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Research Article

A Nonhomogeneous Dirichlet Problem for a Nonlinear Pseudoparabolic Equation Arising in the Flow of Second-Grade Fluid

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We study the following initial-boundary value problem $\{u_t-(\mu(t)+\alpha(t)(\partial/\partial t))(\partial^2 u/\partial x^2+(\gamma/x)(\partial u/\partial x))+f(u)=f_1(x,t),\ 1<x<R,\ t>0;\ u(1,t)=g_1(t),\ u(R,t)=g_R(t);\ u(x,0)=\tilde{u}_0(x)\}$, where $\gamma>0$, $R>1$ are given constants and $f, f_1, g_1, g_R, \tilde{u}_0, \alpha$, and μ are given functions. In Part 1, we use the Galerkin method and compactness method to prove the existence of a unique weak solution of the problem above on $(0,T)$, for every $T>0$. In Part 2, we investigate asymptotic behavior of the solution as $t\to+\infty$. In Part 3, we prove the existence and uniqueness of a weak solution of problem $\{u_t-(\mu(t)+\alpha(t)(\partial/\partial t))(\partial^2 u/\partial x^2+(\gamma/x)(\partial u/\partial x))+f(u)=f_1(x,t),\ 1<x<R,\ t>0;\ u(1,t)=g_1(t),\ u(R,t)=g_R(t)\}$ associated with a "(η,T)-periodic condition" $u(x,0)=\eta u(x,T)$, where $0<|\eta|\le 1$ is given constant.

1. Introduction

In this paper, we consider the following nonlinear pseudoparabolic equation:

$$u_t-\left(\mu(t)+\alpha(t)\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\gamma}{x}\frac{\partial u}{\partial x}\right)+f(u)=f_1(x,t),\quad 1<x<R,\ t>0, \tag{1}$$

associated with the boundary conditions

$$u(1,t)=g_1(t),\qquad u(R,t)=g_R(t) \tag{2}$$

and the initial condition

$$u(x,0)=\tilde{u}_0(x), \tag{3}$$

or the "(η,T)-periodic condition"

$$u(x,0)=\eta u(x,T), \tag{4}$$

where $\gamma>0$, $R>1$, and $0<|\eta|\le 1$ are given constants and f, f_1, g_1, g_R, $\tilde{u}_0$, α, and μ are given functions satisfying conditions specified later.

In the case of $\gamma=1$, $\mu(t)=\mu>0$, and $\alpha(t)=\alpha>0$ being the constants, the initial-boundary value problems (1)–(3) are classical and have a long history of applications and mathematical development. We refer to the monographs of Al'shin et al. [1] and of Carroll and Showalter [2] for references and results on pseudoparabolic or Sobolev type equations. We also refer to [3] for asymptotic behavior and to[4] for nonlinear problems. Problems of this type arise in material science and physics, which have been extensively studied, and several results concerning existence, regularity, and asymptotic behavior have been established.

Equation (1) arises within frameworks of mathematical models in engineering and physical sciences (see [5–11] for references therein and interesting results on second grade fluids or a fourth grade fluid or other unsteady flows). It is well known that fluid solid mixtures are generally considered as second-grade fluids and are modeled as fluids with variable physical parameters; thus, an analysis is performed for a second-grade fluid with space dependent viscosity, elasticity, and density.

In [9], some unsteady flow problems of a second-grade fluid were considered. The flows are generated by the sudden application of a constant pressure gradient or by the impulsive motion of a boundary. Here, the velocities of the flows are described by the partial differential equations and exact analytic solutions of these differential equations are obtained. Suppose that the second-grade fluid is in a circular cylinder and is initially at rest, and the fluid starts suddenly due to the motion of the cylinder parallel to its length. The axis of the cylinder is chosen as the z-axis. Using cylindrical polar coordinates, the governing partial differential equation is

$$\begin{gathered}\frac{\partial w}{\partial t}=\left(\nu+\alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)w(r,t)-Nw,\\ 0<r<a,\ t>0,\\ w(a,t)=W,\quad t>0,\\ w(r,0)=0,\quad 0\le r<a,\end{gathered}\tag{5}$$

where $w(r,t)$ is the velocity along the z-axis, ν is the kinematic viscosity, α is the material parameter, and N is the imposed magnetic field. In the boundary and initial conditions, W is the constant velocity at $r = a$ and a is the radius of the cylinder.

In [6], two types of time-dependent flows were investigated. An eigenfunction expansion method was used to find the velocity distribution. The obtained solutions satisfy the boundary and initial conditions and the governing equation. Remarkably, some exact analytic solutions are possible for flows involving second-grade fluid with variable material properties in terms of trigonometric and Chebyshev functions.

In [5], Mahmood et al. have considered the longitudinal oscillatory motion of second-grade fluid between two infinite coaxial circular cylinders, oscillating along their common axis with given constant angular frequencies Ω_1 and Ω_2. Velocity field and associated tangential stress of the motion were determined by using Laplace and Hankel transforms. In order to find exact analytic solutions for the flow of second-grade fluid between two longitudinally oscillating cylinders, the following problem was studied:

$$\begin{gathered}\frac{\partial v}{\partial t}=\left(\mu+\alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t),\\ R_1<r<R_2,\ t>0,\\ v(R_1,t)=V_1\sin(\Omega_1 t),\\ v(R_2,t)=V_2\sin(\Omega_2 t),\\ v(r,0)=0,\quad R_1\le r\le R_2,\end{gathered}\tag{6}$$

where $0 < R_1 < R_2$, μ, α, V_1, V_2, Ω_1, and Ω_2 are positive constants. The solutions obtained have been presented under series form in terms of Bessel functions $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$, $J_2(x)$, and $Y_2(x)$, satisfying the governing equation and all imposed initial and boundary conditions.

The nonlinear parabolic problems of the form (1)–(3), with/without the term $u_{rr} + (\gamma/r)u_r$, were also studied in [12, 13] and references therein. In [12], by using the Galerkin and compactness method in appropriate Sobolev spaces with weight, the authors proved the existence of a unique weak solution of the following initial and boundary value problem for nonlinear parabolic equation:

$$\begin{gathered}u_t-a(t)\left(u_{rr}+\frac{\gamma}{r}u_r\right)+F(r,u)=f(r,t),\\ 0<r<1,\ 0<t<T,\\ \left|\lim_{r\to 0_+}r^{\gamma/2}u_r(r,t)\right|<+\infty,\\ u_r(1,t)+h(t)\left(u(1,t)-\overline{u}_0\right)=0,\\ u(r,0)=u_0(r).\end{gathered}\tag{7}$$

Furthermore, asymptotic behavior of the solution as $t \to +\infty$ was studied. In [13], the following nonlinear heat equation associated with Dirichlet-Robin conditions was investigated:

$$\begin{gathered}u_t-\frac{\partial}{\partial x}\left[\mu(x,t)u_x\right]+f(u)=f_1(x,t),\\ (x,t)\in(0,1)\times(0,T),\\ u_x(0,t)=h_0u(0,t)+g_0(t),\\ -u_x(1,t)=h_1u(1,t)+g_1(t),\\ u(x,0)=u_0(x).\end{gathered}\tag{8}$$

Condition (4), which we call "(η, T)-periodic condition," is known as a drifted periodic condition (see [14]). Indeed, if $u(t) = \eta u(t + T)$, $\forall t \ge 0$, in the case of $0 < |\eta| \le 1$, then we have

$$u(t+T)=\frac{1}{\eta}u(t)=u(t)+\left(\frac{1}{\eta}-1\right)u(t),\quad \forall t\ge 0,\tag{9}$$

which means

$$u(t+T)=u(t)+\delta(t),\quad \forall t\ge 0,\tag{10}$$

with $\delta(t) = (1/\eta - 1)u(t)$ satisfying the condition

$$\delta(t)=\eta\delta(t+T),\quad \forall t\ge 0.\tag{11}$$

Note that (11) holds by the fact that

$$\begin{aligned}\eta\delta(t+T)&=\eta\left[\left(\frac{1}{\eta}-1\right)u(t+T)\right]=\left(\frac{1}{\eta}-1\right)u(t)\\ &=\delta(t),\quad \forall t\ge 0.\end{aligned}\tag{12}$$

With $\eta = 1$, (4) leads to T-periodic condition

$$u(x,0)=u(x,T),\tag{13}$$

and with $\eta = -1$, we have the antiperiodic condition

$$u(x,0) = -u(x,T). \qquad (14)$$

The present paper is concerned with the second-grade fluid in a circular cylinder associated with the initial condition (3) or a drifted periodic condition (10). The extensive study of such flows is motivated by both their fundamental interest and their practical importance (see [9]).

This paper is a continuation of paper [15] dealing with the nonlinear pseudoparabolic equation (1) associated with the mixed inhomogeneous condition, in the case of $\gamma = 1$, $\mu(t) = \mu > 0$, $\alpha(t) = \alpha > 0$ being the constants. It consists of five sections. First, preliminaries are done in Section 2. Under appropriate conditions, the existence of a unique weak solution of problems (1)–(3) is proved in Section 3. Next, an asymptotic behavior of the solution of problems (1)–(3), as $t \to +\infty$, is discussed in Section 4. Finally, Section 5 is devoted to the establishment, the existence, and uniqueness of a weak solution of problems (1), (2), and (4).

Because of mathematical context, the results obtained here generalize relatively the ones in [12, 13, 15], by improving the techniques used as before and with appropriate modifications.

2. Preliminaries

Put $\Omega = (1,R)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We omit the definitions of the usual function spaces: $C^m(\overline{\Omega})$, $L^p(\Omega)$, and $W^{m,p}(\Omega)$. We define $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$ and $H^m = W^{m,2}(\Omega)$, $1 \le p \le \infty$, $m = 0, 1, \ldots$. The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $(\cdot,\cdot)$ the scalar product in L^2. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X) 1 \le p \le \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty \quad \text{for } 1 \le p < \infty, \qquad (15)$$
$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Let $u(t), u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, and $u_{xx}(t)$ denote $u(x,t)$, $(\partial u/\partial t)(x,t)$, $(\partial^2 u/\partial t^2)(x,t)$, $(\partial u/\partial x)(x,t)$, $(\partial^2 u/\partial x^2)(x,t)$, respectively.

On H^1, we shall use the following norm:

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \qquad (16)$$

We put

$$H_0^1 = \left\{v \in H^1(\Omega) : v(1) = v(R) = 0\right\}. \qquad (17)$$

H_0^1 is a closed subspace of H^1 and on H_0^1, two norms $\|v\|_{H^1}$ and $\|v_x\|$ are equivalent.

Note that L^2 and H^1 are also the Hilbert spaces with respect to the corresponding scalar products

$$\langle u, v\rangle = \int_1^R x^\gamma u(x) v(x)\, dx, \quad \langle u, v\rangle + \langle u_x, v_x\rangle, \qquad (18)$$

respectively. The norms in L^2 and H^1 induced by the corresponding scalar products are denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. H_0^1 is continuously and densely embedded in L^2. Identifying L^2 with $(L^2)'$ (the dual of L^2), we have $H_0^1 \hookrightarrow L^2 \hookrightarrow (H_0^1)' = H^{-1}$; on the other hand, the notation $\langle\cdot,\cdot\rangle$ is used for the pairing between H_0^1 and H^{-1}.

We then have the following lemmas, the proofs of which can be found in [16].

Lemma 1. *We have the following inequalities:*

$$\begin{aligned}&\text{(i)}\ \|v\| \le \|v\|_0 \le \sqrt{R^\gamma}\,\|v\|, \quad \forall v \in L^2,\\ &\text{(ii)}\ \|v\|_{H^1} \le \|v\|_1 \le \sqrt{R^\gamma}\,\|v\|_{H^1}, \quad \forall v \in H^1.\end{aligned} \qquad (19)$$

Lemma 2. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact.*

Lemma 3. *The imbedding $H_0^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\begin{aligned}&\text{(i)}\ \|v\|_{C^0(\overline{\Omega})} \le \sqrt{R-1}\,\|v_x\| \quad \forall v \in H_0^1,\\ &\text{(ii)}\ \|v\| \le \frac{R-1}{\sqrt{2}}\,\|v_x\| \quad \forall v \in H_0^1,\\ &\text{(iii)}\ \|v\|_0 \le \sqrt{\frac{R^\gamma}{2}}\,(R-1)\,\|v_x\|_0 \quad \forall v \in H_0^1.\end{aligned} \qquad (20)$$

Remark 4. On L^2, two norms $v \mapsto \|v\|$ and $v \mapsto \|v\|_0$ are equivalent. So there are two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v\|_1$ on H^1 and four norms $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v\|_1$, $v \mapsto \|v_x\|$, and $v \mapsto \|v_x\|_0$ on H_0^1.

