z_2\times Z_3\times S^1$ is given by $\rho z = z; \kappa z = z$. Obviously, the maximal isotropy subgroup is $Z_2\times Z_3$, which corresponds to standard Hopf bifurcation and is preserved. Thus, all neurons in two rings are synchronous:

(1) $(x(t), x(t), x(t), x(t), x(t), x(t))$.

Similar to the analysis in Case 2, $\Delta_2 = 0$ implies that the purely imaginary eigenvalues associated with Hopf bifurcation are double. $\sum(\rho e^{i2\pi/3},\kappa)$ and $\sum(\rho e^{i4\pi/3},\kappa)$ are maximal isotropy subgroups of $Z_2\times Z_3$ which are generated by $\rho z = e^{i2\pi/3}z; \kappa z = z$ and $\rho z = e^{i4\pi/3}z; \kappa z = z$. Two types of symmetric periodic solutions are generated:

(2) $(x(t), x(t+(T/3)), x(t+(2T/3)), x(t), x(t+(T/3)), x(t(2T/3)))$;

(3) $(x(t), x(t+(2T/3)), x(t+(4T/3)), x(t), x(t+(2T/3)), x(t(4T/3)))$.

For Case 3, $\Delta_4 = 0$ means the purely imaginary eigenvalues associated with Hopf bifurcation are simple, and the

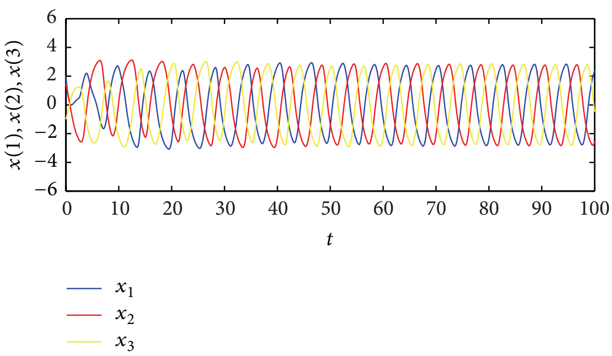

FIGURE 2: Three adjacent neurons $x_1(t), x_2(t), x_3(t)$ are $2T/3$ out of phase with $\tau = 1.45$ and initial condition $(2, 1.5, -1, -0.4, -1.5, 1.8)$.

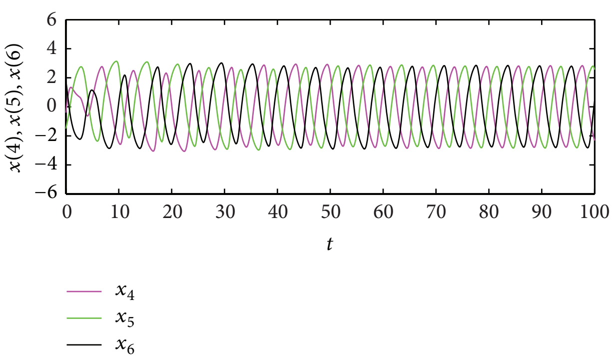

FIGURE 3: Three adjacent neurons $x_4(t), x_5(t), x_6(t)$ are $2T/3$ out of phase with $\tau = 1.45$ and initial condition $(2, 1.5, -1, -0.4, -1.5, 1.8)$.

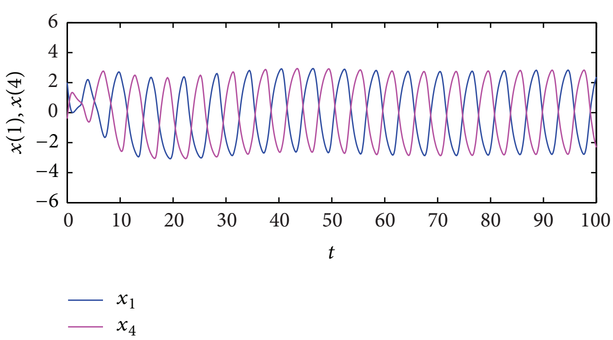

FIGURE 4: Two neurons $x_1(t), x_4(t)$ in different rings are $T/2$ out of phase with $\tau = 1.45$ and initial condition $(2, 1.5, -1, -0.4, -1.5, 1.8)$.

