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Research Article

Research on Geometric Mappings in Complex Systems Analysis

Yanyan Cui,[1,2] Chaojun Wang,[1] and Sifeng Zhu[1]

[1] *College of Mathematics and Statistics, Zhoukou Normal University, Zhoukou, Henan 466001, China*
[2] *College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei 050016, China*

Correspondence should be addressed to Yanyan Cui; cui9907081@163.com

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We mainly discuss the properties of a new subclass of starlike functions, namely, almost starlike functions of complex order λ, in one and several complex variables. We get the growth and distortion results for almost starlike functions of complex order λ. By the properties of functions with positive real parts and considering the zero of order k, we obtain the coefficient estimates for almost starlike functions of complex order λ on D. We also discuss the invariance of almost starlike mappings of complex order λ on Reinhardt domains and on the unit ball B in complex Banach spaces. The conclusions contain and generalize some known results.

1. Introduction

The growth, distortion theorems, and coefficient estimates for univalent functions are important research contents in geometric function theory of one and several complex variables. In one complex variable we have the following known growth and distortion theorems.

Theorem 1 (see [1]). *Let f be a normalized biholomorphic function on the unit disk D in $\mathbb{C}$. Then*

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}, \quad \forall z \in D,$$
$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad \forall z \in D. \tag{1}$$

The distortion theorem for univalent functions on D was introduced by Koebe. In the process of generalizing the distortion theorem for univalent functions on D to the unit ball in $\mathbb{C}^n$, Cartan et al. [2, 3] found the distortion theorem did not hold for general biholomorphic mappings. Cartan et al. suggested restricting the mappings by geometric properties, such as starlikeness and convexity. So many people began to discuss the growth and distortion theorems for biholomorphic mappings with special geometric properties. Starlike mappings and convex mappings are discussed most [4–7]. While they discussed starlike mappings and convex mappings, they introduced some subclasses of those mappings. Furthermore, they always discussed these new subclasses in one complex variable firstly. The growth and distortion results [8] for starlike functions are the same as Theorem 1. We have the following results with respect to convex functions.

Theorem 2 (see [9]). *Let f be a convex function on D with $f(0)=0$ and $f'(0)=1$. Then*

$$\frac{|z|}{1+|z|} \le |f(z)| \le \frac{|z|}{1-|z|}, \quad \forall z \in D,$$
$$\frac{1}{(1+|z|)^2} \le |f'(z)| \le \frac{1}{(1-|z|)^2}, \quad \forall z \in D. \tag{2}$$

In 1936, Robertson [10] obtained the growth, covering theorems and coefficient estimates for starlike functions of order α.

Theorem 3 (see [10]). *Let $f(z)=z+\sum_{j=2}^{n} a_n z^n$ be a starlike function of order α on D. Then*

$$\frac{|z|}{(1+|z|)^{2(1-\alpha)}} \le |f(z)| \le \frac{|z|}{(1-|z|)^{2(1-\alpha)}}, \quad \forall z \in D,$$
$$|a_n| \le \frac{1}{(n-1)!}\prod_{k=2}^{n}(k-2\alpha), \quad n=2,3,\dots. \tag{3}$$

In 1966, Boyd [11] obtained the coefficient estimates for the starlike function f of order α, where $\alpha \in (0,1)$ and $z = 0$ is the zero of order $k+1$ of $f(z)-z$. In 1975, Silverman [12] discussed the distortion theorems and coefficient estimates for univalent analytic functions with negative coefficients. In 1991, Srivastava and Owa [13] obtained the distortion theorems and coefficient estimates for a subclass $S^*(\alpha,\beta,\gamma)$ of starlike functions.

Recently, there are many nice results about the growth, distortion theorems, and coefficient estimates for subclasses of starlike functions and convex functions. Obviously we hope there exist similar results in several complex variables. In the process of discussing the properties of subclasses of starlike mappings and convex mappings, we need the specific examples of the mappings. It is easy to find specific examples of these new subclasses in $\mathbb{C}$, while it is very difficult in $\mathbb{C}^n$.

In 1995, Roper and Suffridge [14] introduced an operator

$$\phi_n(f)(z) = \left(f(z_1), \sqrt{f'(z_1)}z_0\right)', \quad (4)$$

where $z = (z_1, z_0) \in B^n$, $z_1 \in D$, $z_0 = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$, $f(z_1) \in H(D)$, and $\sqrt{f'(0)} = 1$. They proved the operator preserves convexity and starlikeness on B^n. Graham and Kohr [15] proved the Roper-Suffridge operator preserves the properties of Bloch mappings on B^n. Thus, we can construct lots of convex mappings and starlike mappings on B^n by corresponding functions on D by the Roper-Suffridge operator. So the Roper-Suffridge operator plays an important role in several complex variables. Later, many people generalized the Roper-Suffridge operator on different domains and different spaces so as to construct biholomorphic mappings with specific geometric properties in several complex variables. Therefore we can discuss the properties of biholomorphic mappings in several variables preferably. In recent years, there are many results about the generalized Roper-Suffridge operators (such as [16–18]).

In this paper, we mainly discuss almost starlike functions of complex order λ in one and several complex variables. In Sections 2 and 3, we discuss the growth, distortion theorems, and coefficient estimates for almost starlike functions of complex order λ on D, respectively. In Section 4, we discuss the invariance of almost starlike mappings of complex order λ on Reinhardt domains and on the unit ball B in complex Banach spaces. Thereby, we can construct lots of almost starlike mappings of complex order λ in several complex variables through almost starlike functions of complex order λ on D. The conclusions generalize some known results.

To get the main results, we need the following definitions.

Definition 4 (see [19]). Let $\Omega \subset \mathbb{C}^n$ be a bounded starlike and circular domain whose Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let $F(z)$ be a normalized locally biholomorphic mapping on Ω. Let

$$\Re\left[(1-\lambda)\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right] \geq -\Re\lambda, \quad z \in \Omega\setminus\{0\}, \quad (5)$$

where $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda \leq 0$. Then $F(z)$ is called an almost starlike mapping of complex order λ on Ω.

For $n = 1$, the condition in Definition 4 reduces to

$$\Re\left[(1-\lambda)\frac{F(z)}{zF'(z)}\right] \geq -\Re\lambda. \quad (6)$$

Definition 5 (see [19]). Let $F(z)$ be a normalized locally biholomorphic mapping on the unit ball B in complex Banach spaces. Let

$$\Re\left\{(1-\lambda)T_x\left[(DF(x))^{-1}F(x)\right]\right\} \geq -\Re\lambda\|x\|, \quad x \in B\setminus\{0\}, \quad (7)$$

where $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda \leq 0$. Then $F(z)$ is called an almost starlike mapping of complex order λ on B.

Setting $\lambda = \alpha/(\alpha-1)$, $\alpha \in [0,1)$ in Definitions 4 and 5, we obtain the definition of almost starlike mappings of order α.