Consider $a(\cdot,\cdot)$ is the symmetric bilinear form on $H_0^1 \times H_0^1$ defined by

$$a(u,w) = \langle u_x, w_x\rangle, \quad \forall u, w \in H_0^1. \qquad (21)$$

Then, the symmetric bilinear form $a(\cdot,\cdot)$ is continuous on $H_0^1 \times H_0^1$ and coercive on H_0^1.

We have also the following lemma.

Lemma 5. *There exists the Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of the eigenfunctions w_j corresponding to the eigenvalue $\overline{\lambda}_j$ such that*

$$\begin{aligned}&0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \cdots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \cdots,\\ &\lim_{j\to+\infty} \overline{\lambda}_j = +\infty,\\ &a(w_j, w) = \overline{\lambda}_j \langle w_j, w\rangle \quad \forall w \in H_0^1, \ j = 1, 2, \ldots.\end{aligned} \qquad (22)$$

Furthermore, the sequence $\{w_j/\sqrt{\overline{\lambda}_j}\}$ is also the Hilbert orthonormal base of H_0^1 with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have w_j satisfying the following boundary value problem:

$$-\left(w_{jxx}+\frac{\gamma}{x}w_{jx}\right)=\overline{\lambda}_j w_j, \quad in\ (1,R), \qquad (23)$$
$$w_j(1)=w_j(R)=0, \quad w_j\in C^{\infty}([1,R]).$$

The proof of Lemma 5 can be found in [17, p. 87, Theorem 7.7], with $H=L^2$ and $V=H_0^1$ and $a(\cdot,\cdot)$ as defined by (21).

3. The Existence and the Uniqueness

Now, we consider problems (1)–(3) in which γ is a positive constant and make the following assumptions:

- (H_1) $\tilde{u}_0\in H^1$.
- (H_2) $g_1, g_R\in W^{1,1}(0,T)$, $\tilde{u}_0(1)-g_1(0)=\tilde{u}_0(R)-g_R(0)=0$.
- (H_3) $\alpha\in W^{1,1}(0,T)$, $\alpha(t)\geq\alpha_*>0$, $\forall t\in[0,T]$.
- (H_4) $\mu\in W^{1,1}(0,T)$, $\mu(t)\geq\mu_*>0$, $\forall t\in[0,T]$.
- (H_5) $f_1\in L^1(0,T;L^2)$.
- (H_6) $f\in C^0(\mathbb{R};\mathbb{R})$ satisfies the condition that there exists positive constant δ such that $(y-z)(f(y)-f(z))\geq -\delta|y-z|^2$, for all $y,z\in\mathbb{R}$.

In case $g_1\neq 0$ or $g_R\neq 0$, it is clearly that problems (1)–(3) reduce to a problem with homogeneous boundary conditions by the suitable transformation. Indeed, putting $\varphi(x,t)=((x-1)/(R-1))g_R(t)+((R-x)/(R-1))g_1(t)$, by the transformation $v(x,t)=u(x,t)-\varphi(x,t)$, problems (1)–(3) reduce to the following problem:

$$v_t-\left(\mu(t)+\alpha(t)\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\gamma}{x}\frac{\partial v}{\partial x}\right)+f(v+\varphi)$$
$$=\overline{f}_1(x,t), \quad 1<x<R,\ t>0, \qquad (24)$$
$$v(1,t)=v(R,t)=0,$$
$$v(x,0)=\tilde{v}_0(x),$$

where

$$\overline{f}_1(x,t)=f_1(x,t)-\frac{1}{R-1}\left[(x-1)g_R'(t)+(R-x)g_1'(t)\right]$$
$$+\frac{\gamma}{(R-1)x}\left[\mu(t)\left(g_R(t)-g_1(t)\right)+\alpha(t)\left(g_R'(t)-g_1'(t)\right)\right], \qquad (25)$$
$$\tilde{v}_0(x)=\tilde{u}_0(x)-\varphi(x,0), \quad \tilde{v}_0\in H_0^1$$

and $\tilde{u}_0$, g_1, and g_R satisfy the condition $\tilde{u}_0(1)-g_1(0)=\tilde{u}_0(R)-g_R(0)=0$.

The weak formulation of the initial-boundary value problem (24) can be given in the following manner: Find $v\in L^{\infty}(0,T;H_0^1)$ with $tv_t\in L^2(0,T;H_0^1)$, such that v satisfies the following variational equation:

$$\frac{d}{dt}\left[\langle v(t),w\rangle+\alpha(t)a(v(t),w)\right]+\left(\mu(t)-\alpha'(t)\right)a(v(t),w)+\left\langle f(v(t)+\varphi(t)),w\right\rangle=\left\langle\overline{f}_1(t),w\right\rangle,$$
$$\forall w\in H_0^1,\ \text{a.e.},\ t\in(0,T), \qquad (26)$$
$$v(0)=\tilde{v}_0,$$

where $a(\cdot,\cdot)$ is the symmetric bilinear form on $H_0^1\times H_0^1$ defined by (21).

Then, we have the following theorem.

Theorem 6. *Let $T>0$ and (H_1)–(H_6) hold. Then, problem (24) has a unique weak solution v such that*

$$v\in L^{\infty}\left(0,T;H_0^1\right), \qquad (27)$$
$$tv_t\in L^2\left(0,T;H_0^1\right).$$

Moreover, if (H_5) is replaced by $f_1\in L^2(Q_T)$, then the solution v satisfies

$$v\in L^{\infty}\left(0,T;H_0^1\right), \qquad (28)$$
$$v_t\in L^2\left(0,T;H_0^1\right).$$

Proof. The proof consists of several steps.

Step 1 (the Faedo-Galerkin approximation (introduced by Lions [18])). Consider the basis $\{w_j\}$ for H_0^1 as in Lemma 5. We find the approximate solution of problem (24) in the form

$$v_m(t)=\sum_{j=1}^{m}c_{mj}(t)w_j, \qquad (29)$$

where the coefficients c_{mj} satisfy the system of linear differential equations

$$\left\langle v_m'(t),w_j\right\rangle+\alpha(t)a\left(v_m'(t),w_j\right)+\mu(t)a\left(v_m(t),w_j\right)+\left\langle f\left(v_m(t)+\varphi(t)\right),w_j\right\rangle=\left\langle\overline{f}_1(t),w_j\right\rangle,$$
$$1\leq j\leq m, \qquad (30)$$
$$v_m(0)=v_{0m},$$

where

$$v_{0m}=\sum_{j=1}^{m}\alpha_{mj}w_j\longrightarrow\tilde{v}_0 \quad \text{strongly in } H_0^1. \qquad (31)$$

The system of (30) can be rewritten in the form

$$\begin{aligned} &c_{mj}'(t)+\frac{\overline{\lambda}_j\mu(t)}{1+\overline{\lambda}_j\alpha(t)}c_{mj}(t)\\ &\quad+\frac{1}{1+\overline{\lambda}_j\alpha(t)}\left\langle f\left(v_m(t)+\varphi(t)\right),w_j\right\rangle\\ &\quad=\frac{1}{1+\overline{\lambda}_j\alpha(t)}\left\langle \overline{f}_1(t),w_j\right\rangle,\\ &c_{mj}(0)=\alpha_{mj},\quad 1\le j\le m. \end{aligned} \tag{32}$$

It is clear that for each m there exists a solution $v_m(t)$ in the form of (29) which satisfies (30) almost everywhere on $0\le t\le \widetilde{T}_m$ for some $\widetilde{T}_m$, $0<\widetilde{T}_m\le T$. The following estimates allow one to take $\widetilde{T}_m=T$ for all m.

Step 2 (a priori estimates)

(a) The First Estimate. Multiplying the jth equation of (30) by $c_{mj}(t)$ and summing up with respect to j, afterwards, integrating by parts with respect to the time variable from 0 to t, we get after some rearrangements:

$$\begin{aligned} &\left\|v_m(t)\right\|_0^2+\alpha(t)\left\|v_{mx}(t)\right\|_0^2\\ &\quad=\left\|v_{0m}\right\|_0^2+\alpha(0)\left\|v_{0mx}\right\|_0^2\\ &\qquad-\int_0^t\left(2\mu(s)-\alpha'(s)\right)\left\|v_{mx}(s)\right\|_0^2ds\\ &\qquad-2\int_0^t\left\langle f\left(v_m(s)+\varphi(s)\right),v_m(s)\right\rangle ds\\ &\qquad+2\int_0^t\left\langle \overline{f}_1(s),v_m(s)\right\rangle ds. \end{aligned} \tag{33}$$

By $v_{0m}\to\widetilde{v}_0$ strongly in H_0^1, we have

$$\left\|v_{0m}\right\|_0^2+\alpha(0)\left\|v_{0mx}\right\|_0^2\le\overline{S}_0,\quad\forall m, \tag{34}$$

where $\overline{S}_0$ always indicates a bound depending on $\widetilde{v}_0$.

Put

$$S_m(t)=\left\|v_m(t)\right\|_0^2+\alpha_*\left\|v_{mx}(t)\right\|_0^2. \tag{35}$$

By the assumptions (H_3)–(H_6), we estimate without difficulty the following terms in (33) as follows:

$$\begin{aligned} &-\int_0^t\left(2\mu(s)-\alpha'(s)\right)\left\|v_{mx}(s)\right\|_0^2ds\\ &\quad\le\frac{1}{\alpha_*}\int_0^t\left|2\mu(s)-\alpha'(s)\right|S_m(s)\,ds;\\ &-2\int_0^t\left\langle f\left(v_m(s)+\varphi(s)\right),v_m(s)\right\rangle ds\\ &\quad\le 2\delta\int_0^t\left\|v_m(s)\right\|_0^2ds\\ &\qquad+2\int_0^t\left\|f\left(\varphi(s)\right)\right\|_0\left\|v_m(s)\right\|_0ds\\ &\quad\le(2\delta+1)\int_0^t\left\|v_m(s)\right\|_0^2ds+\int_0^T\left\|f\left(\varphi(s)\right)\right\|_0^2ds;\\ &2\int_0^t\left\langle \overline{f}_1(s),v_m(s)\right\rangle ds\\ &\quad\le\left\|\overline{f}_1\right\|_{L^1(0,T;L^2)}+\int_0^t\left\|\overline{f}_1(s)\right\|_0\left\|v_m(s)\right\|_0^2ds\\ &\quad\le\left\|\overline{f}_1\right\|_{L^1(0,T;L^2)}+\int_0^t\left\|\overline{f}_1(s)\right\|_0S_m(s)\,ds. \end{aligned} \tag{36}$$

Hence, it follows from (33), (34), and (36) that

$$S_m(t)\le C_T^{(1)}+\int_0^t d_T^{(1)}(s)\,S_m(s)\,ds, \tag{37}$$

where

$$\begin{aligned} &C_T^{(1)}=\overline{S}_0+\left\|\overline{f}_1\right\|_{L^1(0,T;L^2)}+\int_0^T\left\|f\left(\varphi(s)\right)\right\|_0^2ds,\\ &d_T^{(1)}(s)=1+2\delta+\left\|\overline{f}_1(s)\right\|_0+\frac{1}{\alpha_*}\left|2\mu(s)-\alpha'(s)\right|,\\ &\qquad d_T^{(1)}\in L^1(0,T). \end{aligned} \tag{38}$$

By Gronwall's lemma, we obtain from (37) that

$$S_m(t)\le C_T^{(1)}\exp\left(\int_0^t d_T^{(1)}(s)\,ds\right)\le C_T, \tag{39}$$

for all $m\in\mathbb{N}$, for all t, $0\le t\le\widetilde{T}_m\le T$; that is, $\widetilde{T}_m=T$, where C_T always indicates a bound depending on T.