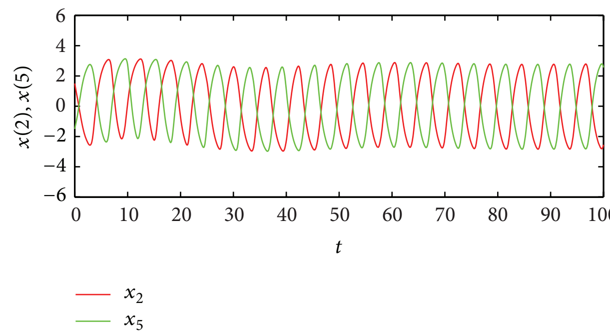

FIGURE 5: Two neurons $x_2(t), x_5(t)$ in different rings are $T/2$ out of phase with $\tau = 1.45$ and initial condition $(2, 1.5, -1, -0.4, -1.5, 1.8)$.

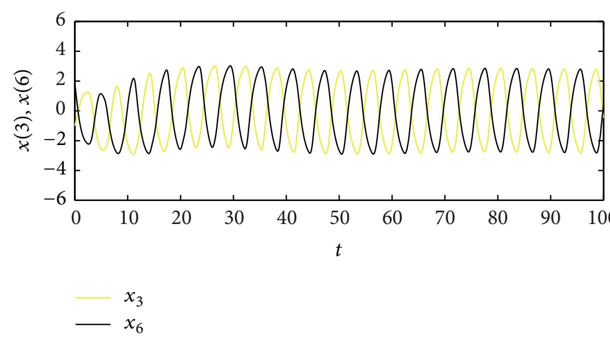

FIGURE 6: Two neurons $x_3(t), x_6(t)$ in different rings are $T/2$ out of phase with $\tau = 1.45$ and initial condition $(2, 1.5, -1, -0.4, -1.5, 1.8)$.

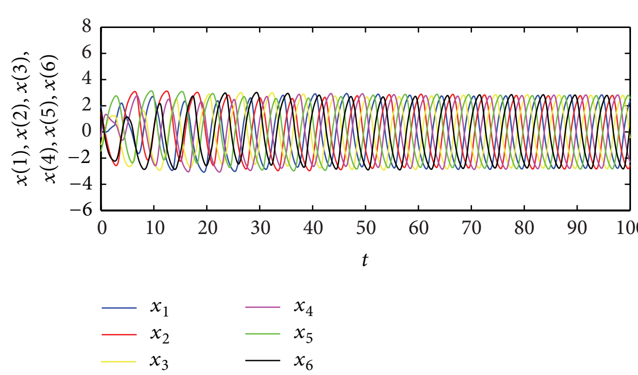

FIGURE 7: Neurons in different rings are $T/2$ out of phase with each other, and each neuron is $2T/3$ out of phase with the adjacent neuron when $\tau = 1.45$ and initial condition $(2, 1.5, -1, -0.4, -1.5, 1.8)$.

maximal isotropy subgroup is $\sum(\rho, -\kappa)$ and the symmetric periodic solutions have the form

(4) $(x(t), x(t), x(t), -x(t), -x(t), -x(t))$.

That means neurons in different rings are $T/2$ out of phase with each other, and all neurons are $2T/3$ out of phase with the adjacent behaving identically in the same ring.

The fourth case, $\Delta_5 = 0$, gives purely imaginary with double. The maximal isotropy subgroup has two types: $\sum(\rho e^{i2\pi/3}, -\kappa)$ and $\sum(\rho e^{i4\pi/3}, -\kappa)$, so the symmetric periodic solutions have the form

(5) $(x(t), x(t + (T/3)), x(t + (2T/3)), x(t + (T/2)), x(t + (5T/6)), x(t(7T/6)))$;

(6) $(x(t), x(t + (2T/3)), x(t + (T/3)), x(t + (T/2)), x(t + (7T/6)), x(t + (5T/6)))$.

In summary, we write the results in Table 2.

4. Computer Simulation

To illustrate the analytical results found, in the following we consider the following particular case of (1).

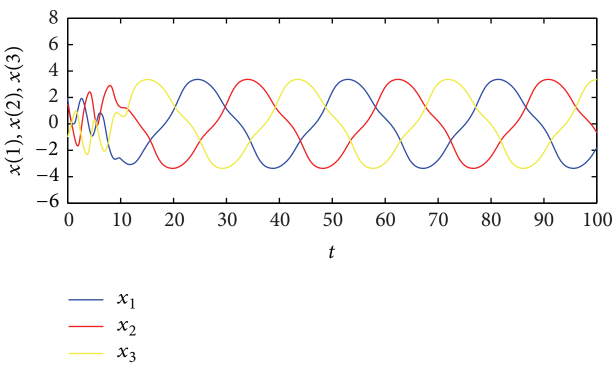

FIGURE 8: Three adjacent neurons $x_1(t)$, $x_2(t)$, $x_3(t)$ are $2T/3$ out of phase with $\tau = 3.3$ and initial condition (2, 1.5, −1, −0.4, −1.5, 1.8).