2. Growth and Distortion Results

Lemma 6 (see [20]). *$|(z-z_1)/(z-z_2)| = k$ $(0 < k \neq 1,\ z_1 \neq z_2)$ indicates a circle in $\mathbb{C}$ whose center is z_0 and whose radius is ρ, where*

$$z_0 = \frac{z_1 - k^2 z_2}{1-k^2}, \qquad \rho = \frac{k|z_1 - z_2|}{|1-k^2|}. \quad (8)$$

Theorem 7. *Let $f(z)$ be an almost starlike function of complex order λ on D with $\lambda \in \mathbb{C}\setminus\{-1\}$, $\operatorname{Re}\lambda \leq 0$. Then*

$$\begin{aligned}
&|z|\left|\frac{1-\lambda}{1+\lambda}\right|^{2(1+\Re\lambda)/|1+\lambda|^2}\left(\left|\frac{1-\lambda}{1+\lambda}\right|-|z|\right)^{(|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2} \\
&\cdot\left(\left|\frac{1-\lambda}{1+\lambda}\right|+|z|\right)^{(-|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2} \leq |f(z)| \leq |z| \\
&\cdot\left|\frac{1-\lambda}{1+\lambda}\right|^{2(1+\Re\lambda)/|1+\lambda|^2} \\
&\cdot\left(\left|\frac{1-\lambda}{1+\lambda}\right|-|z|\right)^{(-|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2} \\
&\cdot\left(\left|\frac{1-\lambda}{1+\lambda}\right|+|z|\right)^{(|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2}.
\end{aligned} \quad (9)$$

Proof. Since $f(z)$ is an almost starlike function of complex order λ on D, then

$$\Re\left[(1-\lambda)\frac{f(z)}{zf'(z)}\right] \geq \Re(-\lambda), \quad (10)$$

which follows

$$\left|2\Re(-\lambda)\frac{1}{(1-\lambda)(f(z)/zf'(z))} - 1\right| \leq 1. \quad (11)$$

Let

$$g(z)=\begin{cases}\dfrac{zf'(z)}{(1-\lambda)f(z)}, & z\in D\setminus\{0\},\\ \dfrac{1}{1-\lambda}, & z=0.\end{cases}\tag{12}$$

Therefore $|2\Re(-\lambda)g(z)-1|\le 1$. Let $p(z)=2\Re(-\lambda)g(z)-1$. We have $|p(z)|\le 1$ and $p(0)=2\Re(-\lambda)/(1-\lambda)-1$. Let

$$h(z)=\frac{p(z)-p(0)}{1-\overline{p(0)}p(z)},\quad z\in D.\tag{13}$$

Then $h(0)=0$ and $|h(z)|<1$. By Schwarz lemma we obtain $|h(z)|\le|z|$ which follows

$$\left|\frac{2\Re(-\lambda)g(z)-2\Re(-\lambda)/(1-\lambda)}{1-\left(2\Re(-\lambda)/\left(1-\overline{\lambda}\right)-1\right)[2\Re(-\lambda)g(z)-1]}\right|\le|z|.\tag{14}$$

For

$$\begin{aligned}1&-\left(\frac{2\Re(-\lambda)}{1-\overline{\lambda}}-1\right)[2\Re(-\lambda)g(z)-1]\\&=1+\frac{1+\lambda}{1-\overline{\lambda}}[2\Re(-\lambda)g(z)-1]\\&=\frac{-2\Re\lambda[(1+\lambda)g(z)+1]}{1-\overline{\lambda}},\end{aligned}\tag{15}$$

then, by (14), we obtain

$$\left|\frac{(1-\lambda)g(z)-1}{(1+\lambda)g(z)+1}\right|\le|z|,\tag{16}$$

which follows

$$\left|\frac{zf'(z)/f(z)-1}{zf'(z)/f(z)+(1-\lambda)/(1+\lambda)}\right|\le\left|\frac{1+\lambda}{1-\lambda}\right||z|.\tag{17}$$

By Lemma 6 we get that $zf'(z)/f(z)$ is in a circle whose center is a and whose radius is ρ, where

$$\begin{aligned}a&=\frac{1+|(1+\lambda)/(1-\lambda)|^2|z|^2((1-\lambda)/(1+\lambda))}{1-|(1+\lambda)/(1-\lambda)|^2|z|^2},\\ \rho&=\frac{|(1+\lambda)/(1-\lambda)||z||1+(1-\lambda)/(1+\lambda)|}{\left|1-|(1+\lambda)/(1-\lambda)|^2|z|^2\right|}\\&=\frac{2|z||1-\lambda|}{|1-\lambda|^2-|1+\lambda|^2|z|^2}.\end{aligned}\tag{18}$$

So we obtain

$$\Re a-\rho\le\Re\frac{zf'(z)}{f(z)}\le\Re a+\rho.\tag{19}$$

By direct computation we have

$$\begin{aligned}\Re a-\rho&=\frac{1+|(1+\lambda)/(1-\lambda)|^2|z|^2\,\Re((1-\lambda)/(1+\lambda))}{1-|(1+\lambda)/(1-\lambda)|^2|z|^2}-\frac{2|z||1-\lambda|}{|1-\lambda|^2-|1+\lambda|^2|z|^2}\\&=\frac{1+|(1+\lambda)/(1-\lambda)|^2|z|^2\left(\left(1-|\lambda|^2\right)/\left(1+|\lambda|^2+2\Re\lambda\right)\right)}{1-|(1+\lambda)/(1-\lambda)|^2|z|^2}-\frac{2|z||1-\lambda|}{|1-\lambda|^2-|1+\lambda|^2|z|^2}\\&=\frac{|1+\lambda|^2\left(\left(1-|\lambda|^2\right)/\left(1+|\lambda|^2+2\Re\lambda\right)\right)|z|^2-2|1-\lambda||z|+|1-\lambda|^2}{|1-\lambda|^2-|1+\lambda|^2|z|^2}.\end{aligned}\tag{20}$$

Similarly, we can get

$$\Re a+\rho=\frac{|1+\lambda|^2\left(\left(1-|\lambda|^2\right)/\left(1+|\lambda|^2+2\Re\lambda\right)\right)|z|^2+2|1-\lambda||z|+|1-\lambda|^2}{|1-\lambda|^2-|1+\lambda|^2|z|^2}=\frac{m|z|^2+2|1-\lambda||z|+|1-\lambda|^2}{|1-\lambda|^2-|1+\lambda|^2|z|^2},\tag{21}$$

where $m=|1+\lambda|^2((1-|\lambda|^2)/(1+|\lambda|^2+2\Re\lambda))$. Let

$$f(x)=\frac{mx^2+2|1-\lambda|x+|1-\lambda|^2}{|1-\lambda|^2-|1+\lambda|^2x^2},\quad x\in[0,1)\tag{22}$$