(b) The Second Estimate. Multiplying the jth equation of (30) by $2t^2c_{mj}'(t)$ and summing up with respect to j, we have

$$\begin{aligned} &2\left\|tv_m'(t)\right\|_0^2+2\alpha(t)\left\|tv_{mx}'(t)\right\|_0^2\\ &\qquad+\frac{d}{dt}\left[\mu(t)\left\|tv_{mx}(t)\right\|_0^2\right]\\ &\quad=\left(t^2\mu(t)\right)'\left\|v_{mx}(t)\right\|_0^2\\ &\qquad-2t^2\left\langle f\left(v_m(t)+\varphi(t)\right),v_m'(t)\right\rangle\\ &\qquad+2t^2\left\langle \overline{f}_1(t),v_m'(t)\right\rangle. \end{aligned} \tag{40}$$

Integrating (40), we get

$$\begin{aligned} &2\int_0^t \left\|sv'_m(s)\right\|_0^2 ds + 2\int_0^t \alpha(s)\left\|sv'_{mx}(s)\right\|_0^2 ds \\ &\quad + \mu(t)\left\|tv_{mx}(t)\right\|_0^2 \\ &\quad = \int_0^t \left(s^2\mu(s)\right)'\left\|v_{mx}(s)\right\|_0^2 ds \\ &\quad - 2\int_0^t \left\langle sf\left(v_m(s)+\varphi(s)\right), sv'_m(s)\right\rangle ds \\ &\quad + 2\int_0^t \left\langle s\overline{f}_1(s), sv'_m(s)\right\rangle ds. \end{aligned} \tag{41}$$

We shall estimate the terms of (41) as follows:

$$\begin{aligned} &\int_0^t \left(s^2\mu(s)\right)'\left\|v_{mx}(s)\right\|_0^2 ds \\ &\quad \le \frac{1}{\alpha_*}\int_0^t \left|\left(s^2\mu(s)\right)'\right| S_m(s)\, ds \\ &\quad \le \frac{C_T}{\alpha_*}\int_0^T \left|\left(s^2\mu(s)\right)'\right| ds; \end{aligned} \tag{42}$$

$$\begin{aligned} &2\int_0^t \left\langle s\overline{f}_1(s), sv'_m(s)\right\rangle ds \\ &\quad \le 2\int_0^T \left\|s\overline{f}_1(s)\right\|_0^2 ds + \frac{1}{2}\int_0^t \left\|sv'_m(s)\right\|_0^2 ds. \end{aligned} \tag{43}$$

On the other hand, we have

$$\begin{aligned} \left|v_m(x,s)\right| + \left|\varphi(x,s)\right| &\le \left\|v_m(s)\right\|_{C^0(\overline{\Omega})} + \left|g_1(s)\right| \\ &\quad + \left|g_R(s)\right| \\ &\le \sqrt{\frac{(R-1)C_T}{\alpha_*}} + \left\|g_1\right\|_{C^0([0,T])} \\ &\quad + \left\|g_R\right\|_{C^0([0,T])} \equiv \overline{C}_T, \end{aligned} \tag{44}$$

and hence

$$\begin{aligned} &2\int_0^t \left\langle sf\left(v_m(s)+\varphi(s)\right), sv'_m(s)\right\rangle ds \\ &\quad \le 2\int_0^t \left\|sf\left(v_m(s)+\varphi(s)\right)\right\|_0^2 ds \\ &\quad + \frac{1}{2}\int_0^t \left\|sv'_m(s)\right\|_0^2 ds \\ &\quad \le 2\int_0^t s^2 ds \int_1^R x^\gamma \sup_{|z|\le\overline{C}_T} f^2(z)\, dx \\ &\quad + \frac{1}{2}\int_0^t \left\|sv'_m(s)\right\|_0^2 ds \\ &\quad \le 2T^2\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right)\sup_{|z|\le\overline{C}_T} f^2(z)\, dx \\ &\quad + \frac{1}{2}\int_0^t \left\|sv'_m(s)\right\|_0^2 ds. \end{aligned} \tag{45}$$

It follows from (41)–(43) and (45) that

$$\begin{aligned} &\int_0^t \left\|sv'_m(s)\right\|_0^2 ds + 2\alpha_*\int_0^t \left\|sv'_{mx}(s)\right\|_0^2 ds \\ &\quad + \mu_*\left\|tv_{mx}(t)\right\|_0^2 \le \frac{C_T}{\alpha_*}\int_0^T \left|\left(s^2\mu(s)\right)'\right| ds \\ &\quad + 2\int_0^T \left\|s\overline{f}_1(s)\right\|_0^2 ds \\ &\quad + 2T^2\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right)\sup_{|z|\le\overline{C}_T} f^2(z)\, dx \le C_T, \end{aligned} \tag{46}$$

for all $m \in \mathbb{N}$, for all $t \in [0,T]$, where C_T always indicates a bound depending on T.

By $(tv_{mx})' = tv'_{mx} + v_{mx}$ and (39) and (46), we deduce that

$$\begin{aligned} \left\|(tv_{mx})'\right\|_{L^2(Q_T)} &\le \left\|tv'_{mx}\right\|_{L^2(Q_T)} + \left\|v_{mx}\right\|_{L^2(Q_T)} \\ &\le \sqrt{\int_0^T \left\|sv'_{mx}(s)\right\|_0^2 ds} \\ &\quad + \sqrt{T}\left\|v_m\right\|_{L^2(0,T;H_0^1)} \le C_T. \end{aligned} \tag{47}$$

Step 3 (the limiting process). By (39), (46), and (47), we deduce that there exists a subsequence of $\{v_m\}$, still denoted by $\{v_m\}$ such that

$$\begin{aligned} v_m &\longrightarrow v \quad \text{in } L^\infty\left(0,T;H_0^1\right) \text{ weakly}^*, \\ (tv_m)' &\longrightarrow (tv)' \quad \text{in } L^2\left(0,T;H_0^1\right) \text{ weakly}. \end{aligned} \tag{48}$$

Using a compactness lemma ([18], Lions, p. 57), applied to (48), we can extract from the sequence $\{v_m\}$ a subsequence still denoted by $\{v_m\}$, such that

$$tv_m \longrightarrow tv \quad \text{strongly in } L^2(Q_T). \tag{49}$$

By the Riesz-Fischer theorem, we can extract from $\{v_m\}$ a subsequence still denoted by $\{v_m\}$, such that

$$v_m(x,t) \longrightarrow v(x,t) \quad \text{a.e. } (x,t) \text{ in } Q_T. \tag{50}$$

Because f is continuous, it gives

$$\begin{aligned} f\left(v_m(x,t)+\varphi(x,t)\right) &\longrightarrow f\left(v(x,t)+\varphi(x,t)\right) \\ &\qquad \text{a.e. } (x,t) \text{ in } Q_T. \end{aligned} \tag{51}$$

On the other hand, by (H_6), it follows from (44) that

$$\left|f\left(v_m(x,t)+\varphi(x,t)\right)\right| \le \sup_{|z|\le\overline{C}_T}\left|f(z)\right| \le C_T, \tag{52}$$

where C_T is a constant independent of m.

Using the dominated convergence theorem, (51) and (52) yield

$$f(v_m+\varphi) \longrightarrow f(v+\varphi) \quad \text{strongly in } L^2(Q_T). \tag{53}$$

Passing to the limit in (30) by (31), (48), and (53), we obtain

$$\begin{aligned}&\frac{d}{dt}[\langle v(t),w\rangle + \alpha(t)a(v(t),w)]\\&\quad+(\mu(t)-\alpha'(t))a(v(t),w)\\&\quad+\langle f(v(t)+\varphi(t)),w\rangle = \langle \overline{f}_1(t),w\rangle,\\&\qquad\qquad\forall w\in H_0^1,\ \text{a.e.},\ t\in(0,T),\\&v(0)=\widetilde{v}_0.\end{aligned}\tag{54}$$

Step 4 (uniqueness of the solution). First, we shall need the following lemma.

Lemma 7. *Let v be the weak solution of the following problem:*

$$\begin{aligned}&v_t-\left(\mu(t)+\alpha(t)\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\gamma}{x}\frac{\partial v}{\partial x}\right)=\widetilde{f}(x,t),\\&\qquad 1<x<R,\ 0<t<T,\\&\qquad v(1,t)=v(R,t)=0,\\&\qquad v(x,0)=\widetilde{v}_0(x),\\&v\in L^\infty(0,T;H_0^1),\ tv_t\in L^2(0,T;H_0^1),\ \mu,\alpha\in W^{1,1}(0,T).\end{aligned}\tag{55}$$

Then,

$$\begin{aligned}&\|v(t)\|_0^2+\alpha(t)\|v_x(t)\|_0^2\\&\quad+\int_0^t(2\mu(s)-\alpha'(s))\|v_x(s)\|_0^2\,ds\\&\quad\ge\|\widetilde{v}_0\|_0^2+\alpha(0)\|\widetilde{v}_{0x}\|_0^2+2\int_0^t\langle\widetilde{f}(s),v(s)\rangle\,ds.\end{aligned}\tag{56}$$

Furthermore, if $\widetilde{v}_0=0$, then the equality in (56) holds.

Lemma 7 is a slight improvement of a lemma used in [12] (or it can be found in Lions's book [18]).

Now, we will prove the uniqueness of the solutions.

Let v_1 and v_2 be two weak solutions of (24). Then, $v=v_1-v_2$ is a weak solution of (55) with the right-hand side function replaced by $\widetilde{f}(x,t)=-f(v_1+\varphi)+f(v_2+\varphi)$ and $\widetilde{v}_0=0$. Using Lemma 7, we have equality

$$\begin{aligned}\sigma_1(t)=&-\int_0^t(2\mu(s)-\alpha'(s))\|v_x(s)\|_0^2\,ds\\&-2\int_0^t\langle f(v_1+\varphi)-f(v_2+\varphi),v(s)\rangle\,ds,\end{aligned}\tag{57}$$

where

$$\sigma_1(t)=\|v(t)\|_0^2+\alpha(t)\|v_x(t)\|_0^2.\tag{58}$$

By (H_6), we obtain

$$\begin{aligned}&2\int_0^t\langle f(v_1+\varphi)-f(v_2+\varphi),v(s)\rangle\,ds\\&\quad\ge-2\delta\int_0^t\|v(s)\|_0^2\,ds\ge-2\delta\int_0^t\sigma_1(s)\,ds;\\&-\int_0^t(2\mu(s)-\alpha'(s))\|v_x(s)\|_0^2\,ds\\&\quad\le\frac{1}{\alpha_*}\int_0^t|2\mu(s)-\alpha'(s)|\sigma_1(s)\,ds.\end{aligned}\tag{59}$$

It follows from (57)–(59) that

$$\sigma_1(t)\le\int_0^t\left(2\delta+\frac{1}{\alpha_*}|2\mu(s)-\alpha'(s)|\right)\sigma_1(s)\,ds.\tag{60}$$

By Gronwall's lemma, $v=0$.

Assume now that (H_5) is replaced by $f_1\in L^2(Q_T)$; then we only have to show that $\{v_m'\}$ is bounded in $L^2(0,T;H_0^1)$.