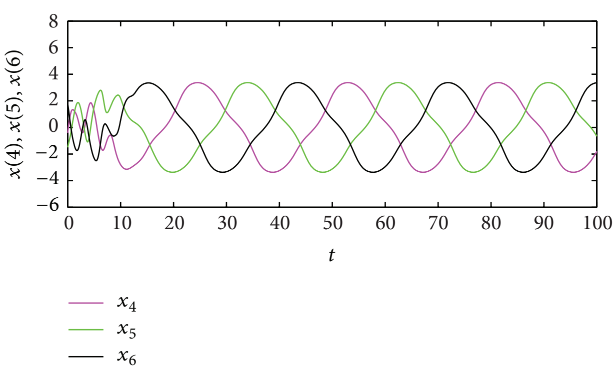

FIGURE 9: Three adjacent neurons $x_4(t)$, $x_5(t)$, $x_6(t)$ are $2T/3$ out of phase with $\tau = 3.3$ and initial condition (2, 1.5, −1, −0.4, −1.5, 1.8).

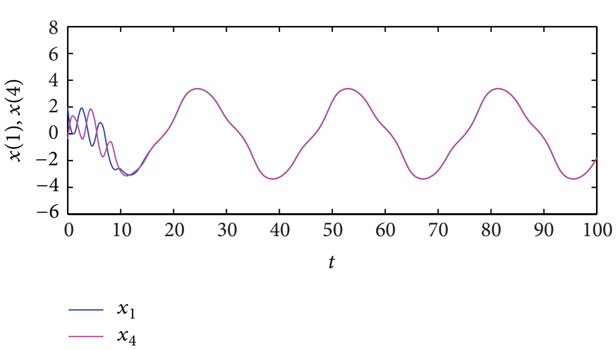

FIGURE 10: Two neurons $x_1(t)$, $x_4(t)$ in different rings behave identically with $\tau = 3.3$ and initial condition (2, 1.5, −1, −0.4, −1.5, 1.8).

Let $b = -0.5$, $c = 2$. Then $\omega_2^+ = \omega_5^+ = 1.421$, $\tau_0^{2+} = 3.585$, $\tau_0^{5+} = 1.375$.

From Table 2, the spatiotemporal patterns of bifurcating periodic oscillations alternate according to the change of the propagation time delay. See Figures 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and 13.

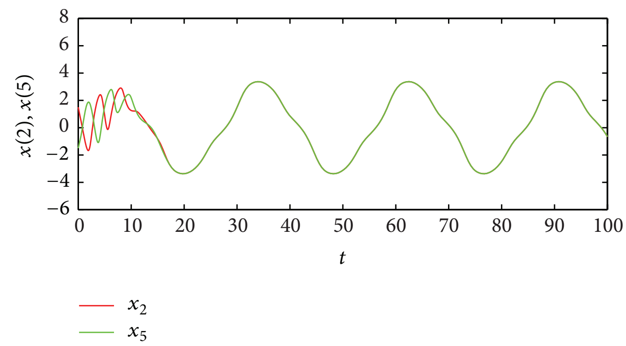

FIGURE 11: Two neurons $x_2(t)$, $x_5(t)$ in different rings behave identically with $\tau = 3.3$ and initial condition (2, 1.5, −1, −0.4, −1.5, 1.8).

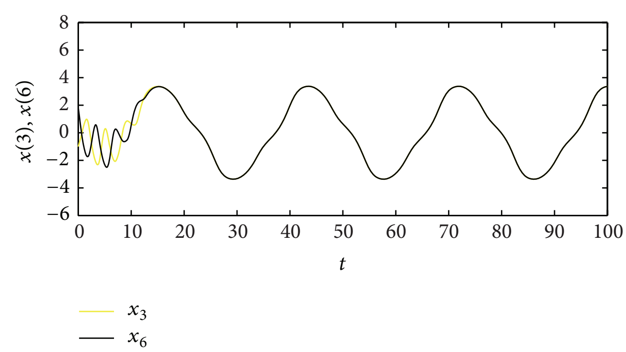

FIGURE 12: Two neurons $x_3(t)$, $x_6(t)$ in different rings behave identically with $\tau = 3.3$ and initial condition (2, 1.5, −1, −0.4, −1.5, 1.8).