Immediately, we have

$$f'(x)=\frac{P}{\left(|1-\lambda|^2-|1+\lambda|^2x^2\right)^2},\tag{23}$$

where

$$P = (2mx + 2|1-\lambda|)\left(|1-\lambda|^2 - |1+\lambda|^2 x^2\right) + \left(mx^2 + 2|1-\lambda| x + |1-\lambda|^2\right) 2|1+\lambda|^2 x = 2|1-\lambda||1+\lambda|^2 x^2 + 2|1-\lambda|^2\left(m + |1+\lambda|^2\right) x + 2|1-\lambda|^3 = 2|1-\lambda|\left[|1+\lambda|^2 x^2 + |1-\lambda||1+\lambda|^2 \cdot \frac{2(1+\Re\lambda)}{1+|\lambda|^2+2\Re\lambda} x + |1-\lambda|^2\right] = 2|1-\lambda||1+\lambda|^2 \cdot \left[\left(x + \frac{|1-\lambda|(1+\Re\lambda)}{1+|\lambda|^2+2\Re\lambda}\right)^2 + \frac{|1-\lambda|^2}{|1+\lambda|^2} - \frac{|1-\lambda|^2(1+\Re\lambda)^2}{\left(1+|\lambda|^2+2\Re\lambda\right)^2}\right] = 2|1-\lambda||1+\lambda|^2 \left\{\left(x + \frac{|1-\lambda|(1+\Re\lambda)}{1+|\lambda|^2+2\Re\lambda}\right)^2 + \frac{|1-\lambda|^2}{|1+\lambda|^2\left(1+|\lambda|^2+2\Re\lambda\right)^2}\left[\left(1+|\lambda|^2+2\Re\lambda\right)^2 - |1+\lambda|^2(1+\Re\lambda)^2\right]\right\} = 2|1-\lambda||1+\lambda|^2\left[\left(x + \frac{|1-\lambda|(1+\Re\lambda)}{1+|\lambda|^2+2\Re\lambda}\right)^2 + \frac{|1-\lambda|^2(\Im\lambda)^2}{|1+\lambda|^2\left(1+|\lambda|^2+2\Re\lambda\right)^2}\right] \ge 0, \tag{24}$$

which follows $f'(x) \ge 0$. Therefore $f(x)$ is monotone increasing. So

$$\Re a + \rho = \frac{m|z|^2 + 2|1-\lambda||z| + |1-\lambda|^2}{|1-\lambda|^2 - |1+\lambda|^2|z|^2} \le \frac{mr^2 + 2|1-\lambda| r + |1-\lambda|^2}{|1-\lambda|^2 - |1+\lambda|^2 r^2}, \tag{25}$$

where $z = re^{i\theta}, \theta \in [0, 2\pi)$. By (19) we have $\Re(zf'(z)/f(z)) = r(\partial/\partial r)\ln|f(z)| \le \Re a + \rho$. Therefore

$$\frac{\partial}{\partial r}\ln|f(z)| \le \frac{1}{r}\frac{mr^2 + 2|1-\lambda| r + |1-\lambda|^2}{|1-\lambda|^2 - |1+\lambda|^2 r^2} = \frac{1}{r\left(|1-\lambda|^2 - |1+\lambda|^2 r^2\right)}\left[\frac{|\lambda|^2-1}{1+|\lambda|^2+2\Re\lambda}\left(|1-\lambda|^2 - |1+\lambda|^2 r^2\right) + \frac{1-|\lambda|^2}{1+|\lambda|^2+2\Re\lambda}|1-\lambda|^2 + |1-\lambda|^2 + 2|1-\lambda| r\right] = \frac{1}{r}\left[\frac{|\lambda|^2-1}{1+|\lambda|^2+2\Re\lambda} + 2|1-\lambda|\frac{r + (1+\Re\lambda)|1-\lambda|/\left(1+|\lambda|^2+2\Re\lambda\right)}{|1-\lambda|^2 - |1+\lambda|^2 r^2}\right]. \tag{26}$$

Integrating both ends of (26) on $[\varepsilon, r]$, we obtain

$$\int_\varepsilon^r \frac{\partial}{\partial t}\ln\left|f\left(te^{i\theta}\right)\right| dt \le \frac{|\lambda|^2-1}{1+|\lambda|^2+2\Re\lambda}\int_\varepsilon^r \frac{dt}{t} + 2|1-\lambda|\left[\int_\varepsilon^r \frac{dt}{|1-\lambda|^2 - |1+\lambda|^2 t^2} + \frac{(1+\Re\lambda)|1-\lambda|}{1+|\lambda|^2+2\Re\lambda}\int_\varepsilon^r \frac{1}{t}\frac{dt}{|1-\lambda|^2 - |1+\lambda|^2 t^2}\right] = \frac{|\lambda|^2-1}{1+|\lambda|^2+2\Re\lambda}\ln t\Big|_\varepsilon^r + 2|1-\lambda|\frac{1}{2\left|1-\lambda^2\right|} \cdot \ln\left|\frac{t + |(1-\lambda)/(1+\lambda)|}{t - |(1-\lambda)/(1+\lambda)|}\right|\Bigg|_\varepsilon^r + \frac{2(1+\Re\lambda)|1-\lambda|^2}{1+|\lambda|^2+2\Re\lambda} \cdot \frac{1}{2|1-\lambda|^2}\ln\frac{t^2}{\left||1-\lambda|^2 - |1+\lambda|^2 t^2\right|}\Bigg|_\varepsilon^r. \tag{27}$$

Therefore

$$\ln|f(z)| - \ln\left|f\left(\varepsilon e^{i\theta}\right)\right| \le \frac{|\lambda|^2-1}{1+|\lambda|^2+2\Re\lambda}(\ln r - \ln\varepsilon) + \frac{1}{|1+\lambda|}\left(\ln\left|\frac{r + |(1-\lambda)/(1+\lambda)|}{r - |(1-\lambda)/(1+\lambda)|}\right| - \ln\left|\frac{\varepsilon + |(1-\lambda)/(1+\lambda)|}{\varepsilon - |(1-\lambda)/(1+\lambda)|}\right|\right) + \frac{(1+\Re\lambda)}{1+|\lambda|^2+2\Re\lambda}\left(\ln\frac{r^2}{\left||1-\lambda|^2 - |1+\lambda|^2 r^2\right|} - \ln\frac{\varepsilon^2}{\left||1-\lambda|^2 - |1+\lambda|^2 \varepsilon^2\right|}\right). \tag{28}$$

Since

$$\ln\left|f\left(\varepsilon e^{i\theta}\right)\right| - \frac{|\lambda|^2-1}{1+|\lambda|^2+2\Re\lambda}\ln\varepsilon - \frac{1+\Re\lambda}{1+|\lambda|^2+2\Re\lambda}\ln\varepsilon^2 = \ln\frac{\left|f\left(\varepsilon e^{i\theta}\right)\right|}{\varepsilon} \longrightarrow 0 \quad (\varepsilon \longrightarrow 0), \tag{29}$$

setting $\varepsilon \to 0$ in (28), we have

$$\begin{aligned}\ln\left|f(z)\right| \le{}& \frac{|\lambda|^2-1}{1+|\lambda|^2+2\Re\lambda}\ln r+\frac{1}{|1+\lambda|}\\&\cdot\ln\left|\frac{r+|(1-\lambda)/(1+\lambda)|}{r-|(1-\lambda)/(1+\lambda)|}\right|\\&+\frac{1+\Re\lambda}{1+|\lambda|^2+2\Re\lambda}\left(\ln\frac{r^2}{\left||1-\lambda|^2-|1+\lambda|^2r^2\right|}\right.\\&\left.+\ln|1-\lambda|^2\right).\end{aligned} \tag{30}$$