Indeed, multiplying the jth equation of (30) by $c_{mj}'(t)$ and summing up with respect to j, afterwards, integrating with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned}X_m(t)=&X_m(0)+\int_0^t\mu'(s)\|v_{mx}(s)\|_0^2\,ds\\&+2\int_0^t\langle\overline{f}_1(s),v_m'(s)\rangle\,ds\\&-2\int_0^t\langle f(v_m(s)+\varphi(s)),v_m'(s)\rangle\,ds,\end{aligned}\tag{61}$$

where

$$\begin{aligned}X_m(t)=&2\int_0^t\left(\|v_m'(s)\|_0^2+\alpha(s)\|v_{mx}'(s)\|_0^2\right)ds\\&+\mu(t)\|v_{mx}(t)\|_0^2.\end{aligned}\tag{62}$$

By the same estimates as above, we obtain

$$X_m(0)=\mu(0)\|v_{0mx}\|_0^2\le\frac{\mu(0)}{\alpha_*}\overline{S}_0;$$

$$\begin{aligned}&\int_0^t\mu'(s)\|v_{mx}(s)\|_0^2\,ds\le\int_0^t\frac{|\mu'(s)|}{\alpha_*}S_m(s)\,ds\\&\quad\le\frac{C_T}{\alpha_*}\int_0^T|\mu'(s)|\,ds;\end{aligned}$$

$$\begin{aligned}&2\int_0^t\langle\overline{f}_1(s),v_m'(s)\rangle\,ds\\&\quad\le2\|\overline{f}_1\|_{L^2(Q_T)}^2+\frac{1}{2}\int_0^t\|v_m'(s)\|_0^2\,ds\\&\quad\le2\|\overline{f}_1\|_{L^2(Q_T)}^2+\frac{1}{4}X_m(t);\end{aligned}$$

$$
\begin{aligned}
&-2\int_0^t \left\langle f\left(v_m(s)+\varphi(s)\right), v_m'(s)\right\rangle ds \\
&\le 2T\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right)\sup_{|z|\le \overline{C}_T} f^2(z)+\frac{1}{2}\int_0^t \left\|v_m'(s)\right\|_0^2 ds \\
&\le 2T\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right)\sup_{|z|\le \overline{C}_T} f^2(z)+\frac{1}{4}X_m(t).
\end{aligned} \tag{63}
$$

This implies

$$
\begin{aligned}
X_m(t) \le{}& \frac{2}{\alpha_*}\left(\mu(0)\overline{S}_0 + C_T\int_0^T \left|\mu'(s)\right| ds\right) \\
&+4\left\|\overline{f}_1\right\|_{L^2(Q_T)}^2 \\
&+4T\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right)\sup_{|z|\le \overline{C}_T} f^2(z) = \overline{C}_T^{(1)}.
\end{aligned} \tag{64}
$$

Then, the sequence $\{v_m'\}$ is bounded in $L^2(0,T;H_0^1)$.

Applying a similar argument used as above, the limit v of the sequence $\{v_m\}$ in suitable function spaces is a unique weak solution of (24) satisfying (28).

Therefore, Theorem 6 is proved. □

4. Asymptotic Behavior of the Solution as $t \longrightarrow +\infty$

In this part, let $T > 0$, (H_1)–(H_6) hold. Then, there exists a unique solution $u = v + \varphi$ of problems (1)–(3) such that

$$
\begin{gathered}
u-\varphi = v \in L^\infty\left(0,T;H_0^1\right), \\
t\left(u_t-\varphi_t\right) = tv_t \in L^2\left(0,T;H_0^1\right).
\end{gathered} \tag{65}
$$

We shall study asymptotic behavior of the solution $u(t)$ as $t \to +\infty$.

We make the following supplementary assumptions on the functions $g_1(t)$, $g_R(t)$, $\alpha(t)$, $\mu(t)$, and $f_1(x,t)$:

(H_2') $g_1, g_R \in W^{1,1}(\mathbb{R}_+)$, $\widetilde{u}_0(1)-g_1(0)=\widetilde{u}_0(R)-g_R(0)=0$, and there exist the positive constants $\overline{C}_1$, $\overline{C}_R$, $\overline{\gamma}_1$, and $\overline{\gamma}_R$, such that

$$
\left|g_i(t)\right|+\left|g_i'(t)\right| \le \overline{C}_i e^{-\overline{\gamma}_i t}, \quad \forall t\ge 0,\ i\in\{1,R\}. \tag{66}
$$

(H_3') $\alpha \in W^{1,1}(\mathbb{R}_+)$, $\alpha(t) \ge \alpha_* > 0$, $\forall t \ge 0$, $\alpha(0) \le \alpha(T)$.

(H_4') $\mu \in W^{1,1}(\mathbb{R}_+)$, and there exist the positive constants μ_*, $\overline{\mu}_*$, μ_∞, $\overline{C}_\mu$, and $\overline{\gamma}_\mu$, such that

$$
\begin{gathered}
\mu(t) \ge \mu_* > 0, \quad \forall t\ge 0, \\
\left|\mu(t)-\mu_\infty\right| \le \overline{C}_\mu e^{-\overline{\gamma}_\mu t}, \quad \forall t \ge 0, \\
2\mu(t)-\alpha'(t) \ge 2\overline{\mu}_* > 0, \quad \forall t\ge 0.
\end{gathered} \tag{67}
$$

(H_5') $f_1 \in L^\infty(0,\infty;L^2)$, and there exist the positive constants C_1, γ_1 and the function $f_{1\infty} \in L^2$, such that $\|f_1(t)-f_{1\infty}\|_0 \le C_1 e^{-\gamma_1 t}$, $\forall t \ge 0$.

(H_6') $f \in C^0(\mathbb{R};\mathbb{R})$ and there exists a positive constant δ, with $0<\delta<2\mu_\infty/R^\gamma(R-1)^2$, such that $(y-z)(f(y)-f(z)) \ge -\delta|y-z|^2$, for all $y,z\in\mathbb{R}$.

First, we consider the following stationary problem:

$$
\begin{gathered}
-\mu_\infty\left(\frac{\partial^2 u}{\partial x^2}+\frac{\gamma}{x}\frac{\partial u}{\partial x}\right)+f(u)=f_{1\infty}(x), \\
1<x<R, \\
u(1,t)=u(R,t)=0.
\end{gathered} \tag{68}
$$

The weak solution of problem (68) is obtained from the following variational problem. Find $u_\infty \in H_0^1$ such that

$$
\mu_\infty a\left(u_\infty, w\right)+\left\langle f\left(u_\infty\right), w\right\rangle = \left\langle f_{1\infty}, w\right\rangle, \tag{69}
$$

for all $w \in H_0^1$, where $a(\cdot,\cdot)$ is the symmetric bilinear form on $H_0^1\times H_0^1$ defined by (21).

We then have the following theorem.

Theorem 8. *Let (H_4')–(H_6') hold. Then, there exists a unique solution u_∞ of the variational problem (69) such that $u_\infty \in H_0^1$.*

Proof. Consider the basis $\{w_j\}$ for H_0^1 as in Lemma 5. Put

$$
y_m = \sum_{j=1}^m d_{mj} w_j, \tag{70}
$$

where d_{mj} satisfies the following nonlinear equation system:

$$
\begin{gathered}
\mu_\infty a\left(y_m, w_j\right)+\left\langle f\left(y_m\right), w_j\right\rangle = \left\langle f_{1\infty}, w_j\right\rangle, \\
1\le j\le m.
\end{gathered} \tag{71}
$$

By Brouwer's lemma (see Lions [18], Lemma 4.3, p. 53), it follows from the hypotheses (H_4')–(H_6') that systems (70) and (71) have a solution y_m.

Multiplying the jth equation of system (71) by d_{mj}, and then summing up with respect to j, we have

$$
\mu_\infty a\left(y_m, y_m\right)+\left\langle f\left(y_m\right), y_m\right\rangle = \left\langle f_{1\infty}, y_m\right\rangle. \tag{72}
$$

By using (H_6), we obtain

$$
\begin{aligned}
\left\langle f\left(y_m\right), y_m\right\rangle ={}& \int_1^R x^\gamma\left(f\left(y_m(x)\right)-f(0)\right) y_m(x)\,dx \\
&+\int_1^R x^\gamma f(0)\, y_m(x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\geq -\delta \int_1^R x^\gamma y_m^2(x)\,dx \\
&\quad + \int_1^R x^\gamma f(0)\, y_m(x)\,dx \\
&\geq -\delta \|y_m\|_0^2 - \varepsilon_1 \|y_m\|_0^2 \\
&\quad - \frac{1}{4\varepsilon_1}\int_1^R x^\gamma f^2(0)\,dx \\
&= -(\delta+\varepsilon_1)\|y_m\|_0^2 \\
&\quad - \frac{1}{4\varepsilon_1}\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right) f^2(0), \\
&\qquad \forall \varepsilon_1 > 0.
\end{aligned} \tag{73}
$$

By using inequalities (20)(iii) and (73), we obtain from (72) that

$$
\begin{aligned}
\mu_\infty \|y_{mx}\|_0^2 &\leq (\delta+\varepsilon_1)\|y_m\|_0^2 \\
&\quad + \frac{1}{4\varepsilon_1}\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right) f^2(0) \\
&\quad + \|f_{1\infty}\|_0 \|y_m\|_0 \\
&\leq (\delta+\varepsilon_1)\|y_m\|_0^2 \\
&\quad + \frac{1}{4\varepsilon_1}\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right) f^2(0) \\
&\quad + \frac{1}{4\varepsilon_1}\|f_{1\infty}\|_0^2 + \varepsilon_1 \|y_m\|_0^2 \\
&= (\delta+2\varepsilon_1)\|y_m\|_0^2 \\
&\quad + \frac{1}{4\varepsilon_1}\left[\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right) f^2(0) + \|f_{1\infty}\|_0^2\right] \\
&= (\delta+2\varepsilon_1)\frac{R^\gamma}{2}(R-1)^2 \|y_{mx}\|_0^2 \\
&\quad + \frac{1}{4\varepsilon_1}\left[\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right) f^2(0) + \|f_{1\infty}\|_0^2\right].
\end{aligned} \tag{74}
$$

By $0 < \delta < 2\mu_\infty/R^\gamma(R-1)^2$, choose $\varepsilon_1 > 0$ such that $(\delta+2\varepsilon_1)(R^\gamma/2)(R-1)^2 < \mu_\infty$.

Hence, we deduce from (74) that

$$
\begin{aligned}
\|y_{mx}\|_0 &\leq \sqrt{\frac{\left(\left(R^{\gamma+1}-1\right)/(\gamma+1)\right) f^2(0) + \|f_{1\infty}\|_0^2}{4\varepsilon_1\left[\mu_\infty - (\delta+2\varepsilon_1)(R^\gamma/2)(R-1)^2\right]}} \\
&= \widetilde{D}_1,
\end{aligned} \tag{75}
$$

and $\widetilde{D}_1$ is a constant independent of m.

By means of (75) and Lemma 3, the sequence $\{y_m\}$ has a subsequence still denoted by $\{y_m\}$ such that

$$
\begin{aligned}
y_m &\longrightarrow u_\infty \quad \text{in } H_0^1 \text{ weakly}, \\
y_m &\longrightarrow u_\infty \quad \text{in } C^0([1,R]) \text{ strongly}.
\end{aligned} \tag{76}
$$

On the other hand, by $(76)_2$ and the continuity of f, we have

$$
f(y_m) \longrightarrow f(u_\infty) \quad \text{in } C^0([1,R]) \text{ strongly}. \tag{77}
$$

Passing to the limit in (71), we find without difficulty from (76) and (77) that u_∞ satisfies the equation

$$
\mu_\infty a\left(u_\infty, w_j\right) + \left\langle f(u_\infty), w_j\right\rangle = \left\langle f_{1\infty}, w_j\right\rangle. \tag{78}
$$

Equation (78) holds for every $j = 1, 2, \ldots$; that is, (69) is true.

The solution of problem (69) is unique, which can be shown by the same arguments as in the proof of Theorem 6. □

Now we consider asymptotic behavior of the solution $u(t)$ as $t \to +\infty$.

We then have the following theorem.

Theorem 9. *Let (H_1), (H_2')–(H_5'), and (H_6) hold. Let f satisfy the following condition, in addition,*

(H_6'') $\forall M > 0$, $\exists k_M > 0 : |f(y) - f(z)| \leq k_M|y-z|$, $\forall y, z \in [-M, M]$.

And let $\delta > 0$ in (H_6) satisfy the following condition, in addition,

(H_6''') $0 < \delta < (2/R^\gamma(R-1)^2)\min\{\mu_\infty, \overline{\mu}_*\}$.

Then we have

$$
\left\|u(t) - u_\infty\right\|_1 \leq \overline{C}e^{-\overline{\gamma}t}, \quad \forall t \geq 0, \tag{79}
$$

where $\overline{\gamma} > 0$, $\overline{C} > 0$ are constants independent of t.

