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Research Article

Arbitrary Order Fractional Difference Operators with Discrete Exponential Kernels and Applications

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Recently, Abdeljawad and Baleanu have formulated and studied the discrete versions of the fractional operators of order $0 < \alpha \leq 1$ with exponential kernels initiated by Caputo-Fabrizio. In this paper, we extend the order of such fractional difference operators to arbitrary positive order. The extension is given to both left and right fractional differences and sums. Then, existence and uniqueness theorems for the Caputo (CFC) and Riemann (CFR) type initial difference value problems by using Banach contraction theorem are proved. Finally, a Lyapunov type inequality for the Riemann type fractional difference boundary value problems of order $2 < \alpha \leq 3$ is proved and the ordinary difference Lyapunov inequality then follows as α tends to 2 from right. Illustrative examples are discussed and an application about Sturm-Liouville eigenvalue problem in the sense of this new fractional difference calculus is given.

1. Introduction

In the last few decades, the continuous and discrete fractional differential equations have received considerable interest due to their importance in many scientific fields; see, by way of example not exhaustive enumeration, [1–7].

In [8], the authors introduced a fractional derivative with an exponential kernel which tends to the ordinary derivative as α tends to 1. More properties of this fractional derivative have been studied in [9], where the correspondent fractional integral operator was formulated. Then, the authors in [7] defined the left and right fractional derivatives with exponential kernel in the Riemann sense and formulated the right fractional derivatives in the sense of Caputo-Fabrizio with complete investigation to the correspondent fractional integrals and all the discrete versions with integration and summation by parts applied in the fractional and discrete fractional variational calculus. Then, very recently, the same authors proved an interesting monotonicity result in the sense of this new fractional difference calculus in [10].

In the same direction, for the purpose of providing more fractional derivatives with different nonsingular kernels, the authors in [11] defined a fractional operator with Mittag-Leffler kernel and in [12, 13] the complete details and discrete versions have been studied. The exponential kernel fractional derivatives and hence their discrete counterparts are quite different from the Mittag-Leffler kernel fractional operators. For example, the integral operator corresponding to exponential kernel fractional derivatives consists of a multiple of the function f added to a multiple of the integration of f, whereas the Mittag-Leffler kernel correspondent integral operator consists of a multiple of f and a Riemann-Liouville fractional integral of the same order. Also, the monotonicity coefficient of the CFR fractional difference operator of order $0 < \alpha \leq 1$ is α as shown in [10], whereas for the discrete Mittag-Leffler CFR operator is α^2 as proven in [14].

Motivated, by what we mentioned above, we extend the order of fractional difference type operators with discrete exponential kernels to arbitrary positive order, prove existence and uniqueness theorems for the fractional initial value difference problems, and finally prove a Lyapunov type inequality for the CFR fractional difference operators of order $2 < \alpha \leq 3$. The ordinary discrete Lyapunov inequality is then confirmed as α tends to 2 from the right not as in the case of the classical fractional difference as α tends to 2 from the left [15]. For various fractional Lyapunov extensions we refer, for example, to [16–29]. All the authors there were motivated by the following theorem on ordinary Lyapunov inequality.

Theorem 1 (see [30]). *If the boundary value problem*

$$y''(t) + q(t)\,y(t) = 0, \quad t \in (a,b), \; y(a) = y(b) = 0, \quad (1)$$

has a nontrivial solution, where q is a real continuous function; then

$$\int_a^b |q(s)|\,ds > \frac{4}{b-a}. \quad (2)$$

Notice that inequality (2) is known as the classical Lyapunov inequality. It is worth mentioning that Cheng [31] had pointed out that Lyapunov neither stated nor proved Theorem 1 but he only stated the following result.

Theorem 2 (see [26]). *Let $q(t)$ be a nontrivial, continuous, and nonnegative function with period ω and let*

$$\int_0^\omega q(s)\,ds \le \frac{4}{\omega}. \quad (3)$$

Then the roots of the characteristic equation corresponding to Hills equation

$$x''(t) + q(t)\,x(t) = 0, \quad -\infty < t < \infty, \quad (4)$$

are purely imaginary with modulus one.

