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Research Article

Worst-Case Investment and Reinsurance Optimization for an Insurer under Model Uncertainty

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In this paper, we study optimal investment-reinsurance strategies for an insurer who faces model uncertainty. The insurer is allowed to acquire new business and invest into a financial market which consists of one risk-free asset and one risky asset whose price process is modeled by a Geometric Brownian motion. Minimizing the expected quadratic distance of the terminal wealth to a given benchmark under the "worst-case" scenario, we obtain the closed-form expressions of optimal strategies and the corresponding value function by solving the Hamilton-Jacobi-Bellman (HJB) equation. Numerical examples are presented to show the impact of model parameters on the optimal strategies.

1. Introduction

In recent years, insurance companies are playing the more active role in the financial market. To avoid their risk, insurers are apt to invest in the financial market, purchase reinsurance from the reinsurer and acquire new business (acting as a reinsurer for other insurers). Tremendous literature on optimal investment and/or reinsurance problem for insurers has been investigated. For example, Browne [1] studied the optimal investment problem for an insurer whose surplus is described by a drifted Brownian motion under the criterion of maximizing the expected exponential utility of the terminal wealth or minimizing the ruin probability. To the same question, Hipp and Plum [2] showed a different result when insurance business is modeled by a compound Poisson process. Schmidli [3, 4] investigated optimal strategies under the criterion of minimizing the ruin probability.

Mean-variance portfolio selection theory was first proposed by Markowitz [5] and it has been used to investigate optimal investment and reinsurance problems in the actuarial literature [6–13].

In addition, it is important to consider model uncertainty due to the uncertainties in the global economy and the debit crisis exiting in financial markets and insurance industries. Meanwhile a growing number of scholars have paid attention to incorporating model uncertainty into optimization problems. For example, Talay and Zheng [14] viewed the Trader and the Market as two players and formulated the model risk control problem as a zero-sum stochastic differential game problem. Mataramvura and Øksendal [15] investigated the risk-minimizing portfolio selection problem in a jump diffusion market. Zhang and Siu [16] studied optimal investment and reinsurance problems via the game theoretic approach and derived closed-form solutions when the objective function is the expected exponential utility of terminal wealth and the expected discounted penalty of ruin, respectively. Lin et al. [17] discussed an optimal portfolio selection problem of an insurer who faces model uncertainty and obtained closed-form solutions to the game problems in both the jump diffusion risk process and its diffusion approximation for the case of an exponential utility by using techniques of stochastic linear-quadratic control.

In this paper, we are concerned about optimal investment and reinsurance strategies for an insurer with model uncertainty. Suppose that the insurer can purchase proportional reinsurance/acquire new business and invest in a simplified continuous-time financial market with a bank account and a risky stock. We measure the risk faced by the insurer by means of the deviation of the surplus to a fixed terminal target (a given benchmark). The insurer's objective is to find an optimal investment-reinsurance strategy under the criterion of minimizing the expected quadratic distance of the terminal wealth to the given benchmark in the "worst-case" scenario. Different from those in Zhang and Siu [16], we propose here a mean-variance portfolio optimization problem under model uncertainty.

The rest of this paper is organized as follows. In Section 2, we give the assumptions and formulate the model dynamics. In Section 3, we derive optimal investment-reinsurance strategy and the corresponding value function with the help of stochastic control theory. In Section 4, we provide some numerical analyses to demonstrate our results. Section 5 concludes the paper.

2. Model Formulation

Throughout this paper, we work on a filtered complete probability space $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{0\le t\le T})$, where $\mathcal{P}$ is a reference probability measure from which a family of real-world probability measures absolutely continuous with respect to $\mathcal{P}$ are generated; $\{\mathcal{F}_t\}_{0\le t\le T}$ is the filtration generated by all Brownian motions standing for the information available up to time t. The finite constant $T > 0$ is a final time horizon.

2.1. Surplus Process. In what follows, we consider an insurer whose surplus process $\{R(t), t \in [0,T]\}$ is approximated by a diffusion model. We firstly go back to the classical risk model as follows:

$$R(t) = x_0 + ct - \sum_{i=1}^{N_t} Y_i. \tag{1}$$

Here $\{Y_i\}$ is the size of the ith claim and $\{Y_i,\ i = 1,2,\ldots\}$ are independent and identically distributed (i.i.d.) nonnegative random variables with finite first-order moment μ_∞ and second-order moment σ_∞^2; $\{N_t\}$, independent of $\{Y_i,\ i = 1,2,\ldots\}$, is a homogeneous Poisson process with intensity λ; $c > 0$ is the premium rate which is assumed to be calculated by the expected value principle, that is, $c = (1+\delta)\lambda\mu_\infty$, where δ is the safety loading of the insurer and x_0 is the initial capital.

According to Grandell [18] or Schmidli [19], the insurer's surplus process can be approximated by the following diffusion model:

$$R(t) = x_0 + qt + \sigma_0 W_0(t), \tag{2}$$

where $q = \delta\lambda\mu_\infty$ is the premium return rate of the insurer; $\sigma_0^2 = \lambda\sigma_\infty^2$ represents the volatility of the insurer's surplus; $\{W_0(t)\}$ is a standard Brownian motion.

2.2. Financial Market. The insurance company is allowed to invest the money into a financial market with a riskless asset and a risky asset. Without loss of generality we assume any fractional units of assets can be traded continuously and no taxes or transaction costs are involved in our model.

The price process of the riskless asset $\{S_0(t) \mid t \in [0,T]\}$ is modeled by

$$dS_0(t) = r_0 S_0(t)\, dt, \quad S_0(0) = 1, \tag{3}$$

where $r_0 > 0$ represents the riskless interest rate.

The price process of the risky asset $\{S_1(t) \mid t \in [0,T]\}$ follows the geometric Brownian process

$$dS_1(t) = S_1(t)\left[r_1 dt + \sigma_1 dW_1(t)\right], \quad S_1(0) = s_1 > 0, \tag{4}$$

where $r_1 > r_0$ is the appreciation rate of the risky asset and $\sigma_1 > 0$ is the volatility of the risky asset; $\{W_1(t) \mid t \in [0,T]\}$ is another one-dimensional standard Brownian motion. For convenience, we assume $\{W_0(t) \mid t \in [0,T]\}$ and $\{W_1(t) \mid t \in [0,T]\}$ are independent Brownian motions.