Proof. Put $Z_m(t) = v_m(t) - y_m$. Let us subtract $(30)_1$ with (71) to obtain

$$
\begin{aligned}
&\left\langle Z_m'(t), w_j\right\rangle + \alpha(t)\, a\left(Z_m'(t), w_j\right) \\
&\quad + \mu(t)\, a\left(Z_m(t), w_j\right) + \left(\mu(t) - \mu_\infty\right) a\left(y_m, w_j\right) \\
&\quad + \left\langle f\left(v_m(t) + \varphi(t)\right) - f(y_m), w_j\right\rangle \\
&\quad = \left\langle \overline{f}_1(t) - f_{1\infty}, w_j\right\rangle, \quad 1 \leq j \leq m, \\
&Z_m(0) = v_{0m} - y_m.
\end{aligned} \tag{80}
$$

By multiplying $(80)_1$ by $c_{mj}(t)-d_{mj}$ and summing up in j, we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\|Z_m(t)\|_0^2+\alpha(t)\|Z_{mx}(t)\|_0^2\right]\\&\quad+\left(2\mu(t)-\alpha'(t)\right)\|Z_{mx}(t)\|_0^2\\&\quad+2\left(\mu(t)-\mu_\infty\right)a\left(y_m,Z_m(t)\right)\\&\quad+2\left\langle f\left(v_m(t)+\varphi(t)\right)-f\left(y_m+\varphi(t)\right),Z_m(t)\right\rangle\\&\quad+2\left\langle f\left(y_m+\varphi(t)\right)-f\left(y_m\right),Z_m(t)\right\rangle\\&=2\left\langle\overline{f}_1(t)-f_1(t),Z_m(t)\right\rangle\\&\quad+2\left\langle f_1(t)-f_{1\infty},Z_m(t)\right\rangle.\end{aligned}\tag{81}$$

By the assumptions (H_2')–(H_5'), (H_6), (H_6''), and (H_6''') and using inequality (20)(iii), and with $\varepsilon_1>0$, we estimate without difficulty the following terms in (81) as follows:

(i) *Estimate* $(2\mu(t)-\alpha'(t))\|Z_{mx}(t)\|_0^2$, as

$$\left(2\mu(t)-\alpha'(t)\right)\|Z_{mx}(t)\|_0^2\geq2\overline{\mu}_*\|Z_{mx}(t)\|_0^2;\tag{82}$$

(ii) *Estimate* $2(\mu(t)-\mu_\infty)a(y_m,Z_m(t))$, as

$$\begin{aligned}&2\left(\mu(t)-\mu_\infty\right)a\left(y_m,Z_m(t)\right)\\&\quad\leq2\left|\mu(t)-\mu_\infty\right|\|y_{mx}\|_0\|Z_{mx}(t)\|_0\\&\quad\leq2\overline{C}_\mu e^{-\overline{\gamma}_\mu t}\widetilde{C}\|Z_{mx}(t)\|_0\\&\quad\leq\frac{1}{\varepsilon_1}\widetilde{C}^2\overline{C}_\mu^2e^{-2\overline{\gamma}_\mu t}+\varepsilon_1\|Z_{mx}(t)\|_0^2.\end{aligned}\tag{83}$$

(iii) *Estimate* $2\langle f(v_m(t)+\varphi(t))-f(y_m+\varphi(t)),Z_m(t)\rangle$, as

$$\begin{aligned}&2\left\langle f\left(v_m(t)+\varphi(t)\right)-f\left(y_m+\varphi(t)\right),Z_m(t)\right\rangle\\&\quad\geq-2\delta\|Z_m(t)\|_0^2\geq-2\delta\frac{R^\gamma}{2}(R-1)^2\|Z_{mx}(t)\|_0^2.\end{aligned}\tag{84}$$

(iv) *Estimate* $2\langle f(y_m+\varphi(t))-f(y_m),Z_m(t)\rangle$. Note that, from the inequalities

$$\begin{aligned}\left|\varphi(x,t)\right|&\leq\left|g_1(t)\right|+\left|g_R(t)\right|\leq\overline{C}_1+\overline{C}_R,\\\|y_m\|_{C^0(\overline{\Omega})}&\leq\sqrt{R-1}\|y_{mx}\|\leq\sqrt{R-1}\widetilde{D}_1,\\\|y_m+\varphi\|_{C^0(\overline{\Omega})}&\leq\sqrt{R-1}\widetilde{D}_1+\overline{C}_1+\overline{C}_R=M_1,\end{aligned}\tag{85}$$

and (H_6''), we deduce that

$$\begin{aligned}\left|f\left(y_m+\varphi(t)\right)-f\left(y_m\right)\right|&\leq k_{M_1}\left|\varphi(x,t)\right|\\&\leq k_{M_1}\left(\left|g_1(t)\right|+\left|g_R(t)\right|\right)=k_{M_1}\Psi(t),\end{aligned}\tag{86}$$

where

$$\Psi(t)=\overline{C}_1e^{-\overline{\gamma}_1t}+\overline{C}_Re^{-\overline{\gamma}_Rt}.\tag{87}$$

Hence,

$$\begin{aligned}&\left\|f\left(y_m+\varphi(t)\right)-f\left(y_m\right)\right\|_0^2\\&\quad\leq\frac{1}{\gamma+1}\left(R^{\gamma+1}-1\right)k_{M_1}^2\Psi^2(t).\end{aligned}\tag{88}$$

Thus,

$$\begin{aligned}&2\left\langle f\left(y_m+\varphi(t)\right)-f\left(y_m\right),Z_m(t)\right\rangle\\&\quad\leq\frac{1}{\varepsilon_1}\left\|f\left(y_m+\varphi(t)\right)-f\left(y_m\right)\right\|_0^2+\varepsilon_1\|Z_m(t)\|_0^2\\&\quad\leq\frac{1}{\varepsilon_1}\frac{1}{\gamma+1}\left(R^{\gamma+1}-1\right)k_{M_1}^2\Psi^2(t)\\&\qquad+\varepsilon_1\frac{R^\gamma}{2}(R-1)^2\|Z_{mx}(t)\|_0^2.\end{aligned}\tag{89}$$

(v) *Estimate* $\langle\overline{f}_1(t)-f_1(t),Z_m(t)\rangle$. We have

$$\begin{aligned}\overline{f}_1(x,t)-f_1(x,t)&=-\frac{1}{R-1}\left[(x-1)g_R'(t)\right.\\&\quad\left.+(R-x)g_1'(t)\right]\\&\quad+\frac{\gamma}{(R-1)x}\left[\mu(t)\left(g_R(t)-g_1(t)\right)\right.\\&\quad\left.+\alpha(t)\left(g_R'(t)-g_1'(t)\right)\right].\end{aligned}\tag{90}$$

Hence,

$$\begin{aligned}&\left|\overline{f}_1(x,t)-f_1(x,t)\right|\leq\left|g_1'(t)\right|+\left|g_R'(t)\right|\\&\quad+\frac{\gamma}{R-1}\left(\alpha(t)+\mu(t)\right)\\&\quad\cdot\left[\left|g_1(t)\right|+\left|g_1'(t)\right|+\left|g_R(t)\right|+\left|g_R'(t)\right|\right]\leq\overline{C}_1e^{-\overline{\gamma}_1t}\\&\quad+\overline{C}_Re^{-\overline{\gamma}_Rt}+\frac{\gamma}{R-1}\left(\|\alpha\|_{L^\infty(\mathbb{R}_+)}+\|\mu\|_{L^\infty(\mathbb{R}_+)}\right)\\&\quad\cdot\left[\overline{C}_1e^{-\overline{\gamma}_1t}+\overline{C}_Re^{-\overline{\gamma}_Rt}\right]\\&\quad=\left[1+\frac{\gamma}{R-1}\left(\|\alpha\|_{L^\infty(\mathbb{R}_+)}+\|\mu\|_{L^\infty(\mathbb{R}_+)}\right)\right]\\&\quad\cdot\left[\overline{C}_1e^{-\overline{\gamma}_1t}+\overline{C}_Re^{-\overline{\gamma}_Rt}\right]\\&\quad=\left[1+\frac{\gamma}{R-1}\left(\|\alpha\|_{L^\infty(\mathbb{R}_+)}+\|\mu\|_{L^\infty(\mathbb{R}_+)}\right)\right]\Psi(t)\\&\quad\equiv\widetilde{C}(\alpha,\mu)\Psi(t).\end{aligned}\tag{91}$$

It follows that

$$\left\|\overline{f}_1(t)-f_1(t)\right\|_0^2\leq\frac{1}{\gamma+1}\left(R^{\gamma+1}-1\right)\widetilde{C}^2(\alpha,\mu)\Psi^2(t).\tag{92}$$

Thus,

$$\begin{aligned} &2\left\langle \overline{f}_1(t)-f_1(t),Z_m(t)\right\rangle \\ &\quad\le \frac{1}{\varepsilon_1}\left\|\overline{f}_1(t)-f_1(t)\right\|_0^2+\varepsilon_1\left\|Z_m(t)\right\|_0^2 \\ &\quad\le \frac{1}{\varepsilon_1}\frac{1}{\gamma+1}\left(R^{\gamma+1}-1\right)\widetilde{C}^2(\alpha,\mu)\Psi^2(t) \\ &\qquad+\varepsilon_1\frac{R^\gamma}{2}(R-1)^2\left\|Z_{mx}(t)\right\|_0^2. \end{aligned} \tag{93}$$

(vi) *Estimate* $\langle f_1(t)-f_{1\infty},Z_m(t)\rangle$, as

$$\begin{aligned} &2\left\langle f_1(t)-f_{1\infty},Z_m(t)\right\rangle \\ &\quad\le \frac{1}{\varepsilon_1}\left\|f_1(t)-f_{1\infty}\right\|_0^2+\varepsilon_1\left\|Z_m(t)\right\|_0^2 \\ &\quad\le \frac{1}{\varepsilon_1}C_1^2e^{-2\gamma_1 t}+\varepsilon_1\frac{R^\gamma}{2}(R-1)^2\left\|Z_{mx}(t)\right\|_0^2. \end{aligned} \tag{94}$$

It follows from (81)–(84), (89), (93), and (94) that

$$\begin{aligned} &\frac{d}{dt}\left[\left\|Z_m(t)\right\|_0^2+\alpha(t)\left\|Z_{mx}(t)\right\|_0^2\right] \\ &\quad+\left[2\overline{\mu}_*-\varepsilon_1-(2\delta+3\varepsilon_1)\frac{R^\gamma}{2}(R-1)^2\right]\left\|Z_{mx}(t)\right\|_0^2 \\ &\quad\le \frac{1}{\varepsilon_1}\left(\widetilde{C}^2\overline{C}_\mu^2e^{-2\overline{\gamma}_\mu t}+C_1^2e^{-2\gamma_1 t}\right) \\ &\quad+\frac{1}{\varepsilon_1}\frac{R^{\gamma+1}-1}{\gamma+1}\left(k_{M_1}^2+\widetilde{C}^2(\alpha,\mu)\right)\Psi^2(t)\equiv\widetilde{\psi}(t). \end{aligned} \tag{95}$$

By $0<\delta<(2/R^\gamma(R-1)^2)\min\{\mu_\infty,\overline{\mu}_*\}\le 2\overline{\mu}_*/R^\gamma(R-1)^2$, choose $\varepsilon_1>0$ such that $2\widetilde{\gamma}=2\overline{\mu}_*-\varepsilon_1-(2\delta+3\varepsilon_1)(R^\gamma/2)(R-1)^2>0$.