For the classical fractional calculus which is behind many extensions, we refer the reader to [32–35] and for the sake of comparison with the classical discrete fractional case we refer to [36] and the references therein. In addition, for the discrete fractional operators and their duality we refer to [37–39].

The article will be organised as follows: In the remaining part of this section we shall give some basics about the discrete CFC and CFR fractional differences and their correspondent sums as used in [7, 10]. In Section 2, we extend the order of CFC and CFR fractional differences and their correspondent sums to arbitrary positive order and give some illustrative examples. In Section 3, we prove some existence and uniqueness theorems by means of Banach fixed point theorem and give some illustrative examples. In Section 4, we prove a Lyapunov type inequality for a fractional CFR difference boundary value problem of order $2 < \alpha \le 3$ and give an application to the fractional difference Sturm-Liouville Eigenvalue problem (SLEP) to enrich the applicability of our proven Lyapunov inequality in the frame of fractional difference operators with discrete exponential kernels.

2. Preliminaries

Definition 3 (see [36]). For $\alpha > 0$, $a \in \mathbb{R}$, $\rho(s) = s-1$, and f a real-valued function defined on $\mathbb{N}_a = \{a, a+1, \ldots\}$, the left Riemann-Liouville fractional sum of order $\alpha > 0$ is defined by

$$\left({}_a\nabla^{-\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \left(t-\rho(s)\right)^{\overline{\alpha-1}} f(s). \quad (5)$$

This is fractionalising of the n-iterated nabla sum

$$\left({}_a\nabla^{-n} f\right)(t) = \frac{1}{(n-1)!} \sum_{a+1}^{t} \left(t-\rho(s)\right)^{\overline{n-1}} f(s). \quad (6)$$

The right fractional integral ending at b, where usually we assume that $a \equiv b \pmod 1$, is defined by

$$\left(\nabla_b^{-\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \left(s-\rho(t)\right)^{\overline{\alpha-1}} f(s), \quad (7)$$

where $t^{\overline{\alpha}} = \Gamma(t+\alpha)/\Gamma(t)$ and $\Gamma(t)$ is the well-known gamma special function of t.

Definition 4 (see [7, 10]). For $\alpha \in (0,1)$ and f defined on $\mathbb{N}_a$, or ${}_b\mathbb{N} = \{b, b-1, \ldots\}$ in right case, we have the following definitions:

(i) The left (nabla) CFC fractional difference is given by

$$\begin{aligned}\left({}^{\mathrm{CFC}}_{\;\;a}\nabla^{\alpha} f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \sum_{s=a+1}^{t} \left(\nabla_s f\right)(s)\,(1-\alpha)^{t-\rho(s)} \\ &= B(\alpha) \sum_{s=a+1}^{t} \left(\nabla_s f\right)(s)\,(1-\alpha)^{t-s}.\end{aligned} \quad (8)$$

(ii) The right (nabla) CFC fractional difference has the following form:

$$\begin{aligned}\left({}^{\mathrm{CFC}}\nabla_b^{\alpha} f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \sum_{s=t}^{b-1} \left(-\Delta_s f\right)(s)\,(1-\alpha)^{s-\rho(t)} \\ &= B(\alpha) \sum_{s=t}^{b-1} \left(-\Delta_s f\right)(s)\,(1-\alpha)^{s-t}.\end{aligned} \quad (9)$$

(iii) The left (nabla) CFR fractional difference is written as

$$\begin{aligned}\left({}^{\mathrm{CFR}}_{\;\;a}\nabla^{\alpha} f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \nabla_t \sum_{s=a+1}^{t} f(s)\,(1-\alpha)^{t-\rho(s)} \\ &= B(\alpha)\,\nabla_t \sum_{s=a+1}^{t} f(s)\,(1-\alpha)^{t-s}.\end{aligned} \quad (10)$$

(iv) The right (nabla) CFR fractional difference is given by

$$\begin{aligned}\left({}^{\mathrm{CFR}}\nabla_b^{\alpha} f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \left(-\Delta_t\right) \sum_{s=t}^{b-1} f(s)\,(1-\alpha)^{s-\rho(t)} \\ &= B(\alpha)\left(-\Delta_t\right) \sum_{s=t}^{b-1} f(s)\,(1-\alpha)^{s-t},\end{aligned} \quad (11)$$

where $B(\alpha)$ is a normalization positive constant depending on α satisfying $B(0) = B(1) = 1$, $(\nabla g)(t) = g(t) - g(t-1)$, and $(\Delta g)(t) = g(t+1) - g(t)$.