2.3. Wealth Process. As mentioned before, the insurer is allowed to invest in the financial market and purchase proportional reinsurance or acquire new business as described in Bäuerle [6]. We denote by $\pi(t)$ the money amount invested in the risky asset and by $a(t)$ the retention level of reinsurance. For each $t \in [0,T]$, $\vartheta(t) := (\pi(t), a(t))$ represents the strategy adopted by the insurance company and denote by $\{X^\vartheta(t) \mid t \in [0,T]\}$ the surplus of the insurance company with the above notation ϑ. Thus the wealth process $X^\vartheta(t)$ of the insurer can be described as

$$\begin{aligned} dX^\vartheta(t) &= \left[\lambda\mu_\infty\left(\eta a(t) - (\eta-\delta)\right) + r_0 X^\vartheta(t) + r\pi(t)\right] dt \\ &\quad + \sigma_0 a(t)\, dW_0(t) + \sigma_1 \pi(t)\, dW_1(t), \end{aligned} \tag{5}$$

$$X^\vartheta(0) = x_0,$$

where $r = r_1 - r_0 > 0$, $\eta > \delta$ is the safety loading of the reinsurance/new business.

A strategy $\vartheta = \{(\pi(t), a(t)),\ t \in [0,T]\}$ is said to be admissible if it is $\mathcal{F}_t$-progressively measurable, $a(t) \ge 0$, $E[\int_0^T (\sigma_0^2 a^2(t) + \sigma_1^2 \pi^2(t)) ds] < +\infty$, and (5) has a unique strong solution. Denote by $\mathcal{A}$ the set of all admissible strategies.

2.4. Model Uncertainty. For each $t \in [0,T]$, define the enlarged σ-field $\mathcal{G}(t)$, the minimal σ-field generated by $\mathcal{F}^{W_0}(t)$ and $\mathcal{F}^{W_1}(t)$. Denote $\mathcal{G}(t) := \mathcal{F}^{W_0}(t) \vee \mathcal{F}^{W_1}(t)$ and denote $\mathbf{G} := \{\mathcal{G}(t) \mid t \in [0,T]\}$.

Now specify the space of admissible controls by the market. Define $\{\theta(t) \mid t \in [0,T]\}$, which is satisfying the following conditions:

(1) $\{\theta(t) \mid t \in [0,T]\}$ is $\mathbf{G}$-progressively measurable;

(2) $\theta(t) := \theta(t,\omega) < 1$, for a.a. $(t,\omega) \in [0,T] \times \Omega$;

(3) $\int_0^T \theta^2(t) dt < \infty$, $\mathcal{P}$-a.s..

Write Θ for the space of all processes satisfying the above three conditions.

For each $\theta \in \Theta$, define a real-valued, $\mathbf{G}$-adapted process $\{\Lambda^{\theta}(t) \mid t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ by

$$\Lambda^{\theta}(t) := \exp\left\{-\int_0^t \theta(u)\, dW_0(u) - \int_0^t \theta(u)\, dW_1(u) - \int_0^t \theta^2(u)\, du\right\}. \tag{6}$$

By Ito's differentiation rule,

$$\begin{aligned} d\Lambda^{\theta}(t) &= \Lambda^{\theta}(t)\left[-\theta(t)\, dW_0(t) - \theta(t)\, dW_1(t)\right], \\ \Lambda^{\theta}(0) &= 1, \ \mathcal{P}\text{-a.s.} \end{aligned} \tag{7}$$

So, Λ^{θ} is a $(\mathbf{G}, \mathcal{P})$-(local)-martingale. Hence,

$$\mathrm{E}\left[\Lambda^{\theta}(T)\right] = 1. \tag{8}$$

For each $\theta \in \Theta$, a new (real-world) probability measure $\mathcal{P}^{\theta}$ absolutely continuous with respect to $\mathcal{P}$ on $\mathcal{G}(T)$ is defined by

$$\left.\frac{d\mathcal{P}^{\theta}}{d\mathcal{P}}\right|_{\mathcal{G}(T)} := \Lambda^{\theta}(T). \tag{9}$$

Then the family $\mathcal{P}(\Theta)$ of real-world probability measures with index set Θ is generated.

3. The Basic Problem (MV)

As in Mataramvura and Øksendal [15], we define $\{U(t) \mid t \in [0, T]\}$, a vector process, by

$$\begin{aligned} dU(t) &= \left(dU_0(t), dU_1(t), dU_2(t)\right)' \\ &= \left(dU_0(t), dU_1^{\vartheta}(t), dU_2^{\theta}(t)\right)' \\ &= \left(dt, dX^{\vartheta}(t), d\Lambda^{\theta}(t)\right)', \\ U(0) &= u = (s, u_1, u_2)' = (s, x, \Lambda)'. \end{aligned} \tag{10}$$

Suppose the market is the leader of the game and aims to select an admissible strategy $\{\theta(t) \mid t \in [0, T]\}$ which represents the "worst-case" scenario of the minimal expected quadratic deviation between the terminal surplus and some preset targets. That leads to the Min-Max problem

$$\begin{aligned} \Psi(u) &= \inf_{\vartheta \in \mathcal{A}} \sup_{\theta \in \Theta} \left\{\mathrm{E}_{\theta}^{u}\left[\left(X^{\vartheta}(T) - k\right)^2\right]\right\} \\ &= \mathrm{E}_{\theta^*}^{u}\left[\left(X^{\vartheta^*}(T) - k\right)^2\right], \end{aligned} \tag{11}$$

where $k \in \mathcal{R}_+$ is a certain predefined benchmark. Here it is reasonable to assume $k \geq x_0 e^{r_0 T}$, which states that the insurer's terminal wealth k cannot be less than the amount earned by the insurer if all the wealth was invested in the risk-free asset. The market selects an optimal probability scenario $\{\theta^*(t) \mid t \in [0, T]\}$ to maximize the minimal expected quadratic deviation and the insurer reacts antagonistically by choosing an optimal portfolio processes $\{\vartheta^*(t) \mid t \in [0, T]\}$ to minimize the expected quadratic deviation of $X^{\vartheta}(T)$ to k.