Put $\overline{\gamma}_0=\min\{\gamma_1,\overline{\gamma}_1,\overline{\gamma}_R,\overline{\gamma}_\mu\}$, and we have $\widetilde{\psi}(t)\le\overline{C}_0e^{-2\overline{\gamma}_0 t}$ for all $t\ge 0$, as

$$\begin{aligned} &\frac{d}{dt}\left[\left\|Z_m(t)\right\|_0^2+\alpha(t)\left\|Z_{mx}(t)\right\|_0^2\right]+2\widetilde{\gamma}\left\|Z_{mx}(t)\right\|_0^2 \\ &\quad\le\widetilde{\psi}(t)\le\overline{C}_0e^{-2\overline{\gamma}_0 t}. \end{aligned} \tag{96}$$

By

$$\begin{aligned} \left\|Z_{mx}(t)\right\|_0^2&=\frac{1}{2}\left\|Z_{mx}(t)\right\|_0^2+\frac{1}{2}\left\|Z_{mx}(t)\right\|_0^2 \\ &\ge\frac{1}{2}\frac{1}{\alpha(t)}\alpha(t)\left\|Z_{mx}(t)\right\|_0^2 \\ &\quad+\frac{1}{2}\frac{2}{R^\gamma(R-1)^2}\left\|Z_m(t)\right\|_0^2 \\ &\ge\frac{1}{2}\frac{1}{\|\alpha\|_{L^\infty(\mathbb{R}_+)}}\alpha(t)\left\|Z_{mx}(t)\right\|_0^2 \\ &\quad+\frac{1}{2}\frac{2}{R^\gamma(R-1)^2}\left\|Z_m(t)\right\|_0^2 \\ &\ge\beta_1\left(\alpha(t)\left\|Z_{mx}(t)\right\|_0^2+\left\|Z_m(t)\right\|_0^2\right), \end{aligned} \tag{97}$$

where $\beta_1=(1/2)\min\{1/\|\alpha\|_{L^\infty(\mathbb{R}_+)},2/R^\gamma(R-1)^2\}$, it follows from (96) and (97) that

$$\begin{aligned} &\frac{d}{dt}\left[\left\|Z_m(t)\right\|_0^2+\alpha(t)\left\|Z_{mx}(t)\right\|_0^2\right] \\ &\quad+2\widetilde{\gamma}\beta_1\left(\left\|Z_m(t)\right\|_0^2+\alpha(t)\left\|Z_{mx}(t)\right\|_0^2\right)\le\widetilde{\psi}(t) \\ &\quad\le\overline{C}_0e^{-2\overline{\gamma}_0 t}. \end{aligned} \tag{98}$$

Choose $\overline{\gamma}>0$ such that $\overline{\gamma}<\min\{\overline{\gamma}_0,2\widetilde{\gamma}\beta_1\}$, and then we have from (98) that

$$\begin{aligned} &\frac{d}{dt}\left[\left\|Z_m(t)\right\|_0^2+\alpha(t)\left\|Z_{mx}(t)\right\|_0^2\right] \\ &\quad+2\overline{\gamma}\left(\left\|Z_m(t)\right\|_0^2+\alpha(t)\left\|Z_{mx}(t)\right\|_0^2\right)\le\overline{C}_0e^{-2\overline{\gamma}_0 t}. \end{aligned} \tag{99}$$

Hence, we obtain from (99) that

$$\begin{aligned} &\left\|Z_m(t)\right\|_0^2+\alpha_*\left\|Z_{mx}(t)\right\|_0^2\le\left\|Z_m(t)\right\|_0^2+\alpha(t) \\ &\quad\cdot\left\|Z_{mx}(t)\right\|_0^2 \\ &\quad\le\left(\left\|Z_m(0)\right\|_0^2+\alpha(0)\left\|Z_{mx}(0)\right\|_0^2+\frac{\overline{C}_0}{2(\overline{\gamma}_0-\overline{\gamma})}\right) \\ &\quad\cdot e^{-2\overline{\gamma}t}. \end{aligned} \tag{100}$$

Letting $m\to+\infty$ in (100), we obtain

$$\begin{aligned} &\left\|v(t)-u_\infty\right\|_0^2+\alpha_*\left\|v_x(t)-u_{\infty x}\right\|_0^2 \\ &\quad\le\liminf_{m\to+\infty}\left(\left\|v_m(t)-y_m\right\|_0^2+\alpha_*\left\|v_{mx}(t)-y_{mx}\right\|_0^2\right) \\ &\quad\le\left(\left\|\widetilde{v}_0-u_\infty\right\|_0^2+\alpha(0)\left\|\widetilde{v}_{0x}-u_{\infty x}\right\|_0^2\right. \\ &\quad\left.+\frac{\overline{C}_0}{2(\overline{\gamma}_0-\overline{\gamma})}\right)e^{-2\overline{\gamma}t},\quad\forall t\ge 0, \end{aligned} \tag{101}$$

or

$$\left\|v(t)-u_\infty\right\|_1\le\widetilde{D}_2e^{-\overline{\gamma}t},\quad\forall t\ge 0, \tag{102}$$

where

$$\widetilde{D}_2=\sqrt{\frac{1}{\min(1,\alpha_*)}\left(\left\|\widetilde{v}_0-u_\infty\right\|_0^2+\alpha(0)\left\|\widetilde{v}_{0x}-u_{\infty x}\right\|_0^2+\frac{\overline{C}_0}{2(\overline{\gamma}_0-\overline{\gamma})}\right)}. \tag{103}$$

Note that

$$\begin{aligned}|\varphi(x,t)| &\le |g_R(t)| + |g_1(t)| \le \overline{C}_1 e^{-\overline{\gamma}_1 t} + \overline{C}_R e^{-\overline{\gamma}_R t} \\ &= \Psi(t); \\ |\varphi_x(x,t)| &= \frac{|g_R(t) - g_1(t)|}{R-1} \le \frac{1}{R-1}\Psi(t).\end{aligned} \tag{104}$$

Hence,

$$\begin{aligned}\|\varphi(t)\|_1^2 &= \|\varphi(t)\|_0^2 + \|\varphi_x(t)\|_0^2 \\ &\le \frac{1}{\gamma+1}\left(R^{\gamma+1} - 1\right)\left(1 + \frac{1}{(R-1)^2}\right)\Psi^2(t) \\ &\le \widetilde{D}_3^2 e^{-2\overline{\gamma}t}.\end{aligned} \tag{105}$$

It follows from (102) and (105) that

$$\begin{aligned}\|u(t) - u_\infty\|_1 &\le \|v(t) - u_\infty\|_1 + \|\varphi(t)\|_1 \\ &\le \left(\widetilde{D}_2 + \widetilde{D}_3\right) e^{-\overline{\gamma}t}, \quad \forall t \ge 0.\end{aligned} \tag{106}$$

This completes the proof of Theorem 9. □

5. The Existence and the Uniqueness of a (η,T)-Periodic Weak Solution

In this section, we shall consider problems (1), (2), and (4) with $R > 1$, $0 < |\eta| \le 1$ as given constants and μ, α, f, f_1, g_1, and g_R as given functions satisfying the following assumptions:

$(\overline{H}_2)$ $g_1, g_R \in W^{1,1}(0,T)$, and g_1 and g_R are (η,T)-periodic; that is,

$$\begin{aligned}g_1(0) &= \eta g_1(T), \\ g_R(0) &= \eta g_R(T).\end{aligned} \tag{107}$$

$(\overline{H}_3)$ $\alpha \in W^{1,1}(0,T)$, $\alpha(t) \ge \alpha_* > 0$, $\forall t \in [0,T]$, $0 < \alpha(0) \le \alpha(T)$;

$(\overline{H}_4)$ $\mu \in W^{1,1}(0,T)$, $\mu(t) \ge \mu_* > 0$, $\forall t \in [0,T]$, $2\mu(t) - \alpha'(t) \ge 2\overline{\mu}_* > 0$, a.e., $t \in [0,T]$;

$(\overline{H}_5)$ $f_1, f_1' \in L^2(Q_T)$, f_1 is (η,T)-periodic, $f_1(0) = \eta f_1(T)$;

$(\overline{H}_6)$ $f \in C^0(\mathbb{R};\mathbb{R})$ and there exists a positive constant δ, with

$$\begin{aligned}&0 < \delta < \frac{2\overline{\mu}_*}{R^\gamma (R-1)^2}, \\ &\text{such that } (y-z)(f(y) - f(z)) \ge -\delta |y-z|^2, \\ &\qquad\qquad \forall y, z \in \mathbb{R}.\end{aligned} \tag{108}$$

Remark 10. An example of the functions g_1, g_R satisfying $(\overline{H}_2)$ is

$$g_k(t) = \zeta_k e^{pt}, \tag{109}$$

where $p > 0$, ζ_k, and $k \in \{1, R\}$ are constants. It is obvious that $(\overline{H}_2)$ holds, because

$$g_k(t) = e^{-pT}\zeta_k e^{p(t+T)} = \eta g_k(t+T), \tag{110}$$

with $\eta = e^{-pT}$ and

$$g_k(0) = \eta g_k(T), \quad k \in \{1, R\}. \tag{111}$$

Similarly, by the transformation $v(x,t) = u(x,t) - \varphi(x,t)$, with $\varphi(x,t) = ((x-1)/(R-1))g_R(t) + ((R-x)/(R-1))g_1(t)$, $\varphi(x,0) = \eta\varphi(x,T)$, problems (1), (2), and (4) reduce to the following problem:

$$\begin{aligned}&v_t - \left(\mu(t) + \alpha(t)\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\gamma}{x}\frac{\partial v}{\partial x}\right) + f(v+\varphi) \\ &\quad = \overline{f}_1(x,t), \quad 1 < x < R, \ 0 < t < T, \\ &v(1,t) = v(R,t) = 0, \\ &v(x,0) = \eta v(x,T),\end{aligned} \tag{112}$$

where $\overline{f}_1(x,t)$ is defined by $(25)_1$.

The weak formulation of problem (112) can be given in the following manner: Find $v \in L^\infty(0,T;H_0^1)$ with $v_t \in L^2(0,T;H_0^1)$, such that v satisfies the following variational equation:

$$\begin{aligned}&\int_0^T \left\langle v'(t), w(t)\right\rangle dt + \int_0^T \alpha(t)\, a\left(v'(t), w(t)\right) dt \\ &\quad + \int_0^T \mu(t)\, a(v(t), w(t))\, dt \\ &\quad + \int_0^T \left\langle f(v(t) + \varphi(t)), w(t)\right\rangle dt \\ &\quad = \int_0^T \left\langle \overline{f}_1(t), w(t)\right\rangle dt, \quad \forall w \in L^2\left(0,T;H_0^1\right), \\ &v(0) = \eta v(T),\end{aligned} \tag{113}$$

where $a(\cdot,\cdot)$ is the symmetric bilinear form on $H_0^1 \times H_0^1$ defined by (21).

Then, we have the following theorem.

Theorem 11. *Let $T > 0$ and $(\overline{H}_2)$–$(\overline{H}_6)$ hold. Then, problem (112) has a unique weak solution v such that*

$$\begin{aligned}v &\in L^\infty\left(0,T;H_0^1\right), \\ v_t &\in L^2\left(0,T;H_0^1\right).\end{aligned} \tag{114}$$

Proof. The proof consists of several steps.

Step 1 (the Faedo-Galerkin approximation (introduced by Lions [18])). Consider the basis $\{w_j\}$ for H_0^1 as in Lemma 5. Let V_m be the linear space generated by $w_1, w_2, \ldots, w_m$. We consider the following problem.

Find a function $v_m(t)$ in the form (29) satisfying the nonlinear differential equation system $(30)_1$ and the (η, T)-periodic condition:

$$v_m(0) = \eta v_m(T). \tag{115}$$

We consider an initial value problem given by (30), where v_{0m} is given in V_m.

It is clear that, for each m, there exists a solution $v_m(t)$ in the form (29) which satisfies (30) almost everywhere on $0 \le t \le \widetilde{T}_m$ for some $\widetilde{T}_m$, $0 < \widetilde{T}_m \le T$. The following a priori estimates allow us to take $\widetilde{T}_m = T$ for all m.