In [7, 8], it was verified that $({}^{CF}_{a}\nabla^{-\alpha}{}^{CFR}_{a}\nabla^{\alpha}f)(t) = f(t)$ and $({}^{CFR}_{a}\nabla^{\alpha}{}^{CF}_{a}\nabla^{-\alpha}f)(t) = f(t)$. Also, in the right case $({}^{CF}I_{b}^{\alpha}{}^{CFR}\nabla_{b}^{\alpha}f)(t) = f(t)$ and $({}^{CFR}\nabla_{b}^{\alpha}{}^{CF}\nabla_{b}^{-\alpha}f)(t) = f(t)$. From [7, 8] we recall the relation between the CFC and CFR fractional differences as

$$\left({}^{CFC}_{a}\nabla^{\alpha}f\right)(t) = \left({}^{CFR}_{a}\nabla^{\alpha}f\right)(t) - \frac{B(\alpha)}{1-\alpha}f(a)(1-\alpha)^{t-a}, \tag{12}$$

and for the right case by

$$\left({}^{CFC}\nabla_{b}^{\alpha}f\right)(t) = \left({}^{CFR}\nabla_{b}^{\alpha}f\right)(t) - \frac{B(\alpha)}{1-\alpha}f(b)(1-\alpha)^{b-t}. \tag{13}$$

Notice that we extend Definition 4 to arbitrary $\alpha > 0$ in the next section.

Lemma 5 (see [7]). *For $0 < \alpha < 1$, we have*

$$\begin{aligned}\left({}^{CF}_{a}\nabla^{-\alpha}{}^{CFC}_{a}\nabla^{\alpha}f\right)(t) &= f(t) - f(a),\\ \left({}^{CF}\nabla_{b}^{-\alpha}{}^{CFC}\nabla_{b}^{\alpha}f\right)(t) &= f(t) - f(b).\end{aligned} \tag{14}$$

Notation. For a positive integer n, we have

(i) $(\nabla^{n}f)(t) = (\underbrace{\nabla\nabla\cdots\nabla}_{n\text{ times}}f)(t)$.

(ii) $(\Delta^{n}f)(t) = (\underbrace{\Delta\Delta\cdots\Delta}_{n\text{ times}}f)(t)$.

(iii) $({}_{\ominus}\Delta^{n}f)(t) = (-1)^{n}(\Delta^{n}f)(t)$.

3. Higher Order Fractional Differences and Sums

Definition 6. Let $n < \alpha \le n+1$ and f be defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$. Set $\beta = \alpha - n$ and define

$$\begin{aligned}\left({}^{CFC}_{a}\nabla^{\alpha}f\right)(t) &= \left({}^{CFC}_{a}\nabla^{\beta}\nabla^{n}f\right)(t),\\ \left({}^{CFR}_{a}\nabla^{\alpha}f\right)(t) &= \left({}^{CFR}_{a}\nabla^{\beta}\nabla^{n}f\right)(t).\end{aligned} \tag{15}$$

The associated fractional sum is given by

$$\left({}^{CF}_{a}\nabla^{-\alpha}f\right)(t) = \left({}_{a}\nabla^{-n}\,{}^{CF}_{a}\nabla^{-\beta}f\right)(t). \tag{16}$$

Note that if we use the convention that $({}_{a}\nabla^{-0}f)(t) = f(t)$, then for the case $0 < \alpha \le 1$ we have $\beta = \alpha$ and hence we obtain Definition 4. Also, the convention $(\nabla^{0}f)(t) = f(t)$ leads to $({}^{CFR}_{a}\nabla^{\alpha}f)(t)$ and $({}^{CFC}_{a}\nabla^{\alpha}f)(t)$ as in Definition 4 for $0 < \alpha \le 1$.