Therefore

$$\begin{aligned}\left|f(z)\right| \le{}& r^{(|\lambda|^2-1)/(1+|\lambda|^2+2\Re\lambda)}\\&\cdot\left|\frac{r+|(1-\lambda)/(1+\lambda)|}{r-|(1-\lambda)/(1+\lambda)|}\right|^{1/|1+\lambda|}\\&\cdot\left(\frac{|1-\lambda|^2r^2}{\left||1-\lambda|^2-|1+\lambda|^2r^2\right|}\right)^{(1+\Re\lambda)/(1+|\lambda|^2+2\Re\lambda)}\\={}& r\left|\frac{r+|(1-\lambda)/(1+\lambda)|}{r-|(1-\lambda)/(1+\lambda)|}\right|^{1/|1+\lambda|}\\&\cdot\left(\frac{1}{\left|1-|(1+\lambda)/(1-\lambda)|^2r^2\right|}\right)^{(1+\Re\lambda)/(1+|\lambda|^2+2\Re\lambda)}\\={}& r\left|\frac{1-\lambda}{1+\lambda}\right|^{2(1+\Re\lambda)/(1+|\lambda|^2+2\Re\lambda)}\\&\cdot\left|r-\left|\frac{1-\lambda}{1+\lambda}\right|\right|^{-1/|1+\lambda|-(1+\Re\lambda)/(1+|\lambda|^2+2\Re\lambda)}\\&\cdot\left|r+\left|\frac{1-\lambda}{1+\lambda}\right|\right|^{1/|1+\lambda|-(1+\Re\lambda)/(1+|\lambda|^2+2\Re\lambda)}\\={}& r\left|\frac{1-\lambda}{1+\lambda}\right|^{2(1+\Re\lambda)/|1+\lambda|^2}\\&\cdot\left|r-\left|\frac{1-\lambda}{1+\lambda}\right|\right|^{(-|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2}\\&\cdot\left|r+\left|\frac{1-\lambda}{1+\lambda}\right|\right|^{(|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2}=|z|\\&\cdot\left|\frac{1-\lambda}{1+\lambda}\right|^{2(1+\Re\lambda)/|1+\lambda|^2}\\&\cdot\left||z|-\left|\frac{1-\lambda}{1+\lambda}\right|\right|^{(-|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2}\\&\cdot\left(|z|+\left|\frac{1-\lambda}{1+\lambda}\right|\right)^{(|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2}.\end{aligned} \tag{31}$$

Obviously $|(1-\lambda)/(1+\lambda)| \ge 1$ for $\Re\lambda \le 0$. Thus $|z| \le |(1-\lambda)/(1+\lambda)|$. So we obtain

$$\begin{aligned}\left|f(z)\right| \le{}& |z|\left|\frac{1-\lambda}{1+\lambda}\right|^{2(1+\Re\lambda)/|1+\lambda|^2}\\&\cdot\left(\left|\frac{1-\lambda}{1+\lambda}\right|-|z|\right)^{(-|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2}\\&\cdot\left(\left|\frac{1-\lambda}{1+\lambda}\right|+|z|\right)^{(|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2}.\end{aligned} \tag{32}$$

Similarly, by (19) we can get

$$\begin{aligned}\left|f(z)\right| \ge{}& |z|\left|\frac{1-\lambda}{1+\lambda}\right|^{2(1+\Re\lambda)/|1+\lambda|^2}\\&\cdot\left(\left|\frac{1-\lambda}{1+\lambda}\right|-|z|\right)^{(|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2}\\&\cdot\left(\left|\frac{1-\lambda}{1+\lambda}\right|+|z|\right)^{(-|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2}.\end{aligned} \tag{33}$$

□

Theorem 8. *Let $f(z)$ be an almost starlike function of complex order λ on D with $\lambda = -1$. Then*

$$\begin{gathered}1-|z| \le \left|\frac{zf'(z)}{f(z)}\right| \le 1+|z|,\\|z|\,e^{-|z|} \le \left|f(z)\right| \le |z|\,e^{|z|},\\(1-|z|)\,e^{-|z|} \le \left|f'(z)\right| \le (1+|z|)\,e^{|z|}.\end{gathered} \tag{34}$$

Proof. By (14) we have $|2g(z)-1| \le |z|$ which follows $|zf'(z)/f(z)-1| \le |z|$. Therefore

$$\begin{gathered}1-|z| \le \left|\frac{zf'(z)}{f(z)}\right| \le 1+|z|,\\1-|z| \le \Re\frac{zf'(z)}{f(z)} \le 1+|z|.\end{gathered} \tag{35}$$

For $|z| = r$ we have

$$1-r \le r\frac{\partial}{\partial r}\ln\left|f(z)\right| = \Re\frac{zf'(z)}{f(z)} \le 1+r. \tag{36}$$

Similar to Theorem 7, we get

$$\begin{aligned}\int_\varepsilon^r\frac{dt}{t}-\int_\varepsilon^r dt \le{}& \int_\varepsilon^r\frac{\partial}{\partial t}\ln\left|f\left(te^{i\theta}\right)\right|dt\\\le{}& \int_\varepsilon^r\frac{dt}{t}+\int_\varepsilon^r dt.\end{aligned} \tag{37}$$

Let $\varepsilon \to 0$; we obtain $|z|e^{-|z|} \le |f(z)| \le |z|e^{|z|}$. Hence $(1-|z|)e^{-|z|} \le |f'(z)| \le (1+|z|)e^{|z|}$. □

Setting $\lambda = \alpha/(\alpha-1)$ in Theorems 7 and 8, we get the following corollary.

Corollary 9. *Let $f(z)$ be an almost starlike function of order α. Then*

$$\begin{aligned}
&\frac{|z|}{(1+(1-2\alpha)|z|)^{2(1-\alpha)/(1-2\alpha)}} \le |f(z)| \\
&\quad \le \frac{|z|}{(1-(1-2\alpha)|z|)^{2(1-\alpha)/(1-2\alpha)}}, \quad \alpha \in [0,1) \setminus \frac{1}{2}, \\
&|z| e^{-|z|} \le |f(z)| \le |z| e^{|z|}, \quad \alpha = \frac{1}{2}, \\
&1-|z| \le \left|\frac{zf'(z)}{f(z)}\right| \le 1+|z|, \\
&(1-|z|) e^{-|z|} \le |f'(z)| \le (1+|z|) e^{|z|}, \\
&\qquad \alpha = \frac{1}{2}.
\end{aligned} \tag{38}$$

Theorem 10. *Let $f(z)$ be an almost starlike function of complex order λ on D with $\lambda \in \mathbb{C} \setminus \{-1\}$, $\operatorname{Re}\lambda \le 0$. Then*

$$\begin{aligned}
&\frac{1-|z|}{1+|(1+\lambda)/(1-\lambda)||z|} \le \left|\frac{zf'(z)}{f(z)}\right| \\
&\quad \le \frac{1+|z|}{1-|(1+\lambda)/(1-\lambda)||z|}, \\
&(1-|z|) \left|\frac{1-\lambda}{1+\lambda}\right|^{2(1+\Re\lambda)/|1+\lambda|^2+1} \\
&\quad \cdot \left(\left|\frac{1-\lambda}{1+\lambda}\right| - |z|\right)^{(|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2} \\
&\quad \cdot \left(\left|\frac{1-\lambda}{1+\lambda}\right| + |z|\right)^{(-|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2-1} \le |f'(z)| \\
&\quad \le (1+|z|) \left|\frac{1-\lambda}{1+\lambda}\right|^{2(1+\Re\lambda)/|1+\lambda|^2+1} \\
&\quad \cdot \left(\left|\frac{1-\lambda}{1+\lambda}\right| + |z|\right)^{(|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2} \\
&\quad \cdot \left(\left|\frac{1-\lambda}{1+\lambda}\right| - |z|\right)^{(-|1+\lambda|-1-\Re\lambda)/|1+\lambda|^2-1}.
\end{aligned} \tag{39}$$