For each u and $(\theta, \vartheta) \in \Theta \times \mathcal{A}$, we define

$$\begin{aligned} J^{\theta,\vartheta}(u) &= \mathrm{E}_{\theta}^{u}\left[\left(X^{\vartheta}(T) - k\right)^2\right] \\ &= \mathrm{E}^{u}\left[\Lambda^{\theta}(T)\left(X^{\vartheta}(T) - k\right)^2\right]. \end{aligned} \tag{12}$$

Then we can get another statement of the above Min-Max problem (11) by the version of Bayes' rule: to solve the problem, we must find $\Psi(u)$, $\theta^* \in \Theta$, and $\vartheta^* = (\pi^*, a^*) \in \mathcal{A}$ such that

$$\Psi(u) = \inf_{\vartheta \in \mathcal{A}} \sup_{\theta \in \Theta} J^{\theta,\vartheta}(u) = J^{\theta^*,\vartheta^*}(u). \tag{13}$$

Similar to the handling in Øksendal [20] and Øksendal and Sulem [21], we could consider feedback Markov controls and, for some functions, $\theta_0 : [0, T] \times (0, \infty) \times (0, \infty) \to \mathcal{R}$ and $\vartheta_0 : [0, T] \times (0, \infty) \times (0, \infty)$, assume that for each $t \in [0, T]$, $\theta(t) := \theta_0(U(t))$ and $\vartheta(t) := \vartheta_0(U(t))$ under some mild conditions. Then, for each pair $(\theta, \vartheta) \in (\Theta, \mathcal{A})$, the generator of the process $U(\cdot)$ is a partial differential operator $\mathcal{L}^{\theta,\vartheta}(h(u))$.

$$\begin{aligned} &\mathcal{L}^{\theta,\vartheta}(h(u)) \\ &\quad = \frac{\partial h}{\partial t} + \left\{\lambda \mu_{\infty}\left[\eta a - (\eta - \delta)\right] + r_0 u_1 + r\pi\right\} \frac{\partial h}{\partial u_1} \\ &\quad + \frac{1}{2}\left(\sigma_0^2 a^2 + \sigma_1^2 \pi^2\right) \frac{\partial^2 h}{\partial u_1^2} + u_2^2 \theta^2 \frac{\partial^2 h}{\partial u_2^2} \\ &\quad - \left(\sigma_0 a + \sigma_1 \pi\right) \theta u_2 \frac{\partial^2 h}{\partial u_1 \partial u_2}, \end{aligned} \tag{14}$$

where $h(u) := h(\cdot, \cdot, \cdot) \in \mathcal{C}^{1,2,2}([0, T] \times (0, \infty) \times (0, \infty))$.

Theorem 1 (verification theorem). *For the Min-Max problem, if there exist three real value functions ϕ, θ^*, and ϑ^* satisfying the following HJB system:*

$$\begin{aligned} \inf_{\vartheta \in \mathcal{A}} \sup_{\theta \in \Theta} \mathcal{L}^{\theta,\vartheta}\left[\phi(t, u_1, u_2)\right] &= 0, \\ \phi(T, u_1, u_2) &= u_2 (u_1 - k)^2, \end{aligned} \tag{15}$$

then $\Psi(u) = \phi(t, u_1, u_2)$, θ^* corresponds to the "worst-case" scenario, and $\vartheta^* = (\pi^*, a^*)$ is the optimal investment-reinsurance strategy.

The proof of Theorem 1 is similar to the proof of Theorem 3.2 in Mataramvura and Øksendal [15]. So we do not repeat it here.

In order to solve problem (11), we only need to solve the HJB equation (15). Motivated by the terminal boundary condition, we try the following parametric form of the value function:

$$\phi(t,u_1,u_2) = u_2\left[A(t)u_1^2 + B(t)u_1 + C(t)\right], \tag{16}$$

where $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are three suitable functions, and the boundary condition in Theorem 1 implies that $A(T) = 1$, $B(T) = -2k$, and $C(T) = k^2$. The corresponding partial derivatives are

$$\begin{aligned}
\frac{\partial\phi}{\partial t} &= u_2\left[\dot{A}(t)u_1^2 + \dot{B}(t)u_1 + \dot{C}(t)\right],\\
\frac{\partial\phi}{\partial u_1} &= u_2\left[2A(t)u_1 + B(t)\right],\\
\frac{\partial^2\phi}{\partial u_1^2} &= 2A(t)u_2,\\
\frac{\partial\phi}{\partial u_2} &= A(t)u_1^2 + B(t)u_1 + C(t), \quad \frac{\partial^2\phi}{\partial u_2^2} = 0,\\
\frac{\partial^2\phi}{\partial u_1\partial u_2} &= 2A(t)u_1 + B(t).
\end{aligned} \tag{17}$$

Substituting the above derivatives into $\mathscr{L}^{\theta,\vartheta}(\phi(u))$, we have

$$\begin{aligned}
\mathscr{L}^{\theta,\vartheta}(\phi(u)) &= u_2\left\{\dot{A}(t)u_1^2 + \dot{B}(t)u_1 + \dot{C}(t)\right.\\
&\quad + \left[r_0u_1 - \lambda\mu_\infty(\eta-\delta)\right]\left[2A(t)u_1 + B(t)\right]\\
&\quad + \left(\lambda\mu_\infty\eta - \sigma_0\theta\right)\left[2A(t)u_1 + B(t)\right]a + \sigma_0^2A(t)a^2\\
&\quad \left. + \left(r - \sigma_1\theta\right)\left[2A(t)u_1 + B(t)\right]\pi + \sigma_1^2A(t)\pi^2\right\}.
\end{aligned} \tag{18}$$

For fixed θ, by the first-order condition, we can get

$$\widehat{\pi}(\theta) = \frac{\sigma_1\theta - r}{2\sigma_1^2}\left[2u_1 + \frac{B(t)}{A(t)}\right], \tag{19}$$

$$\widehat{a}(\theta) = \frac{\sigma_0\theta - \lambda\mu_\infty\eta}{2\sigma_0^2}\left[2u_1 + \frac{B(t)}{A(t)}\right]. \tag{20}$$