Step 2 (a priori estimates). Multiplying the jth equation of $(30)_1$ by $c_{mj}(t)$ and summing up with respect to j, we get

$$\begin{aligned} &\frac{d}{dt}\left[\|v_m(t)\|_0^2 + \alpha(t)\|v_{mx}(t)\|_0^2\right] \\ &\quad + \left(2\mu(t) - \alpha'(t)\right)\|v_{mx}(t)\|_0^2 \\ &\quad + 2\left\langle f\left(v_m(t) + \varphi(t)\right), v_m(t)\right\rangle \\ &= 2\left\langle \overline{f}_1(t), v_m(t)\right\rangle. \end{aligned} \tag{116}$$

By the same estimates as in Section 3, and with $\varepsilon_1 > 0$, we obtain

$$\begin{aligned} &2\left\langle f\left(v_m(t) + \varphi(t)\right), v_m(t)\right\rangle \\ &\quad = 2\left\langle f\left(v_m(t) + \varphi(t)\right) - f(\varphi(t)), v_m(t)\right\rangle \\ &\qquad + 2\left\langle f(\varphi(t)), v_m(t)\right\rangle \\ &\quad \ge -\left(2\delta + \varepsilon_1\right)\frac{R^\gamma}{2}(R-1)^2\|v_{mx}(t)\|_0^2 \\ &\qquad - \frac{1}{\varepsilon_1}\|f(\varphi(t))\|_0^2; \\ &2\left\langle \overline{f}_1(t), v_m(t)\right\rangle \le \frac{1}{\varepsilon_1}\left\|\overline{f}_1(t)\right\|_0^2 + \varepsilon_1\|v_m(t)\|_0^2 \\ &\quad \le \frac{1}{\varepsilon_1}\left\|\overline{f}_1(t)\right\|_0^2 + \varepsilon_1\frac{R^\gamma}{2}(R-1)^2\|v_{mx}(t)\|_0^2. \end{aligned} \tag{117}$$

Hence, it follows from (116) and (117) that

$$\begin{aligned} &\frac{d}{dt}\left[\|v_m(t)\|_0^2 + \alpha(t)\|v_{mx}(t)\|_0^2\right] \\ &\quad + 2\left[\overline{\mu}_* - \left(\delta + \varepsilon_1\right)\frac{R^\gamma}{2}(R-1)^2\right]\|v_{mx}(t)\|_0^2 \\ &\quad \le \frac{1}{\varepsilon_1}\left(\left\|\overline{f}_1(t)\right\|_0^2 + \|f(\varphi(t))\|_0^2\right). \end{aligned} \tag{118}$$

By $0 < \delta < 2\overline{\mu}_*/R^\gamma(R-1)^2$, choose $\varepsilon_1 > 0$ such that $\widetilde{\gamma} = \overline{\mu}_* - (\delta + \varepsilon_1)(R^\gamma/2)(R-1)^2 > 0$.

It is similar to (97); we get

$$\|v_{mx}(t)\|_0^2 \ge \beta_1\left(\alpha(t)\|v_{mx}(t)\|_0^2 + \|v_m(t)\|_0^2\right), \tag{119}$$

where $\beta_1 = (1/2)\min\{1/\|\alpha\|_{L^\infty(0,T)}, 2/R^\gamma(R-1)^2\}$.

From (118) and (119), it leads to

$$\begin{aligned} &\frac{d}{dt}\left[\|v_m(t)\|_0^2 + \alpha(t)\|v_{mx}(t)\|_0^2\right] \\ &\quad + 2\beta_1\widetilde{\gamma}\left(\|v_m(t)\|_0^2 + \alpha(t)\|v_{mx}(t)\|_0^2\right) \le \overline{f}(t), \end{aligned} \tag{120}$$

in which $\overline{f}(t) = (1/\varepsilon_1)[\|\overline{f}_1(t)\|_0^2 + \|f(\varphi(t))\|_0^2]$.

Integrating (120), we have

$$\begin{aligned} &\|v_m(t)\|_0^2 + \alpha(t)\|v_{mx}(t)\|_0^2 \\ &\quad \le \left(\|v_{0m}\|_0^2 + \alpha(0)\|v_{0mx}\|_0^2 + \int_0^t e^{2\beta_1\widetilde{\gamma}s}\overline{f}(s)\,ds\right) \\ &\quad \cdot e^{-2\beta_1\widetilde{\gamma}t} \le \rho^2 + \left(\|v_{0m}\|_0^2 + \alpha(0)\|v_{0mx}\|_0^2 - \rho^2\right) \\ &\quad \cdot e^{-2\beta_1\widetilde{\gamma}t}, \end{aligned} \tag{121}$$

where $\rho^2 = \sup_{0\le t\le T}\rho_1(t)$, with

$$\rho_1(t) = \begin{cases} \dfrac{1}{e^{2\beta_1\widetilde{\gamma}t} - 1}\displaystyle\int_0^t e^{2\beta_1\widetilde{\gamma}s}\overline{f}(s)\,ds, & 0 < t \le T, \\ \dfrac{1}{2\beta_1\widetilde{\gamma}}\overline{f}(0), & t = 0. \end{cases} \tag{122}$$

Therefore, if we choose v_{0m} such that $\|v_{0m}\|_0^2 + \alpha(0)\|v_{0mx}\|_0^2 \le \rho^2$, we obtain from (121) that

$$\begin{aligned} &\|v_m(t)\|_0^2 + \alpha(t)\|v_{mx}(t)\|_0^2 \le \rho^2, \\ &\text{i.e., } \widetilde{T}_m = T \\ &\forall m. \end{aligned} \tag{123}$$

Let $\overline{B}_m(\rho)$ be a closed ball in the space V_m with the norm $v_{0m} \mapsto \|v_{0m}\|_{V_m} = (\|v_{0m}\|_0^2 + \alpha(0)\|v_{0mx}\|_0^2)^{1/2}$.

By $0 < \alpha(0) \le \alpha(T)$, we obtain

$$\begin{aligned} \|v_m(T)\|_{V_m}^2 &= \|v_m(T)\|_0^2 + \alpha(0)\|v_{mx}(T)\|_0^2 \\ &\le \|v_m(T)\|_0^2 + \alpha(T)\|v_{mx}(T)\|_0^2 \le \rho^2. \end{aligned} \tag{124}$$

Let us define

$$\begin{aligned} \mathscr{F}_m : \overline{B}_m(\rho) &\longrightarrow \overline{B}_m(\rho) \\ v_{0m} &\longmapsto \mathscr{F}_m(v_{0m}) = \eta v_m(T). \end{aligned} \tag{125}$$

We prove that $\mathscr{F}_m$ is a contraction. Let $v_{0m}, \overline{v}_{0m} \in \overline{B}_m(0,\rho)$ and let $y_m(t) = v_m(t) - \overline{v}_m(t)$, where $v_m(t)$ and $\overline{v}_m(t)$ are solutions of system $(30)_1$ on $[0,T]$ satisfying the initial conditions $v_m(0) = v_{0m}$ and $\overline{v}_m(0) = \overline{v}_{0m}$, respectively. Then, $y_m(t)$ satisfies the following differential equation system:

$$\begin{aligned} &\left\langle y_m'(t), w_j\right\rangle + \alpha(t)\,a\left(y_m'(t), w_j\right) \\ &\quad + \mu(t)\,a\left(y_m(t), w_j\right) \\ &\quad + \left\langle f\left(v_m(t) + \varphi(t)\right) - f\left(\overline{v}_m(t) + \varphi(t)\right), w_j\right\rangle \\ &= 0, \end{aligned} \tag{126}$$

where $1 \le j \le m$, with initial condition

$$y_m(0) = v_{0m} - \overline{v}_{0m}. \tag{127}$$

By using the same arguments as before, we can show that

$$\begin{aligned}&\frac{d}{dt}\left[\|y_m(t)\|_0^2 + \alpha(t)\|y_{mx}(t)\|_0^2\right]\\&\quad + 2\beta_2\left[\|y_m(t)\|_0^2 + \alpha(t)\|y_{mx}(t)\|_0^2\right] \le 0,\end{aligned} \tag{128}$$

where $\beta_2 = \beta_1(\overline{\mu}_* - \delta(R^\gamma/2)(R-1)^2) > 0$, $\beta_1 = (1/2)\min\{1/\|\alpha\|_{L^\infty(0,T)}, 2/R^\gamma(R-1)^2\}$.

Integrating inequality (128), we obtain

$$\begin{aligned}&\|y_m(t)\|_0^2 + \alpha(t)\|y_{mx}(t)\|_0^2\\&\quad\le e^{-2\beta_2 t}\left(\|y_m(0)\|_0^2 + \alpha(0)\|y_{mx}(0)\|_0^2\right)\\&\quad= e^{-2\beta_2 t}\|y_m(0)\|_{V_m}^2, \quad \forall t \in [0,T].\end{aligned} \tag{129}$$

By $0 < \alpha(0) \le \alpha(T)$, it follows that

$$\begin{aligned}\|y_m(T)\|_{V_m}^2 &= \|y_m(T)\|_0^2 + \alpha(0)\|y_{mx}(T)\|_0^2\\&\le \|y_m(T)\|_0^2 + \alpha(T)\|y_{mx}(T)\|_0^2\\&\le e^{-2\beta_2 T}\|y_m(0)\|_{V_m}^2\\&= e^{-2\beta_2 T}\|v_{0m} - \overline{v}_{0m}\|_{V_m}^2,\end{aligned} \tag{130}$$

or

$$\begin{aligned}\|\mathcal{F}_m(v_{0m}) - \mathcal{F}_m(\overline{v}_{0m})\|_{V_m} &= \|\eta y_m(T)\|_{V_m}\\&\le |\eta| e^{-\beta_2 T}\|v_{0m} - \overline{v}_{0m}\|_{V_m};\end{aligned} \tag{131}$$

that is, $\mathcal{F}_m$ is a contraction.

Therefore, there exists a unique function $v_{0m} \in \overline{B}_m(\rho)$ such that the solution of the initial value problem (30) is a solution of system $(30)_1$, (115). This solution satisfies inequality (124) a.e., in $[0,T]$.

On the other hand, multiplying the jth equation of $(30)_1$ by $c'_{mj}(t)$ and summing up with respect to j, afterwards, integrating with respect to the time variable from 0 to T, we get after some rearrangements

$$\begin{aligned}&2\int_0^T\left(\|v'_m(t)\|_0^2 + \alpha(t)\|v'_{mx}(t)\|_0^2\right)dt\\&\quad+\int_0^T\frac{d}{dt}\left(\mu(t)\|v_{mx}(t)\|_0^2\right)dt\\&\quad+2\int_0^T\left\langle f(v_m(t)+\varphi(t)), v'_m(t)\right\rangle dt\\&=\int_0^T\mu'(t)\|v_{mx}(t)\|_0^2\,dt\\&\quad+2\int_0^T\left\langle \overline{f}_1(t), v'_m(t)\right\rangle dt.\end{aligned} \tag{132}$$

We estimate without difficulty the following terms in (132):

$$\begin{aligned}&-\int_0^T\frac{d}{dt}\left(\mu(t)\|v_{mx}(t)\|_0^2\right)dt\\&\quad= -\mu(T)\|v_{mx}(T)\|_0^2 + \mu(0)\|v_{mx}(0)\|_0^2\\&\quad\le \mu(0)\|v_{mx}(0)\|_0^2 \le \mu(0)\frac{\rho^2}{\alpha_*}.\end{aligned} \tag{133}$$

Note that

$$\begin{aligned}\|\varphi(t)\|_{C^0([1,R])} &\le |g_1(t)| + |g_R(t)|\\&\le \sup_{0\le t\le T}\left(|g_1(t)| + |g_R(t)|\right),\end{aligned} \tag{134}$$

$$\begin{aligned}&\|v_m(t)+\varphi(t)\|_{C^0([1,R])}\\&\quad\le \|v_m(t)\|_{C^0([1,R])} + \|\varphi(t)\|_{C^0([1,R])}\\&\quad\le \sqrt{R-1}\frac{\rho}{\sqrt{\alpha_*}} + \sup_{0\le t\le T}\left(|g_1(t)| + |g_R(t)|\right)\\&\quad= M_1(T).\end{aligned} \tag{135}$$

Hence,

$$\begin{aligned}&2\int_0^T\left\langle f(v_m(t)+\varphi(t)), v'_m(t)\right\rangle dt\\&\quad\le 2\int_0^T\|f(v_m(t)+\varphi(t))\|_0^2\,dt\\&\qquad+\frac{1}{2}\int_0^T\|v'_m(t)\|_0^2\,dt\\&\quad\le M_2(T) + \frac{1}{2}\int_0^T\|v'_m(t)\|_0^2\,dt,\end{aligned} \tag{136}$$

where

$$M_2(T) = 2T\left(\frac{R^{\gamma+1}-1}{\gamma+1}\right)\sup_{|z|\le M_1(T)} f^2(z). \tag{137}$$

Moreover,

$$\begin{aligned}&2\int_0^T\left\langle \overline{f}_1(t), v'_m(t)\right\rangle dt\\&\quad\le 2\int_0^T\|\overline{f}_1(t)\|_0^2\,dt + \frac{1}{2}\int_0^T\|v'_m(t)\|_0^2\,dt;\\&\int_0^T\mu'(t)\|v_{mx}(t)\|_0^2\,dt \le \frac{\rho^2}{\alpha_*}\int_0^T|\mu'(t)|\,dt.\end{aligned} \tag{138}$$

It follows from (132), (133), (136), and (138) that

$$\begin{aligned}&\int_0^T\|v'_m(t)\|_0^2\,dt + 2\alpha_*\int_0^T\|v'_{mx}(t)\|_0^2\,dt\\&\quad\le \mu(0)\frac{\rho^2}{\alpha_*} + M_2(T) + 2\int_0^T\|\overline{f}_1(t)\|_0^2\,dt\\&\qquad+\frac{\rho^2}{\alpha_*}\int_0^T|\mu'(t)|\,dt \le C_T,\end{aligned} \tag{139}$$

for all $m \in \mathbb{N}$, for all $t \in [0,T]$, where C_T always indicates a bound depending on T.