Proof. By (17) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \left|\frac{1+\lambda}{1-\lambda}\right| |z| \left|\frac{zf'(z)}{f(z)} + \frac{1-\lambda}{1+\lambda}\right|. \tag{40}$$

Since

$$\begin{aligned}
&\left|\frac{zf'(z)}{f(z)}\right| - 1 \le \left|\frac{zf'(z)}{f(z)} - 1\right|, \\
&\left|\frac{zf'(z)}{f(z)} + \frac{1-\lambda}{1+\lambda}\right| \le \left|\frac{zf'(z)}{f(z)}\right| + \left|\frac{1-\lambda}{1+\lambda}\right|,
\end{aligned} \tag{41}$$

then we have

$$\left|\frac{zf'(z)}{f(z)}\right| - 1 \le \left(\left|\frac{zf'(z)}{f(z)}\right| + \left|\frac{1-\lambda}{1+\lambda}\right|\right) \left|\frac{1+\lambda}{1-\lambda}\right| |z|, \tag{42}$$

which follows

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+|z|}{1-|(1+\lambda)/(1-\lambda)||z|}. \tag{43}$$

On the other hand, we also have

$$1 - \left|\frac{zf'(z)}{f(z)}\right| \le \left(\left|\frac{zf'(z)}{f(z)}\right| + \left|\frac{1-\lambda}{1+\lambda}\right|\right) \left|\frac{1+\lambda}{1-\lambda}\right| |z|, \tag{44}$$

which leads to

$$\left|\frac{zf'(z)}{f(z)}\right| \ge \frac{1-|z|}{1+|(1+\lambda)/(1-\lambda)||z|}. \tag{45}$$

Hence

$$\begin{aligned}
&\frac{1-|z|}{1+|(1+\lambda)/(1-\lambda)||z|} \le \left|\frac{zf'(z)}{f(z)}\right| \\
&\quad \le \frac{1+|z|}{1-|(1+\lambda)/(1-\lambda)||z|}.
\end{aligned} \tag{46}$$

Then, by Theorem 7 we obtain the desired conclusion (39). □

Setting $\lambda = \alpha/(\alpha-1)$ in Theorem 10, we get the following result.

Corollary 11. *Let $f(z)$ be an almost starlike function of order α with $\alpha \in [0,1) \setminus 1/2$. Then*

$$\begin{aligned}
&\frac{1-|z|}{1+|1-2\alpha||z|} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+|z|}{1-|1-2\alpha||z|}, \\
&\frac{1-|z|}{1+|1-2\alpha||z|} \frac{|z|}{(1+(1-2\alpha)|z|)^{2(1-\alpha)/(1-2\alpha)}} \\
&\quad \le |f'(z)| \\
&\quad \le \frac{1+|z|}{1-|1-2\alpha||z|} \frac{|z|}{(1-(1-2\alpha)|z|)^{2(1-\alpha)/(1-2\alpha)}}.
\end{aligned} \tag{47}$$

3. Coefficient Estimates of Almost Starlike Functions of Complex Order λ

Lemma 12 (see [21]). *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be a holomorphic function on D with $\Re p(z) > 0$. Then $|c_n| \le 2$ $(n = 1, 2, \ldots)$.*

Lemma 13 (see [22]). *Let $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$ be holomorphic on D. Then $\Re p(z)>0$ if and only if*

$$p(z)\neq\frac{\xi-1}{\xi+1},\quad \xi\in\partial D. \tag{48}$$

Theorem 14. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be an almost starlike function of complex order λ with $\lambda\in\mathbb{C}$, $\operatorname{Re}\lambda\leq 0$ on D. Then*

$$|a_n|\leq\prod_{j=2}^{n-1}\left(\frac{2}{|1-\lambda|}+\frac{j-2}{j-1}\right),\quad n=2,3,\ldots. \tag{49}$$

Proof. Since $f(z)$ is an almost starlike function of complex order λ, then

$$\Re\left[(1-\lambda)\frac{f(z)}{zf'(z)}+\lambda\right]>0,\quad z\in D. \tag{50}$$

Let $p(z)=(1-\lambda)(f(z)/zf'(z))+\lambda$. Then we have

$$zf'(z)(p(z)-\lambda)=(1-\lambda)f(z) \tag{51}$$

and $p(0)=1$ and $\Re p(z)>0$. Let $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$. Then

$$|c_n|\leq 2\quad(n=1,2,\ldots) \tag{52}$$

by Lemma 12. For $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, (51) follows

$$\begin{aligned}&z\left(1+\sum_{n=2}^{\infty}na_nz^{n-1}\right)\left(1-\lambda+\sum_{n=1}^{\infty}c_nz^n\right)\\&\quad=(1-\lambda)\left(z+\sum_{n=2}^{\infty}a_nz^n\right).\end{aligned} \tag{53}$$

Therefore

$$\begin{aligned}&(1-\lambda)z+[c_1+2a_2(1-\lambda)]z^2+\cdots+[c_{n-1}\\&\quad+2a_2c_{n-2}+\cdots+(n-1)a_{n-1}c_1+na_n(1-\lambda)]z^n\\&\quad+\cdots=(1-\lambda)z+a_2(1-\lambda)z^2+\cdots+a_n(1-\lambda)\\&\quad\cdot z^n+\cdots.\end{aligned} \tag{54}$$

So

$$a_n=\frac{c_{n-1}+2a_2c_{n-2}+\cdots+(n-1)a_{n-1}c_1}{(n-1)(\lambda-1)}. \tag{55}$$

By (52) we get

$$|a_n|\leq\frac{2[1+2|a_2|+\cdots+(n-1)|a_{n-1}|]}{(n-1)|1-\lambda|}. \tag{56}$$

By mathematical induction we obtain the desired conclusion. □

Theorem 15. *Let $f(z)$ be an almost starlike function of complex order λ with $\lambda\in\mathbb{C}$, $\operatorname{Re}\lambda\leq 0$ on D, and let $z=0$ be the zero of order $k+1$ ($k\in\mathbb{N}$) of $f(z)-z$. Let $f(z)=z+\sum_{n=k+1}^{\infty}a_nz^n$. Then*

$$|a_{n+1}|\leq\frac{2}{|\lambda+n|}\quad(n=k,k+1,\ldots). \tag{57}$$

Proof. Since $f(z)=z+\sum_{n=k+1}^{\infty}a_nz^n$, similar to Theorem 14, by (51) we obtain

$$\begin{aligned}&z\left(1+\sum_{n=k+1}^{\infty}na_nz^{n-1}\right)\left(1-\lambda+\sum_{n=1}^{\infty}c_nz^n\right)\\&\quad=(1-\lambda)\left(z+\sum_{n=k+1}^{\infty}a_nz^n\right),\end{aligned} \tag{58}$$

which follows

$$\begin{aligned}&(1-\lambda)z+c_1z^2+\cdots+c_{k-1}z^k\\&\quad+[c_k+(k+1)a_{k+1}]z^{k+1}+\cdots\\&\quad+[c_n+(n+1)a_{n+1}]z^{n+1}+\cdots\\&\quad=(1-\lambda)z+(1-\lambda)a_{k+1}z^{k+1}+\cdots\\&\quad+(1-\lambda)a_{n+1}z^{n+1}+\cdots.\end{aligned} \tag{59}$$