Substituting $\widehat{\vartheta} = (\widehat{\pi}(\theta), \widehat{a}(\theta))$ into $\mathscr{L}^{\theta,\vartheta}(\phi(u))$, we have

$$\begin{aligned}
\mathscr{L}^{\theta,\widehat{\vartheta}}(\phi(u)) &= u_2\left\{\dot{A}(t)u_1^2 + \dot{B}(t)u_1 + \dot{C}(t)\right.\\
&\quad + \left[r_0u_1 - \lambda\mu_\infty(\eta-\delta)\right]\left[2A(t)u_1 + B(t)\right]\\
&\quad - \left[\sigma_0\theta - \lambda\mu_\infty\eta\right]^2\frac{\left[2A(t)u_1 + B(t)\right]^2}{4\sigma_0^2A(t)}\\
&\quad \left. - \left[\sigma_1\theta - r\right]^2\frac{\left[2A(t)u_1 + B(t)\right]^2}{4\sigma_1^2A(t)}\right\}.
\end{aligned} \tag{21}$$

Maximizing over θ gives the first-order condition for the maximum point $\theta = \widehat{\theta}$ as follows:

$$\widehat{\theta} = \frac{1}{2}\left(\frac{\lambda\mu_\infty\eta}{\sigma_0} + \frac{r}{\sigma_1}\right) = \frac{r\sigma_0 + \lambda\mu_\infty\eta\sigma_1}{2\sigma_0\sigma_1}. \tag{22}$$

So,

$$\begin{aligned}
\widehat{\pi}(t) &= \widehat{\pi}\left(\widehat{\theta}\right) = \frac{\lambda\mu_\infty\eta\sigma_1 - r\sigma_0}{4\sigma_0\sigma_1^2}\left[2u_1 + \frac{B(t)}{A(t)}\right],\\
\widehat{a}(t) &= \widehat{a}\left(\widehat{\theta}\right) = \frac{r\sigma_0 - \lambda\mu_\infty\eta\sigma_1}{4\sigma_0^2\sigma_1}\left[2u_1 + \frac{B(t)}{A(t)}\right].
\end{aligned} \tag{23}$$

We rewrite $\mathscr{L}^{\widehat{\theta},\widehat{\vartheta}}(\phi(u))$ as follows:

$$\begin{aligned}
\mathscr{L}^{\widehat{\theta},\widehat{\vartheta}}(\phi(u)) &= u_2\left\{\left[\dot{A}(t) + \left(2r_0 - \mathscr{D}\right)A(t)\right]u_1^2\right.\\
&\quad + \left[\dot{B}(t) + \left(r_0 - \mathscr{D}\right)B(t) - 2\Gamma A(t)\right]u_1 + \dot{C}(t)\\
&\quad \left. - \Gamma B(t) - \frac{\mathscr{D}B^2(t)}{4A(t)}\right\},
\end{aligned} \tag{24}$$

where $\mathscr{D} = (r\sigma_0 - \lambda\mu_\infty\eta\sigma_1)^2/4\sigma_0^2\sigma_1^2$ and $\Gamma = \lambda\mu_\infty(\eta-\delta)$.

To ensure $\mathscr{L}^{\widehat{\theta},\widehat{\vartheta}}(\phi(u)) = 0$ holds, we require

$$\begin{aligned}
\dot{A}(t) + \left(2r_0 - \mathscr{D}\right)A(t) &= 0, \quad A(T) = 1,\\
\dot{B}(t) + \left(r_0 - \mathscr{D}\right)B(t) - 2\Gamma A(t) &= 0, \quad B(T) = -2k,\\
\dot{C}(t) - \Gamma B(t) - \frac{\mathscr{D}B^2(t)}{4A(t)} &= 0, \quad C(T) = k^2.
\end{aligned} \tag{25}$$

Then it is easy to obtain

$$\begin{aligned}
A(t) &= e^{(2r_0 - \mathscr{D})(T-t)},\\
B(t) &= -2\left[\frac{\Gamma}{r_0}\left(e^{r_0(T-t)} - 1\right) + k\right]e^{(r_0-\mathscr{D})(T-t)},\\
C(t) &= \left[\frac{\Gamma}{r_0}\left(e^{r_0(T-t)} - 1\right) + k\right]^2 e^{-\mathscr{D}(T-t)}.
\end{aligned} \tag{26}$$

So, for $s \le t \le T$, we have

$$\begin{aligned}
\widehat{\pi}(t) &= \frac{r\sigma_0 - \lambda\mu_\infty\eta\sigma_1}{2\sigma_0\sigma_1^2}\left[\left(k - \frac{\Gamma}{r_0}\right)e^{-r_0(T-t)}\right.\\
&\quad \left. - \left(u_1 - \frac{\Gamma}{r_0}\right)\right] = \frac{r\sigma_0 - \lambda\mu_\infty\eta\sigma_1}{2\sigma_0\sigma_1^2e^{r_0(T-t)}}\left[\left(k - xe^{r_0(T-t)}\right)\right.\\
&\quad \left. + \frac{\Gamma}{r_0}\left(e^{r_0(T-t)} - 1\right)\right],\\
\widehat{a}(t) &= \frac{\lambda\mu_\infty\eta\sigma_1 - r\sigma_0}{2\sigma_0^2\sigma_1}\left[\left(k - \frac{\Gamma}{r_0}\right)e^{-r_0(T-t)}\right.\\
&\quad \left. - \left(u_1 - \frac{\Gamma}{r_0}\right)\right] = \frac{\lambda\mu_\infty\eta\sigma_1 - r\sigma_0}{2\sigma_0^2\sigma_1e^{r_0(T-t)}}\left[\left(k - xe^{r_0(T-t)}\right)\right.\\
&\quad \left. + \frac{\Gamma}{r_0}\left(e^{r_0(T-t)} - 1\right)\right].
\end{aligned} \tag{27}$$

Note that the reinsurance strategy a must be nonnegative. Denote

$$\Delta = \lambda\mu_{\infty}\eta\sigma_1 - r\sigma_0,$$
$$H(t) = \left(k - xe^{r_0(T-t)}\right) + \frac{\Gamma}{r_0}\left(e^{r_0(T-t)} - 1\right). \tag{28}$$

Then for $t \in [0, T]$,

(i) when $\Delta \geq 0, H(t) \geq 0$ or $\Delta \leq 0, H(t) \leq 0$, it is easy to get $\widehat{a}(t) \geq 0$, which can be a candidate of the optimal strategy;