Remark 7. In Definition 6, if we let $\alpha = n+1$ then $\beta = 1$ and hence $({}^{CFR}_{a}\nabla^{\alpha}f)(t) = ({}^{CFR}_{a}\nabla^{1}\nabla^{n}f)(t) = (\nabla^{n+1}f)(t)$. Also, by noting that $({}^{CF}_{a}\nabla^{-1}f)(t) = ({}_{a}\nabla^{-1}f)(t)$, we see that for $\alpha = n+1$ we have $({}^{CF}_{a}\nabla^{-\alpha}f)(t) = ({}_{a}\nabla^{-(n+1)}f)(t)$. Also, for $0 < \alpha \le 1$ we reobtain the concepts defined in Definition 4. Therefore, our generalization to higher order case is confirmed.

Analogously, in the right case we have the following extension.

Definition 8. Let $n < \alpha \le n+1$ and f be defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$. Set $\beta = \alpha - n$. Then $\beta \in (0,1]$ and we define

$$\begin{aligned}\left({}^{CFC}\nabla_{b}^{\alpha}f\right)(t) &= {}^{CFC}\nabla_{b}^{\beta}\left({}_{\ominus}\Delta^{n}f\right)(t),\\ \left({}^{CFR}\nabla_{b}^{\alpha}f\right)(t) &= {}^{CFR}\nabla_{b}^{\beta}\left({}_{\ominus}\Delta^{n}f\right)(t).\end{aligned} \tag{17}$$

The associated fractional integral is given by

$$\left({}^{CF}\nabla_{b}^{-\alpha}f\right)(t) = \left(\nabla_{b}^{-n}\,{}^{CF}\nabla_{b}^{-\beta}f\right)(t). \tag{18}$$

An immediate extension of (12) and (13) by using Definition 6 is the following.

Proposition 9. *For f defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$ and $n < \alpha \le n+1$, we have*

$$\left({}^{CFC}_{a}\nabla^{\alpha}f\right)(t) = \left({}^{CFR}_{a}\nabla^{\alpha}f\right)(t) - \frac{B(\alpha)}{1-\alpha}(\nabla^{n}f)(a)(1-\alpha)^{t-a}, \tag{19}$$

and for the right case

$$\left({}^{CFC}\nabla_{b}^{\alpha}f\right)(t) = \left({}^{CFR}\nabla_{b}^{\alpha}f\right)(t) - \frac{B(\alpha)}{1-\alpha}\left({}_{\ominus}\Delta^{n}f\right)(b)(1-\alpha)^{b-t}. \tag{20}$$

Next proposition explains the action of the arbitrary order sum operator ${}^{CF}_{a}\nabla^{-\alpha}$ on the arbitrary order CFR and CFC differences (and vice versa) and the action of the CFR difference on the CF correspondent sum operator.

Proposition 10. *For $u(t)$ defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$ and for some $n \in \mathbb{N}_0$ with $n < \alpha \le n+1$, we have*

(i) $({}^{CFR}_{a}\nabla^{\alpha}{}^{CF}_{a}\nabla^{-\alpha}u)(t) = u(t)$.

(ii) $({}^{CF}_{a}\nabla^{-\alpha}{}^{CFR}_{a}\nabla^{\alpha}u)(t) = u(t) - \sum_{k=0}^{n-1}((\nabla^{k}u)(a)/k!)(t-a)^{\overline{k}}$.

(iii) $({}^{CF}_{a}\nabla^{-\alpha}{}^{CFC}_{a}\nabla^{\alpha}u)(t) = u(t) - \sum_{k=0}^{n}((\nabla^{k}u)(a)/k!)(t-a)^{\overline{k}}$.