Therefore

$$\begin{aligned}&c_1=\cdots=c_{k-1}=0,\\&c_n=(-\lambda-n)a_{n+1}\quad(n=k,k+1,\ldots).\end{aligned} \tag{60}$$

By (52) we get the desired conclusion. □

Theorem 16. *$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is an almost starlike function of complex order λ with $\lambda\in\mathbb{C}$ and $\operatorname{Re}\lambda\leq 0$ on D if and only if*

$$\begin{aligned}&\sum_{n=2}^{\infty}[(1+(n-1)\lambda)(\xi+1)+n(1-\xi)]a_nz^n\neq-2\\&\qquad(\xi\in\partial D,\ n=2,3,\ldots).\end{aligned} \tag{61}$$

Proof. Let $p(z)=(1-\lambda)(f(z)/zf'(z))+\lambda$. Since $f(z)$ is an almost starlike function of complex order λ, then $p(z)$ is holomorphic on D and $p(0)=0$ and $\Re p(z)>0$. By Lemma 13 we get

$$(1-\lambda)\frac{f(z)}{zf'(z)}+\lambda\neq\frac{\xi-1}{\xi+1},\quad\xi\in\partial D, \tag{62}$$

which follows

$$\begin{aligned}&(1-\lambda)(\xi+1)f(z)+\lambda(\xi+1)zf'(z)\\&\quad\neq(\xi-1)zf'(z).\end{aligned} \tag{63}$$

For $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, we have

$$\begin{aligned}&(1-\lambda)(\xi+1)\left(z+\sum_{n=2}^{\infty}a_nz^n\right)\\&\quad+\lambda(\xi+1)\left(z+\sum_{n=2}^{\infty}na_nz^n\right)\\&\quad\neq(\xi-1)\left(z+\sum_{n=2}^{\infty}na_nz^n\right),\end{aligned} \tag{64}$$

which leads to the desired conclusion. □

Remark 17. Setting $\lambda = \alpha/(\alpha - 1)$ in Theorems 14–16, we get the corresponding results for almost starlike functions of order α.

4. The Invariance of Almost Starlike Mappings of Complex Order λ

In this section, we mainly discuss the invariance of almost starlike mappings of complex order λ on Reinhardt domains and on the unit ball in complex Banach spaces under some generalized Roper-Suffridge operators.

Lemma 18 (see [23]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded and completely Reinhardt domain whose Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold Ω_0. Then for $\forall z = (z_1, \ldots, z_n) \in \Omega \setminus \Omega_0$, we have*

$$\frac{\partial \rho(z)}{\partial z} = \left(\frac{\partial \rho(z)}{\partial z_1}, \frac{\partial \rho(z)}{\partial z_2}, \ldots, \frac{\partial \rho(z)}{\partial z_n}\right),$$
$$\frac{\partial \rho(z)}{\partial z_j} z_j \geq 0, \tag{65}$$
$$\rho(z) = 2\sum_{j=1}^{n} \frac{\partial \rho(z)}{\partial z_j} z_j.$$

Theorem 19. *Let $f(z)$ be an almost starlike function of complex order λ on D with $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda \leq 0$. Let $\Omega \subset \mathbb{C}^n$ be a bounded and completely Reinhardt domain whose Minkowski functional $\rho(z)$ is C^1 on $\overline{\Omega} \setminus \{0\}$. Let*

$$F(z) = \left(rf\left(\frac{z_1}{r}\right), \left(\frac{rf(z_1/r)}{z_1}\right)^{\beta_2} \cdot z_2, \ldots, \left(\frac{rf(z_1/r)}{z_1}\right)^{\beta_n} z_n\right)', \tag{66}$$

where $\beta_j \in [0,1]$, $r = \sup\{|z_1| : z = (z_1, \ldots, z_n)' \in \Omega\}$, and $(rf(z_1/r)/z_1)^{\beta_j}|_{z_1=0} = 1$ $(j = 2, \ldots, n)$. Then $F(z)$ is an almost starlike mapping of complex order λ on Ω.

Proof. For

$$F(z) = \left(rf\left(\frac{z_1}{r}\right), \left(\frac{rf(z_1/r)}{z_1}\right)^{\beta_2} \cdot z_2, \ldots, \left(\frac{rf(z_1/r)}{z_1}\right)^{\beta_n} z_n\right)', \tag{67}$$

immediately we have

$$J_F(z) = \begin{pmatrix} f'\left(\frac{z_1}{r}\right) & 0 & \cdots & 0 \\ \frac{\beta_2}{r}\left[\frac{f'(z_1/r)}{f(z_1/r)} - \frac{r}{z_1}\right]\left(\frac{rf(z_1/r)}{z_1}\right)^{\beta_2} z_2 & \left(\frac{rf(z_1/r)}{z_1}\right)^{\beta_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_n}{r}\left[\frac{f'(z_1/r)}{f(z_1/r)} - \frac{r}{z_1}\right]\left(\frac{rf(z_1/r)}{z_1}\right)^{\beta_n} z_n & 0 & \cdots & \left(\frac{rf(z_1/r)}{z_1}\right)^{\beta_n} \end{pmatrix}. \tag{68}$$

Therefore

$$J_F^{-1}(z) = \begin{pmatrix} \frac{1}{f'(z_1/r)} & 0 & \cdots & 0 \\ \frac{\beta_2}{r}\left[\frac{r}{z_1 f'(z_1/r)} - \frac{1}{f(z_1/r)}\right] z_2 & \left(\frac{rf(z_1/r)}{z_1}\right)^{-\beta_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_n}{r}\left[\frac{r}{z_1 f'(z_1/r)} - \frac{1}{f(z_1/r)}\right] z_n & 0 & \cdots & \left(\frac{rf(z_1/r)}{z_1}\right)^{-\beta_n} \end{pmatrix}. \tag{69}$$

Then

$$J_F^{-1}(z)F(z) = \left(\frac{rf(z_1/r)}{f'(z_1/r)}, \left(1-\beta_2+\beta_2\frac{rf(z_1/r)}{z_1 f'(z_1/r)}\right) \cdot z_2, \ldots, \left(1-\beta_n+\beta_n\frac{rf(z_1/r)}{z_1 f'(z_1/r)}\right)z_n\right)'. \tag{70}$$

By Lemma 18 we get

$$(1-\lambda)\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z) = 2(1-\lambda) \cdot \left[\frac{rf(z_1/r)}{z_1 f'(z_1/r)}\frac{\partial\rho(z)}{\partial z_1}z_1 + \sum_{j=2}^{n}\left(1-\beta_j+\beta_j\frac{rf(z_1/r)}{z_1 f'(z_1/r)}\right)\frac{\partial\rho(z)}{\partial z_j}z_j\right]. \tag{71}$$