(ii) when $\Delta \leq 0, H(t) \geq 0$ or $\Delta \geq 0, H(t) \leq 0$, we may choose $\widehat{a} = 0$ as the optimal reinsurance strategy since (18) is a second-order polynomial in a. Thus the operator $\mathscr{L}^{\theta,\vartheta}(h(v))$ in (14) can be changed as

$$\mathscr{L}^{\theta,\vartheta}(h(u)) = \frac{\partial h}{\partial t} + \left\{\lambda\mu_{\infty}(\delta - \eta) + r_0 u_1 + r\pi\right\}\frac{\partial h}{\partial u_1} + \frac{1}{2}\sigma_1^2\pi^2\frac{\partial^2 h}{\partial u_1^2} + u_2^2\theta^2\frac{\partial^2 h}{\partial u_2^2} - \sigma_1\pi\theta u_2\frac{\partial^2 h}{\partial u_1 \partial u_2}. \tag{29}$$

Parallel to the proof of Theorem 1, we can obtain the same form of expression of the investment strategy $\widehat{\pi}$ in (19) for fixed $\widehat{\theta} = r/\sigma_1$. Then the minimum point $\widehat{\pi}$ is given by $\widehat{\pi} = 0$.

Combining with Theorem 1, the above analyses are summarized as the following theorem.

Theorem 2. *The optimal investment-reinsurance strategy of the Min-Max problem (11) is*

$$(\pi^*, a^*) = \begin{cases} \left(-\dfrac{H(t)\Delta}{2\sigma_0\sigma_1^2 e^{r_0(T-t)}}, \dfrac{H(t)\Delta}{2\sigma_0^2\sigma_1 e^{r_0(T-t)}}\right), & \text{if } H(t)\Delta \geq 0, \\ (0, 0), & \text{if } H(t)\Delta \leq 0. \end{cases} \tag{30}$$

Moreover, the corresponding value function is given by

$$\Psi(u) = u_2\left\{e^{-\mathscr{D}(T-s)}\left[u_1 e^{r_0(T-s)} - \frac{\Gamma}{r_0}\left(e^{r_0(T-s)} - 1\right) - k\right]^2\right\}. \tag{31}$$

Remark 3. We find that (i) under model uncertainty the optimal policy $\vartheta^* = (\pi^*, a^*)$ depends on the volatility of both the insurer's surplus and the risky asset; (ii) if we do not take account of the model risk, the corresponding optimal investment-reinsurance strategies (see Appendix) are different: the volatility of the insurer's surplus has no influence on the investment strategy $\widetilde{\pi}(t)$ (see Appendix (A.11)), and the volatility of the risky asset has no influence on the reinsurance strategy $\widetilde{a}(t)$ (see Appendix (A.12)).

4. Numerical Results

In this section, we present some numerical illustrations and sensitivity analysis of optimal investment-reinsurance strategies. Throughout the numerical analysis, unless otherwise stated, the basic parameters are given by $x_0 = 10$, $\lambda = 1$, $u_1 = 1$, $\eta = 1.5$, $\delta = 1.1$, $r_0 = 0.04$, $r_1 = 0.2$, $\sigma_0 = 1.5$, $\sigma_1 = 1.15$, $k = 15$, $T = 5$, and $N = 300$. In the following numerical examples, we illustrate the effect of different parameters on the optimal strategies by varying the value of one parameter with others fixed each time.

Figure 1 shows the effect of the riskless rate of interest r_0, the appreciation rate of the risky asset r_1, the volatility of the insurer's surplus σ_0, and the volatility of the risky asset σ_1 on the optimal investment strategy $\pi^*(t)$. We compare the investment strategies in Figure 1(a) when $r_0 = 0.01$ and $r_0 = 0.06$ for fixed $t \in [0, 5]$. It is shown that as the risk-free interest rate r_0 increases, the optimal investment strategy $\pi^*(t)$ moves up in any time. And when t is near time 0, the trend is very obvious. From Figures 1(b) and 1(c), we find that $\pi^*(t)$ increases with respect to r_1 and σ_0 for fixed $t \in [0, 5]$ where we compare the investment strategies when $r_1 = 0.05$ and $r_1 = 0.35$ and $\sigma_0 = 1.15$ and $\sigma_0 = 1.5$. It is a different case as in Figure 1(d), where we compare the investment strategies when $\sigma_1 = 0.55$ and $\sigma_1 = 1.65$ for fixed $t \in [0, 5]$.

Figure 2 shows the effect of different parameters on the optimal reinsurance strategy. In Figure 2(a), we compare the reinsurance strategies when $r_0 = 0.01$ and $r_0 = 0.06$ for fixed $t \in [0, 5]$. It is shown that as the risk-free interest rate r_0 increases, the optimal investment strategy $\pi^*(t)$ moves down. In Figure 2(b), we compare the reinsurance strategies when $r_1 = 0.05$ and $r_1 = 0.35$ for fixed $t \in [0, 5]$. In Figure 2(c), we compare the reinsurance strategies when $\sigma_0 = 1.15$ and $\sigma_0 = 1.55$ for fixed $t \in [0, 5]$. In Figure 2(d), we compare the reinsurance strategies when $\sigma_1 = 0.55$ and $\sigma_1 = 1.65$ for fixed $t \in [0, 5]$.