Proof. (i) By Definition 6 and the statement after Definition 4, we have

$$\begin{aligned}\left({}^{CFR}_{a}\nabla^{\alpha}{}^{CF}_{a}\nabla^{-\alpha}u\right)(t) &= {}^{CFR}_{a}\nabla^{\beta}\nabla^{n}{}_{a}\nabla^{-n}\,{}^{CF}_{a}\nabla^{-\beta}u(t)\\ &= {}^{CFR}_{a}\nabla^{\beta}{}^{CF}_{a}\nabla^{-\beta}u(t) = u(t).\end{aligned} \tag{21}$$

(ii) By Definition 6 and the statement after Definition 4, we have

$$\begin{aligned}\left({}^{CF}_{a}\nabla^{-\alpha}{}^{CFR}_{a}\nabla^{\alpha}u\right)(t)&=\left({}_{a}\nabla^{-n}{}^{CF}_{a}\nabla^{-\beta}{}^{CFR}_{a}\nabla^{\beta}\nabla^{n}u\right)(t)\\&={}_{a}\nabla^{-n}\nabla^{n}u(t)\\&=u(t)-\sum_{k=0}^{n-1}\frac{\left(\nabla^{k}u\right)(a)}{k!}(t-a)^{\overline{k}},\end{aligned}\tag{22}$$

where $\beta=\alpha-n$.

(iii) By Lemma 5 applied to $f(t)=(\nabla^{n}u)(t)$, we have

$$\begin{aligned}\left({}^{CF}_{a}\nabla^{-\alpha}{}^{CFC}_{a}\nabla^{\alpha}u\right)(t)&={}_{a}\nabla^{-n}{}_{a}\nabla^{-\beta}{}^{CFC}_{a}\nabla^{\beta}\nabla^{n}u(t)\\&={}_{a}\nabla^{-n}\left[\nabla^{n}u(t)-(\nabla^{n}u)(a)\right]\\&=u(t)-\sum_{k=0}^{n-1}\frac{\left(\nabla^{k}u\right)(a)}{k!}(t-a)^{\overline{k}}\\&\quad-(\nabla^{n}u)(a)\frac{(t-a)^{\overline{n}}}{n!}\\&=u(t)-\sum_{k=0}^{n}\frac{\left(\nabla^{k}u\right)(a)}{k!}(t-a)^{\overline{k}}.\end{aligned}\tag{23}$$

□

Using the facts that ${}_{\ominus}\Delta^{n}\nabla_{b}^{-n}g(t)=g(t)$,

$$\nabla_{b}^{n}{}_{\ominus}\Delta^{n}g(t)=g(t)-\sum_{k=0}^{n-1}\frac{\left({}_{\ominus}\Delta^{k}g\right)(b)}{k!}(b-t)^{\overline{k}},\tag{24}$$

and making use of Lemma 5 and the statement after Definition 4, we can state, for the right case, the following.

Proposition 11. *For $u(t)$ defined on $\mathbb{N}_{a,b}$ and $\alpha\in(n,n+1]$, for some $n\in\mathbb{N}_0$, we have*

(i) $({}^{CFR}\nabla_{b}^{\alpha}{}^{CF}\nabla_{b}^{-\alpha}u)(t)=u(t)$.

(ii) $({}^{CF}\nabla_{b}^{-\alpha}{}^{CFR}\nabla_{b}^{\alpha}u)(t)=u(t)-\sum_{k=0}^{n-1}(({}_{\ominus}\Delta^{k}u)(b)/k!)(b-t)^{\overline{k}}$.

(iii) $({}^{CF}\nabla_{b}^{-\alpha}{}^{CFC}\nabla_{b}^{\alpha}u)(t)=u(t)-\sum_{k=0}^{n}(({}_{\ominus}\Delta^{k}u)(b)/k!)(b-t)^{\overline{k}}$.

Example 12. Consider the initial value problem:

$$\left({}^{CFC}_{a}\nabla^{\alpha}y\right)(t)=F(t),\tag{25}$$

where $F(t)$ is defined on $\mathbb{N}_{a,b}=\mathbb{N}_{a}\cap{}_{b}\mathbb{N}$. Let us consider two cases depending on the order $\alpha>0$:

(i) Assume $0<\alpha\leq1$, $y(a)=c$, and $F(a)=0$. By applying ${}^{CF}_{a}\nabla^{-\alpha}$ and making use of Proposition 10, we get the solution

$$y(t)=c+\frac{1-\alpha}{B(\alpha)}F(t)+\frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}F(s).\tag{26}$$

Notice that the condition $F(a)=0$ verifies the initial condition $y(a)=c$. In addition, when $\alpha\to1$ we obtain the solution of the ordinary difference initial value problem $(\nabla y)(t)=F(t)$, $y(a)=c$.