Step 3 (the limiting process). By (123) and (139), we deduce that there exists a subsequence of $\{v_m\}$, still denoted by $\{v_m\}$ such that

$$\begin{aligned} v_m &\longrightarrow v \quad \text{in } L^\infty\left(0,T;H_0^1\right) \text{ weakly}^*, \\ v_m' &\longrightarrow v' \quad \text{in } L^2\left(0,T;H_0^1\right) \text{ weakly.} \end{aligned} \tag{140}$$

From (115), we obtain

$$v(0) = \eta v(T). \tag{141}$$

Indeed, we prove (141) as follows.

By $\|v_{0mx}\| \le (1/\sqrt{\alpha_*})\|v_{0m}\|_{V_m} \le \rho/\sqrt{\alpha_*}$, and by the imbedding $H_0^1 \hookrightarrow C^0(\overline{\Omega})$ being compact, there exists a subsequence of $\{v_{0m}\}$, still denoted by $\{v_{0m}\}$ such that

$$\begin{aligned} v_{0m} &\longrightarrow \widetilde{v}_0 \quad \text{in } H_0^1 \text{ weakly,} \\ v_{0m} &\longrightarrow \widetilde{v}_0 \quad \text{in } C^0([1,R]) \text{ strongly.} \end{aligned} \tag{142}$$

By $v_m(t) = v_m(0) + \int_0^t v_m'(s)ds$, we deduce from (140) and (142) that

$$v(t) = \widetilde{v}_0 + \int_0^t v'(s)\,ds. \tag{143}$$

This implies

$$v(0) = \widetilde{v}_0, \tag{144}$$

$$\begin{aligned} v_{0m} &\longrightarrow v(0) \quad \text{in } H_0^1 \text{ weakly,} \\ v_{0m} &\longrightarrow v(0) \quad \text{in } C^0([1,R]) \text{ strongly.} \end{aligned} \tag{145}$$

From (115), we obtain

$$\begin{aligned} \left\langle v_m(0), w_j\right\rangle &= \eta\left\langle v_m(T), w_j\right\rangle \\ &= \eta\left(\left\langle v_m(0), w_j\right\rangle + \int_0^T \left\langle v_m'(t), w_j\right\rangle dt\right), \\ &\qquad \forall j \in \mathbb{N}. \end{aligned} \tag{146}$$

By (140), (145), and (146), it yields

$$\begin{aligned} \left\langle v(0), w_j\right\rangle &= \eta\left(\left\langle v(0), w_j\right\rangle + \int_0^T \left\langle v'(t), w_j\right\rangle dt\right) \\ &= \eta\left\langle v(T), w_j\right\rangle, \quad \forall j \in \mathbb{N}. \end{aligned} \tag{147}$$

Therefore,

$$v(0) = \eta v(T). \tag{148}$$

Therefore, (141) is proved.

Using a compactness lemma ([18], Lions, p. 57), applied to (140), we can extract from the sequence $\{v_m\}$ a subsequence still denoted by $\{v_m\}$, such that

$$v_m \longrightarrow v \quad \text{strongly in } L^2(Q_T). \tag{149}$$

By the Riesz-Fischer theorem, we can extract from $\{v_m\}$ a subsequence still denoted by $\{v_m\}$, such that

$$v_m(x,t) \longrightarrow v(x,t) \quad \text{a.e., } (x,t) \text{ in } Q_T, \tag{150}$$

Because f is continuous, then

$$\begin{aligned} &f\left(v_m(x,t) + \varphi(x,t)\right) \longrightarrow \\ &f\left(v(x,t) + \varphi(x,t)\right) \quad \text{a.e., } (x,t) \text{ in } Q_T. \end{aligned} \tag{151}$$

On the other hand,

$$\begin{aligned} \left|f\left(v_m(x,t) + \varphi(x,t)\right)\right| &\le \sup_{|z| \le M_1(T)} \left|f(z)\right|, \\ &\qquad \text{a.e., } (x,t) \text{ in } Q_T, \end{aligned} \tag{152}$$

where $M_1(T)$ is the constant defined by (135).

Using the dominated convergence theorem, (151) and (152) yield

$$f(v_m + \varphi) \longrightarrow f(v + \varphi) \quad \text{strongly in } L^2(Q_T). \tag{153}$$

Denote by $\{\zeta_i,\ i = 1,2,\ldots\}$ the orthonormal base in the real Hilbert space $L^2(0,T)$. The set $\{\zeta_i w_j,\ i,j = 1,2,\ldots\}$ forms an orthonormal base in $L^2(0,T;H_0^1)$. From $(30)_1$ we have

$$\begin{aligned} &\int_0^T \left\langle v_m'(t), w_j\zeta_i(t)\right\rangle dt \\ &\quad + \int_0^T \alpha(t)\,a\left(v_m'(t), w_j\zeta_i(t)\right) dt \\ &\quad + \int_0^T \mu(t)\,a\left(v_m(t), w_j\zeta_i(t)\right) dt \\ &\quad + \int_0^T \left\langle f\left(v_m(t) + \varphi(t)\right), w_j\zeta_i(t)\right\rangle dt \\ &= \int_0^T \left\langle \overline{f}_1(t), w_j\zeta_i(t)\right\rangle dt, \end{aligned} \tag{154}$$

for all i, j, $1 \le j \le m$, $i \in \mathbb{N}$.

For i, j fixed, passing to the limit in (154) by (140) and (153), we get

$$\begin{aligned} &\int_0^T \left\langle v'(t), w_j\zeta_i(t)\right\rangle dt \\ &\quad + \int_0^T \alpha(t)\,a\left(v'(t), w_j\zeta_i(t)\right) dt \\ &\quad + \int_0^T \mu(t)\,a\left(v(t), w_j\zeta_i(t)\right) dt \\ &\quad + \int_0^T \left\langle f\left(v(t) + \varphi(t)\right), w_j\zeta_i(t)\right\rangle dt \\ &= \int_0^T \left\langle \overline{f}_1(t), w_j\zeta_i(t)\right\rangle dt. \end{aligned} \tag{155}$$

Note that (155) holds for every $i, j \in \mathbb{N}$; that is, the equality

$$\begin{aligned}&\int_0^T \left\langle v'(t), w(t)\right\rangle dt + \int_0^T \alpha(t)\, a\left(v'(t), w(t)\right) dt \\ &\quad + \int_0^T \mu(t)\, a(v(t), w(t))\, dt \\ &\quad + \int_0^T \left\langle f\left(v(t)+\varphi(t)\right), w(t)\right\rangle dt \\ &= \int_0^T \left\langle \overline{f}_1(t), w(t)\right\rangle dt, \quad \forall w \in L^2\left(0,T;H_0^1\right),\end{aligned} \tag{156}$$

is fulfilled.

Step 4 (uniqueness of the solutions). Let v_1 and v_2 be two solutions of (113). Then $v = v_1 - v_2$ satisfies the following problem:

$$\begin{aligned}&\int_0^T \left\langle v'(t), w(t)\right\rangle dt + \int_0^T \alpha(t)\, a\left(v'(t), w(t)\right) dt \\ &\quad + \int_0^T \mu(t)\, a(v(t), w(t))\, dt \\ &\quad + \int_0^T \langle f\left(v_1(t)+\varphi(t)\right) \\ &\quad - f\left(v_2(t)+\varphi(t)\right), w(t)\rangle\, dt = 0, \\ &\qquad\qquad \forall w \in L^2\left(0,T;H_0^1\right), \\ &v(0) = \eta v(T), \\ &\qquad v \in L^\infty\left(0,T;H_0^1\right),\ v_t \in L^2\left(0,T;H_0^1\right).\end{aligned} \tag{157}$$

Taking $w = 2v$ in $(157)_1$ and using $(157)_2$, we get

$$\begin{aligned}&\int_0^T \left(2\mu(t) - \alpha'(t)\right)\left\|v_x(t)\right\|_0^2 dt + \int_0^T \frac{d}{dt}\left\|v(t)\right\|_0^2 dt \\ &\quad + \int_0^T \frac{d}{dt}\left(\alpha(t)\left\|v_x(t)\right\|_0^2\right) dt \\ &\quad + 2\int_0^T \langle f\left(v_1(t)+\varphi(t)\right) \\ &\quad - f\left(v_2(t)+\varphi(t)\right), v(t)\rangle\, dt = 0.\end{aligned} \tag{158}$$

By $2\mu(t) - \alpha'(t) \geq \overline{\mu}_* > 0$, we have

$$\begin{aligned}&\int_0^T \left(2\mu(t) - \alpha'(t)\right)\left\|v_x(t)\right\|_0^2 dt \\ &\quad \geq 2\overline{\mu}_* \int_0^T \left\|v_x(t)\right\|_0^2 dt.\end{aligned} \tag{159}$$

On the other hand,

$$\begin{aligned}&\int_0^T \frac{d}{dt}\left\|v(t)\right\|_0^2 dt = \left\|v(T)\right\|_0^2 - \left\|v(0)\right\|_0^2 \\ &\quad = \left(1-\eta^2\right)\left\|v(T)\right\|_0^2 \geq 0; \\ &\int_0^T \frac{d}{dt}\left(\alpha(t)\left\|v_x(t)\right\|_0^2\right) dt \\ &\quad = \alpha(t)\left\|v_x(T)\right\|_0^2 - \alpha(0)\left\|v_x(0)\right\|_0^2 \\ &\quad \geq \alpha(0)\left(1-\eta^2\right)\left\|v_x(T)\right\|_0^2 \geq 0.\end{aligned} \tag{160}$$

Hence,

$$\begin{aligned}&2\overline{\mu}_* \int_0^T \left\|v_x(t)\right\|_0^2 dt \leq -2\int_0^T \langle f\left(v_1(t)+\varphi(t)\right) \\ &\quad - f\left(v_2(t)+\varphi(t)\right), v(t)\rangle\, dt \\ &\quad \leq 2\delta \int_0^T \left\|v(t)\right\|_0^2 dt \leq 2\delta \frac{R^\gamma}{2}(R-1)^2 \\ &\quad \cdot \int_0^T \left\|v_x(t)\right\|_0^2 dt.\end{aligned} \tag{161}$$

By $0 < \delta < 2\overline{\mu}_*/R^\gamma(R-1)^2$, implying $\delta(R^\gamma/2)(R-1)^2 < \overline{\mu}_*$, we deduce from (161) that $\int_0^T \|v_x(t)\|_0^2 dt = 0$; that is, $v = v_1 - v_2 = 0$.

This completes the proof of Theorem 11. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this article. They read and approved the final manuscript.

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