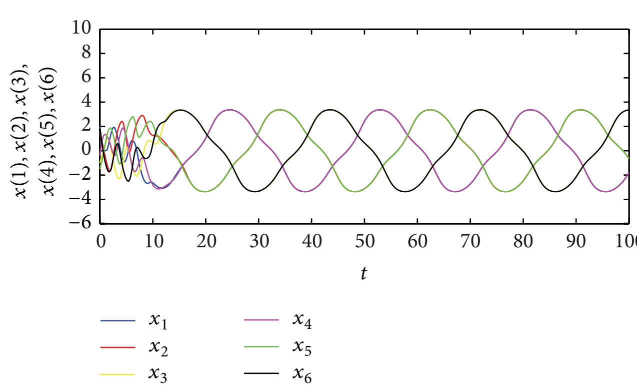

FIGURE 13: Neurons in different rings behave identically, and each neuron is $2T/3$ out of phase with the adjacent neuron when $\tau = 3.3$ and initial condition (2, 1.5, −1, −0.4, −1.5, 1.8).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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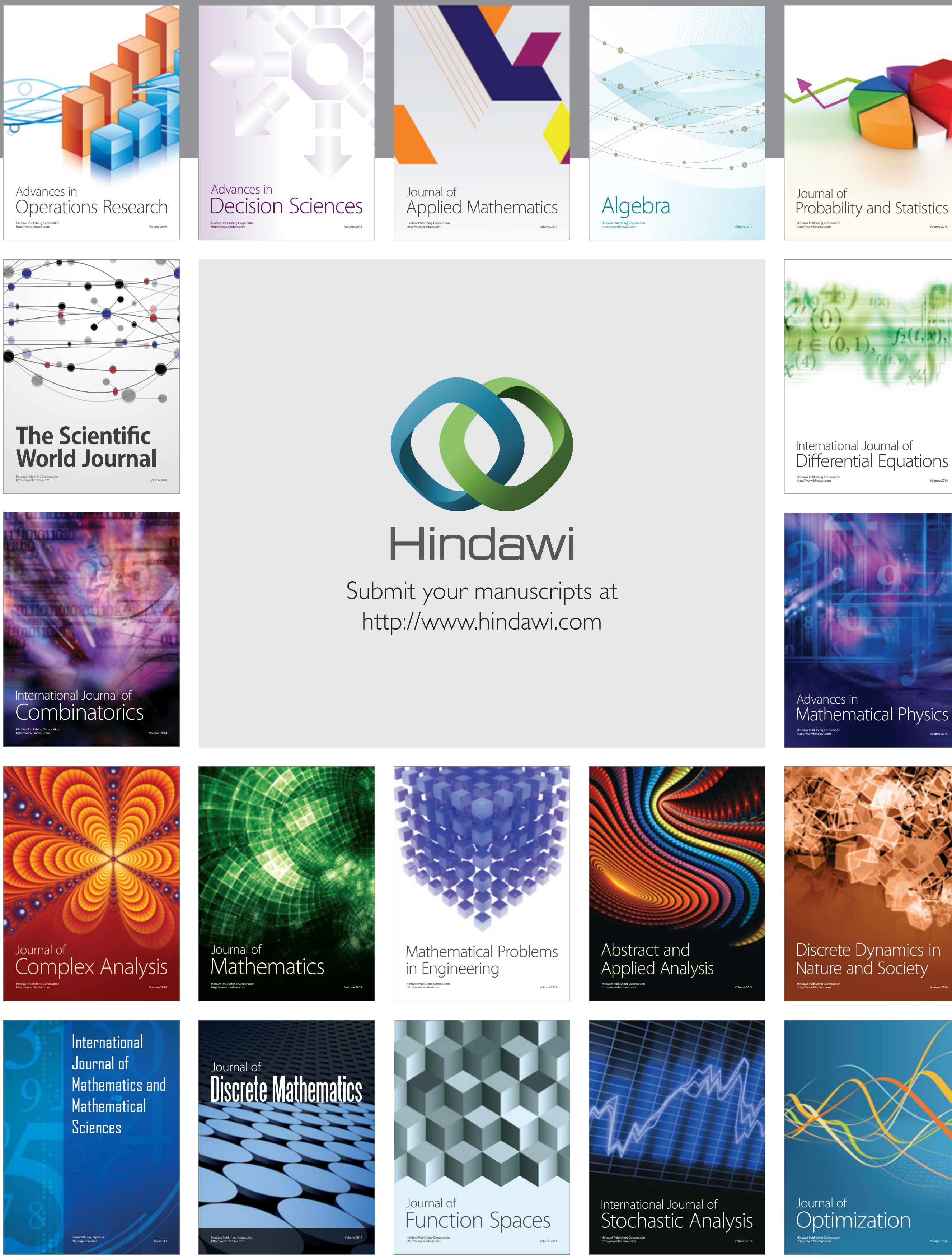
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