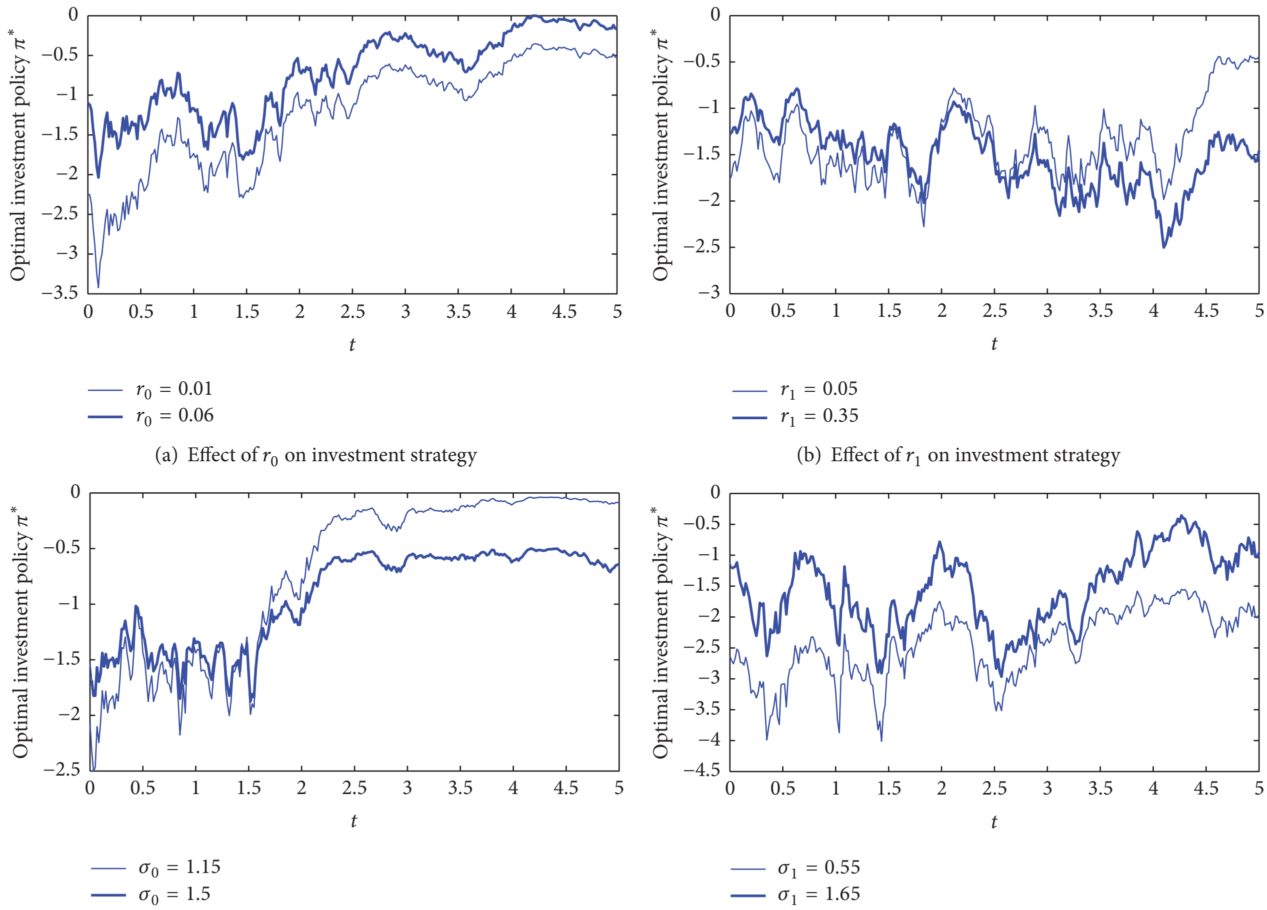
We can find from the above two groups of Figures that, while keeping other parameters unchanged:

(i) The insurer is inclined to invest in the market (compared with buying reinsurance) when the riskless rate of interest r_0 increases.

(ii) When the appreciation rate of the risky asset r_1 increases, the insurer is inclined to invest in the market (compared with buying reinsurance) during the first half of the horizon and to buy reinsurance (compared with investing in the market) during the second half of the horizon.

(iii) As the volatility of the insurers surplus σ_0 increases, the insurer seems to be willing to invest in the market during the first half of the horizon and buy reinsurance during the second half of the horizon.

(iv) The insurer is inclined to invest in the market (compared with buying reinsurance) when the volatility of the risky asset σ_1 increases.



(a) Effect of r_0 on investment strategy

(b) Effect of r_1 on investment strategy

(c) Effect of σ_0 on investment strategy

(d) Effect of σ_1 on investment strategy

FIGURE 1: The effect of parameters on optimal investment strategy.

5. Conclusion

In this paper, we study an optimization problem for an insurer when facing uncertainties. The insurer is allowed to invest into a financial market and purchase proportional reinsurance/acquire new business. The model uncertainty is described by a family of probability measures equivalent to the original probability measure. Compared with Zhang and Siu [16] and Lin et al. [17], we focus on the criterion of minimizing the expected quadratic distance of the insurer's terminal wealth to a given benchmark. Under the "worst-case" scenario, we have derived the optimal investment-reinsurance strategy and the corresponding value function explicitly. In addition, we have presented some numerical illustrations and sensitivity analysis to show the effect of parameters on optimal strategies.

Appendix

Optimal Strategies without Model Risk

The insurer aims to select an optimal investment and reinsurance strategy ϑ which minimizes the expected quadratic distance of the terminal wealth to a given benchmark. That is to find

$$V(x) = \inf_{\vartheta \in \mathscr{A}} \left\{ \mathrm{E}^{x} \left[\left(X^{\vartheta}(T) - k \right)^{2} \right] \right\}. \tag{A.1}$$

Here we also assume $k \geq x_0 e^{r_0 T}$.

Then the generator of the process $V(\cdot)$ is a partial differential operator.

$$\begin{aligned} \widetilde{\mathscr{L}}(h(u)) = {} & \frac{\partial h}{\partial t} \\ & + \left\{ \lambda \mu_{\infty} \left[\eta a - (\eta - \delta) \right] + r_0 u_1 + r\pi \right\} \frac{\partial h}{\partial x} \\ & + \frac{1}{2} \left(\sigma_0^2 a^2 + \sigma_1^2 \pi^2 \right) \frac{\partial^2 h}{\partial x^2}. \end{aligned} \tag{A.2}$$

Thus we can obtain the following HJB equation:

$$\inf_{\vartheta \in \mathscr{A}} \widetilde{\mathscr{L}} \left[\varphi(t, x) \right] = 0, \quad \varphi(T, x) = (x - k)^2. \tag{A.3}$$