Since $f(z)$ is an almost starlike function of complex order λ, then

$$\Re\left[(1-\lambda)\frac{rf(z_1/r)}{z_1 f'(z_1/r)}\right] \geq -\Re\lambda. \tag{72}$$

Therefore, by Lemma 18 we obtain

$$\begin{aligned}\Re\left[(1-\lambda)\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right] &= \Re\left[(1-\lambda) \cdot \frac{rf(z_1/r)}{z_1 f'(z_1/r)}\right]\frac{2\partial\rho(z)}{\partial z_1}z_1 + \sum_{j=2}^{n}\left\{\left(1-\beta_j\right)\Re(1-\lambda) + \beta_j\Re\left[(1-\lambda)\frac{rf(z_1/r)}{z_1 f'(z_1/r)}\right]\right\}\frac{2\partial\rho(z)}{\partial z_j}z_j \geq -\Re\lambda \\ &\cdot \frac{2\partial\rho(z)}{\partial z_1}z_1 + \sum_{j=2}^{n}\left\{\left(1-\beta_j\right)(1-\Re\lambda) + \beta_j(-\Re\lambda)\right\} \cdot \frac{2\partial\rho(z)}{\partial z_j}z_j = (-\Re\lambda)\left[\frac{2\partial\rho(z)}{\partial z_1}z_1 + \frac{2\partial\rho(z)}{\partial z_j}z_j\right] \\ &+ \sum_{j=2}^{n}\left(1-\beta_j\right)\frac{2\partial\rho(z)}{\partial z_j}z_j \geq (-\Re\lambda)\rho(z).\end{aligned} \tag{73}$$

Hence $F(z)$ is an almost starlike mapping of complex order λ on Ω by Definition 4. □

Theorem 20. *Let f_j $(j = 1,2,\ldots,n)$ be almost starlike functions of complex order λ on D with $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda < 0$. Let $\Omega \subset \mathbb{C}^n$ be a bounded starlike and circular domain whose Minkowski functional $\rho(z)$ is C^1 on $\overline{\Omega}\setminus\{0\}$. Let r_j be the radius of the disk $U_j = \{z_j \in \mathbb{C} : z = (z_1,\ldots,z_j,\ldots,z_n)' \in \Omega\}$ whose center is zero. Let*

$$F(z) = z\prod_{j=1}^{n}\left(\frac{r_j f_j(z_j/r_j)}{z_j}\right)^{\lambda_i}, \tag{74}$$

where $\lambda_{ij} \geq 0$, $\sum_{j=1}^{n}\lambda_{ij} = 1$, and $(f_j(z_j)/z_j)^{\lambda_{ij}}|_{z_j=0} = 1$ $(i, j = 1,2,\ldots,n)$. Then $F(z)$ is an almost starlike mapping of complex order λ on Ω.

Proof. Let

$$\zeta_j = \frac{z_j}{r_j}, \qquad g(z) = \prod_{j=1}^{n}\left(\frac{f_j(\zeta_j)}{\zeta_j}\right)^{\lambda_i}, \tag{75}$$

which follows $F(z) = zg(z)$. Then $J_F(z) = J_g(z)z + g(z)$. By direct computation we get

$$J_g(z)z = g(z)\left[\sum_{j=1}^{n}\lambda_j\frac{f_j'(z_j/r_j)z_j}{r_j f_j(z_j/r_j)} - 1\right] = g(z)\left[\sum_{j=1}^{n}\lambda_j\frac{f_j'(\zeta_j)\zeta_j}{f_j(\zeta_j)} - 1\right]. \tag{76}$$

Since f_j are almost starlike functions of complex order λ on D, then

$$\Re\left[(1-\lambda)\frac{f_j(\zeta_j)}{\zeta_j f_j'(\zeta_j)}\right] \geq -\Re\lambda > 0. \tag{77}$$

Therefore

$$\Re\left[\frac{1}{1-\lambda}\left(1+\frac{J_g(z)z}{g(z)}\right)\right] = \Re\left[\frac{1}{1-\lambda}\sum_{j=1}^{n}\lambda_j\frac{f_j'(\zeta_j)\zeta_j}{f_j(\zeta_j)}\right] = \sum_{j=1}^{n}\lambda_j\Re\left[\frac{1}{1-\lambda}\frac{f_j'(\zeta_j)\zeta_j}{f_j(\zeta_j)}\right] > 0. \tag{78}$$

Then $J_g(z)z+g(z) \neq 0$. It is obvious that $F(0) = 0$ and $J_F(0) = I$, so $F(z)$ is a normalized locally biholomorphic mapping on Ω and

$$J_F^{-1}(z)F(z) = \frac{g(z)z}{J_g(z)z+g(z)}. \tag{79}$$

Let

$$q_j(\zeta_j) = 2\operatorname{Re}(-\lambda)\frac{f_j'(\zeta_j)\zeta_j}{(1-\lambda)f_j(\zeta_j)} - 1. \tag{80}$$

(77) follows

$$\left|2\operatorname{Re}(-\lambda)\frac{f_j'(\zeta_j)\zeta_j}{(1-\lambda)f_j(\zeta_j)} - 1\right| \leq 1; \tag{81}$$

that is, $|q_j(\zeta_j)| \leq 1$. For $\operatorname{Re}\lambda < 0$, by (76), (79), and Lemma 18 we obtain

$$\left|2\Re(-\lambda)\frac{1}{(1-\lambda)(2/\rho(z))(\partial\rho/\partial z)(z)J_F^{-1}(z)F(z)} - 1\right| = \left|2\Re(-\lambda)\frac{1}{1-\lambda}\sum_{j=1}^{n}\lambda_j\frac{f_j'(\zeta_j)\zeta_j}{f_j(\zeta_j)} - 1\right| = \left|\sum_{j=1}^{n}\lambda_j q_j(\zeta_j)\right| \leq \sum_{j=1}^{n}\lambda_j = 1, \quad (82)$$

which leads to

$$\Re\left[(1-\lambda)\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right] \geq -\Re\lambda\rho(z). \quad (83)$$

Hence $F(z)$ is an almost starlike mapping of complex order λ on Ω by Definition 4. □

Similar to Theorem 20, we can get the following conclusion.

Theorem 21. *Let f_j $(j = 1, 2, \ldots, n)$ be almost starlike functions of complex order λ on D with $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda < 0$. Let*

$$F(x) = x\prod_{j=1}^{n}\left(\frac{f_j(T_{u_j}(x))}{T_{u_j}(x)}\right)^{\lambda_j}, \quad x \in B, \quad (84)$$

where $\lambda_{ij} \geq 0$, $\sum_{j=1}^{n}\lambda_{ij} = 1$, $(f_j(x_j)/x_j)^{\lambda_{ij}}|_{x_j=0} = 1$ $(x_j = T_{u_j}(x)$, $i, j = 1, 2, \ldots, n)$, and u_j is the unit vector of the complex Banach space X. Then $F(x)$ is an almost starlike mapping of complex order λ on the unit ball B in complex Banach space X.