(ii) Assume $1<\alpha\leq2$, $F(a)=0$, $y(a)=c_1$, and $(\nabla y)(a)=c_2$. By applying ${}^{CF}_{a}\nabla^{-\alpha}$ and making use of Proposition 10 and Definition 6 with $\beta=\alpha-1$, we obtain the solution

$$\begin{aligned}y(t)&=c_1+c_2(t-a)+\frac{2-\alpha}{B(\alpha-1)}\sum_{s=a+1}^{t}K(s)\\&\quad+\frac{\alpha-1}{B(\alpha-1)}\sum_{s=a+1}^{t}(t-\rho(s))F(s).\end{aligned}\tag{27}$$

Notice that the solution $y(t)$ verifies $y(a)=c_1$ without the use of $F(a)=0$. However, it verifies $(\nabla y)(a)=c_2$ under the assumption $F(a)=0$. Also, note that when $\alpha\to2$ we recover the solution of the second-order ordinary difference initial value problem $(\nabla^{2}y)(t)=F(t)$, $y(a)=c_1$, and $(\nabla y)(a)=c_2$.

In the next section, we prove existence and uniqueness theorems for some types of CFC and CFR initial value difference problems.

Example 13. Consider the CFC difference boundary value problem

$$\begin{gathered}\left({}^{CFC}_{a}\nabla^{\alpha}y\right)(t)+q(t)y(t)=0,\\1<\alpha\leq2,\ t\in\mathbb{N}_{a,b},\ y(a)=y(b)=0.\end{gathered}\tag{28}$$

Then $\beta=\alpha-1$ and by Proposition 10 if we apply the operator ${}^{CF}_{a}\nabla^{-\alpha}$, we obtain the solution

$$y(t)=c_1+c_2(t-a)-\left({}^{CF}_{a}\nabla^{-\alpha}q(\cdot)y(\cdot)\right)(t).\tag{29}$$

But

$$\begin{aligned}\left({}^{CF}_{a}\nabla^{-\alpha}q(\cdot)y(\cdot)\right)(t)&=\frac{1-\beta}{B(\beta)}\sum_{s=a+1}^{t}q(s)y(s)\\&\quad+\frac{\beta}{B(\beta)}{}_{a}\nabla^{-2}q(t)y(t).\end{aligned}\tag{30}$$

Hence, the solution has the form

$$\begin{aligned}y(t)&=c_1+c_2(t-a)-\frac{2-\alpha}{B(\alpha-1)}\sum_{s=a+1}^{t}q(s)y(s)\\&\quad-\frac{\alpha-1}{B(\alpha-1)}\sum_{s=a+1}^{t}(t-\rho(s))q(s)y(s).\end{aligned}\tag{31}$$

The boundary conditions imply that $c_1=0$ and

$$\begin{aligned}c_2&=\frac{2-\alpha}{(b-a)B(\alpha-1)}\sum_{s=a+1}^{b}q(s)y(s)\\&\quad+\frac{\alpha-1}{(b-a)B(\alpha-1)}\sum_{s=a+1}^{b}(b-\rho(s))q(s)y(s).\end{aligned}\tag{32}$$

Hence,

$$
\begin{aligned}
y(t) = {} & \frac{(2-\alpha)(t-a)}{(b-a)B(\alpha-1)} \sum_{s=a+1}^{b} q(s)\,y(s) \\
& - \frac{(\alpha-1)(t-a)}{(b-a)B(\alpha-1)} \sum_{s=a+1}^{b} (b-\rho(s))\,q(s)\,y(s) \\
& - \frac{2-\alpha}{B(\alpha-1)} \sum_{s=a+1}^{t} q(s)\,y(s) \\
& - \frac{\alpha-1}{B(\alpha-1)} \sum_{s=a+1}^{t} (t-\rho(s))\,q(s)\,y(s).
\end{aligned}
\tag{33}
$$

4. Existence and Uniqueness Theorems for the Initial Value Problem Types

In this section we prove existence and uniqueness theorems for CFC and CFR type initial value problems.