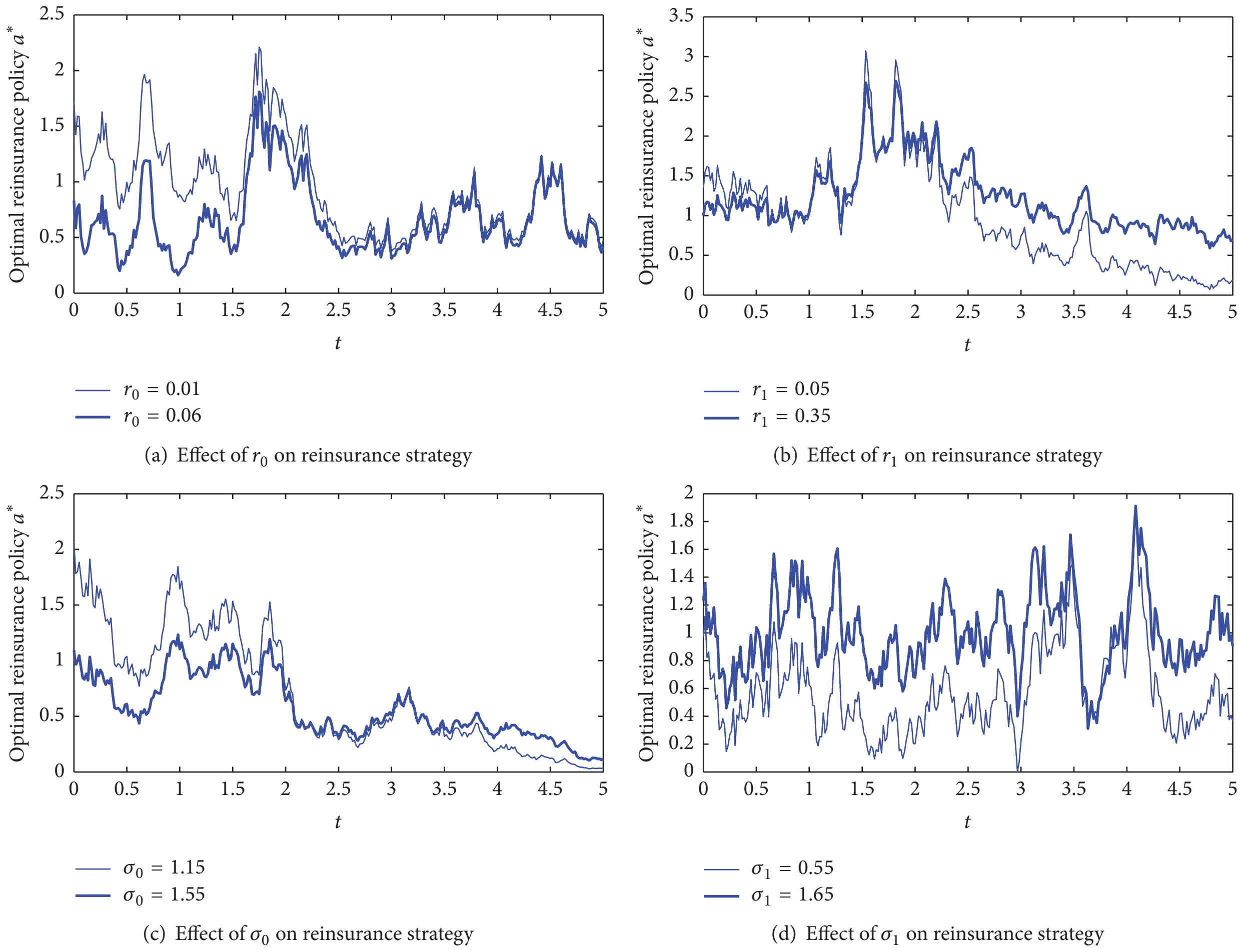


Figure 2: The effect of parameters on optimal reinsurance strategy.

Motivated by the terminal boundary condition, we try the following parametric form of the value function:

$$\varphi(t,x) = m(t)x^2 + n(t)x + l(t), \tag{A.4}$$

with $m(T) = 1, n(T) = -2k, l(T) = k^2$. The corresponding partial derivatives are

$$\begin{aligned} \frac{\partial\varphi}{\partial t} &= \dot{m}(t)x^2 + \dot{n}(t)x + \dot{l}(t), \\ \frac{\partial\varphi}{\partial x} &= 2m(t)x + n(t), \\ \frac{\partial^2\varphi}{\partial x^2} &= 2m(t)x. \end{aligned} \tag{A.5}$$

Substituting the above derivatives into $\widetilde{\mathscr{L}}(\varphi(x))$, we have

$$\begin{aligned} &\widetilde{\mathscr{L}}(\varphi(x)) \\ &\quad = \dot{m}(t)x^2 + \dot{n}(t)x + \dot{l}(t) \\ &\qquad + [r_0 x - \lambda\mu_\infty(\eta-\delta)][2m(t)x + n(t)] \\ &\qquad + \lambda\mu_\infty\eta[2m(t)x + n(t)]a + \sigma_0^2 m(t)a^2 \\ &\qquad + r[2m(t)u_1 + n(t)]\pi + \sigma_1^2 m(t)\pi^2. \end{aligned} \tag{A.6}$$

For fixed θ, by the first-order condition, we can get

$$\begin{aligned} \tilde{\pi}(\theta) &= -\frac{r}{2\sigma_1^2}\left[2x + \frac{n(t)}{m(t)}\right], \\ \tilde{a}(\theta) &= -\frac{\lambda\mu_\infty\eta}{2\sigma_0^2}\left[2x + \frac{n(t)}{m(t)}\right]. \end{aligned} \tag{A.7}$$

Substituting $\tilde{\vartheta} = (\tilde{\pi}(\theta), \tilde{a}(\theta))$ into $\widetilde{\mathscr{L}}(\varphi(x))$, we have

$$\begin{aligned} \widetilde{\mathscr{L}}(\varphi(x)) &= [\dot{m}(t) + (2r_0 - \mathscr{E})m(t)]x^2 \\ &\quad + [\dot{n}(t) + (r_0 - \mathscr{E})n(t) - 2\Gamma m(t)]x \\ &\quad + \dot{l}(t) - \Gamma n(t) - \frac{\mathscr{E}n^2(t)}{4m(t)}, \end{aligned} \tag{A.8}$$

where $\mathscr{E} = (r^2\sigma_0^2 + \lambda^2\mu_\infty^2\eta^2\sigma_1^2)/\sigma_0^2\sigma_1^2$ and $\Gamma = \lambda\mu_\infty(\eta-\delta)$.