Theorem 22. *Let f be an almost starlike function of complex order λ on D with $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda \leq 0$. Let*

$$F(x) = \sum_{j=1}^{n-1}\left(\frac{f(T_{x_1}(x))}{T_{x_1}(x)}\right)^{\beta_j} T_{x_j}(x)x_j + x - \sum_{j=1}^{n-1}T_{x_j}(x)x_j, \quad x \in B, \quad (85)$$

where $x_1 \in \overline{B}$, $\|x_1\| = 1$, $\beta_j \in [0,1]$, $(f(z)/z)^{\beta_j}|_{z=0} = 1$ $(j = 1, 2, \ldots, n-1)$, and $x_1, \ldots, x_n$ are linearly independent and $T_{x_i} \in X^*$ such that $\|T_{x_i}\| = 1$, $T_{x_i}(x_i) = 1$, and $T_{x_i}(x_j) = 0$ $(i \neq j)$ for $\forall x_i$. Then $F(x)$ is an almost starlike mapping of complex order λ on the unit ball B in complex Banach space X.

Proof. From [24] we get $F(x)$ is a normalized biholomorphic mapping on B and

$$\|x\| T_x\left[DF(x)^{-1}F(x)\right] = \|x\|^2 + \sum_{j=1}^{n-1}\beta_j\left|T_{x_j}(x)\right|^2\left[\frac{f(T_{x_1}(x))}{T_{x_1}(x)f'(T_{x_1}(x))} - 1\right]. \quad (86)$$

Since f is an almost starlike function of complex order λ, then

$$\Re\left[(1-\lambda)\frac{f(T_{x_j}(x))}{T_{x_j}(x)f'(T_{x_j}(x))}\right] \geq -\Re\lambda. \quad (87)$$

Therefore

$$\begin{aligned}\Re\left\{(1-\lambda)T_x\left[DF(x)^{-1}F(x)\right]\right\} &= (1-\Re\lambda)\|x\| + \sum_{j=1}^{n-1}\frac{\beta_j\left|T_{x_j}(x)\right|^2}{\|x\|} \cdot \Re\left\{(1-\lambda)\left[\frac{f(T_{x_1}(x))}{T_{x_1}(x)f'(T_{x_1}(x))} - 1\right]\right\} \\ &\geq (1-\Re\lambda)\|x\| + \sum_{j=1}^{n-1}\frac{\beta_j\left|T_{x_j}(x)\right|^2}{\|x\|} \cdot \left[\Re(-\lambda) - (1-\Re\lambda)\right] = -\Re\lambda\|x\| \\ &+ \frac{1}{\|x\|}\left[\|x\|^2 - \sum_{j=1}^{n-1}\beta_j\left|T_{x_j}(x)\right|^2\right] \geq -\Re\lambda\|x\|,\end{aligned} \quad (88)$$

which lead to the desired conclusion by Definition 5. □

Similar to Theorem 22, we can get the following conclusion.

Theorem 23. *Let f be an almost starlike function of complex order λ on D with $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \leq 0$. Let*

$$F(x) = \Phi_\beta(f)(x) = f(T_{x_1}(x))x_1 + \left(\frac{f(T_{x_1}(x))}{T_{x_1}(x)}\right)^{\beta}(x - T_{x_1}(x)x_1), \quad (89)$$

where $0 \leq \beta \leq 1$, $x_1 \in \overline{B}$, $\|x_1\| = 1$, $(f(z_1)/z_1)^{\beta}|_{z_1=0} = 1$, and $f(z_1) \neq 0$ for $z_1 \neq 0$. Then $F(x)$ is an almost starlike mapping of complex order λ on B.

Remark 24. Setting $\lambda = \alpha/(\alpha - 1)$ in Theorems 19–23, we get the corresponding results with respect to almost starlike mappings of order α.

Competing Interests

The authors declare that they have no conflict of interests.

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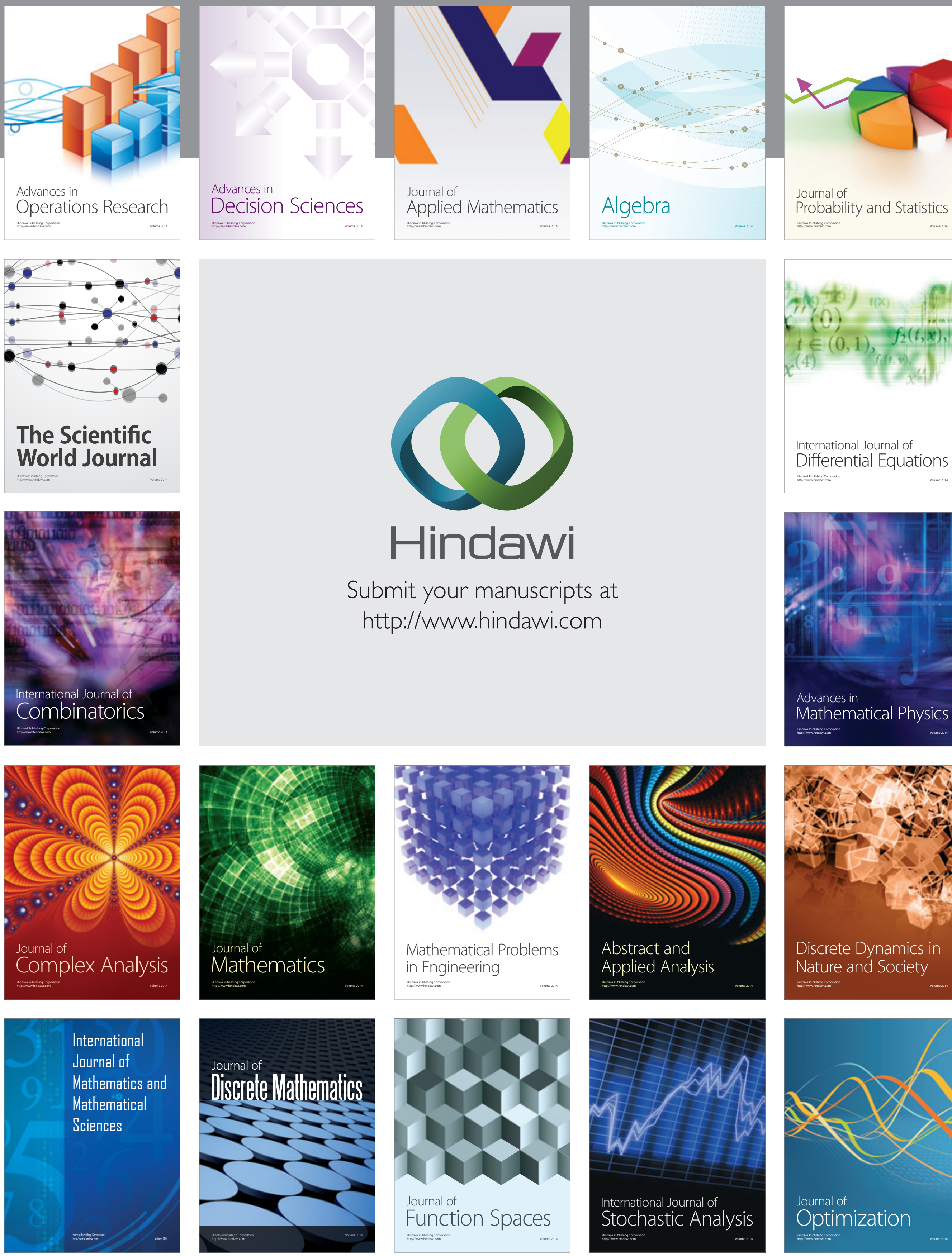
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