To ensure $\widetilde{\mathscr{L}}(\varphi(x)) = 0$ holds, we require

$$\begin{aligned} &\dot{m}(t) + (2r_0 - \mathscr{E})m(t) = 0, \quad m(T) = 1, \\ &\dot{n}(t) + (r_0 - \mathscr{E})n(t) - 2\Gamma m(t) = 0, \quad n(T) = -2k, \\ &\dot{l}(t) - \Gamma n(t) - \frac{\mathscr{E}n^2(t)}{4m(t)} = 0, \quad l(T) = k^2. \end{aligned} \tag{A.9}$$

Then it is easy to obtain

$$\begin{aligned} m(t) &= e^{(2r_0-\mathscr{E})(T-t)}, \\ n(t) &= -2\left[\frac{\Gamma}{r_0}\left(e^{r_0(T-t)}-1\right)+k\right]e^{(r_0-\mathscr{E})(T-t)}, \\ l(t) &= \left[\frac{\Gamma}{r_0}\left(e^{r_0(T-t)}-1\right)+k\right]^2 e^{-\mathscr{E}(T-t)}. \end{aligned} \tag{A.10}$$

So, for $s \le t \le T$, we get the optimal strategies

$$\begin{aligned} \widetilde{\pi}(t) &= \frac{r}{\sigma_1^2}\left[\left(k-\frac{\Gamma}{r_0}\right)e^{-r_0(T-t)}-\left(x-\frac{\Gamma}{r_0}\right)\right] \\ &= \frac{r}{\sigma_1^2 e^{r_0(T-t)}}\left[\left(k-xe^{r_0(T-t)}\right)+\frac{\Gamma}{r_0}\left(e^{r_0(T-t)}-1\right)\right] \end{aligned} \tag{A.11}$$

$$\begin{aligned} \widetilde{a}(t) &= \frac{\lambda\mu_\infty\eta}{\sigma_0^2}\left[\left(k-\frac{\Gamma}{r_0}\right)e^{-r_0(T-t)}-\left(x-\frac{\Gamma}{r_0}\right)\right] \\ &= \frac{\lambda\mu_\infty\eta}{\sigma_0^2 e^{r_0(T-t)}}\left[\left(k-xe^{r_0(T-t)}\right)+\frac{\Gamma}{r_0}\left(e^{r_0(T-t)}-1\right)\right] \end{aligned} \tag{A.12}$$

and the value function

$$V(x) = e^{-\mathscr{E}(T-s)}\left[xe^{r_0(T-s)}-\frac{\Gamma}{r_0}\left(e^{r_0(T-s)}-1\right)-k\right]^2. \tag{A.13}$$

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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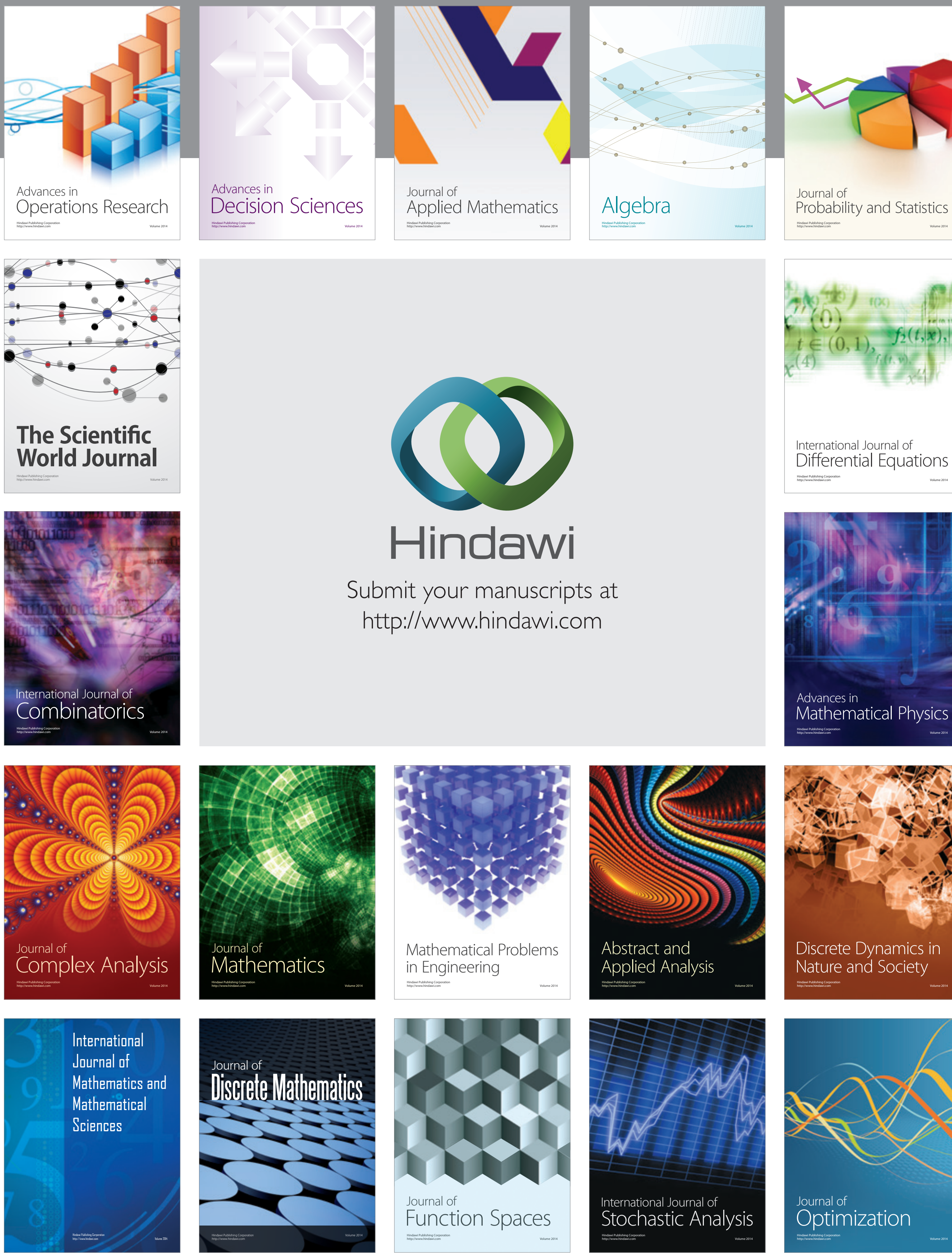
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