Theorem 14. *Consider the system*

$$
\begin{gathered}
\left({}^{CFC}_{\ a}\nabla^{\alpha} y\right)(t) = f(t, y(t)), \\
t \in \mathbb{N}_{a,b},\ 0 < \alpha \leq 1,\ y(a) = c,
\end{gathered}
\tag{34}
$$

such that $f(a, y(a)) = 0$, $A((1-\alpha)/B(\alpha) + \alpha(b-a)/B(\alpha)) < 1$, *and* $|f(t, y_1) - f(t, y_2)| \leq A|y_1 - y_2|$, $A > 0$, *where* $f : \mathbb{N}_{a,b} \times \mathbb{R} \to \mathbb{R}$ *and* $y : \mathbb{N}_{a,b} \to \mathbb{R}$. *Then, system (34) has a unique solution of the form*

$$
y(t) = c + {}^{CF}_{\ a}\nabla^{-\alpha} f(t, y(t)).
\tag{35}
$$

Proof. First, by the help of Proposition 10, (12), and taking into account the fact that $f(a, y(a)) = 0$, it is straight forward to prove that $y(t)$ satisfies system (34) if and only if it satisfies (35).

Let $X = \{x : \max_{t\in\mathbb{N}_{a,b}}|x(t)| < \infty\}$ be the Banach space endowed with the norm $\|x\| = \max_{t\in\mathbb{N}_{a,b}}|x(t)|$. On X define the linear operator

$$
(Tx)(t) = c + {}^{CF}_{\ a}\nabla^{-\alpha} f(t, x(t)).
\tag{36}
$$

Then, for arbitrary $x_1, x_2 \in X$ and $t \in \mathbb{N}_{a,b}$, we have by assumption that

$$
\begin{aligned}
& \left|(Tx_1)(t) - (Tx_2)(t)\right| \\
& \quad = \left|{}^{CF}_{\ a}\nabla^{-\alpha}\left[f(t, x_1(t)) - f(t, x_2(t))\right]\right| \\
& \quad \leq A\left(\frac{1-\alpha}{B(\alpha)} + \frac{\alpha(b-a)}{B(\alpha)}\right)\|x_1 - x_2\|,
\end{aligned}
\tag{37}
$$

and hence T is a contraction. By Banach fixed point theorem, there exists a unique $x \in X$ such that $Tx = x$ and hence the proof is complete. □

Remark 15. Similar existence and uniqueness theorems can be proved for system (34) with higher order by making use of Proposition 10. The condition $f(a, y(a)) = 0$ always can not be avoided as we have seen in Example 12 with $f(t, y(t)) = F(t)$. As a result of Theorem 14, we conclude that the fractional difference linear initial value problem

$$
\begin{gathered}
\left({}^{CFC}_{\ a}\nabla^{\alpha} y\right)(t) = r y(t), \\
r \in \mathbb{R},\ t \in \mathbb{N}_{a,b},\ 0 < \alpha \leq 1,\ y(a) = c,
\end{gathered}
\tag{38}
$$

can have only the trivial solution unless $\alpha = 1$. Indeed, the solution satisfies $y(t) = c + r((1-\alpha)/B(\alpha))y(t) + (\alpha r/B(\alpha))\sum_{s=a+1}^{t} y(s)$. This solution is only verified at a if $(1-\alpha)y(a) = 0$.

Theorem 16. *Consider the system*

$$
\begin{gathered}
\left({}^{CFR}_{\ a}\nabla^{\alpha} y\right)(t) = f(t, y(t)), \\
t \in \mathbb{N}_{a,b},\ 1 < \alpha \leq 2,\ y(a) = c,
\end{gathered}
\tag{39}
$$

such that $(A/B(\alpha-1))((2-\alpha)(b-a)+(\alpha-1)(b-a)^{\overline{2}}/2) < 1$, *and* $|f(t, y_1)-f(t, y_2)| \leq A|y_1-y_2|$, $A > 0$, *where* $f : \mathbb{N}_{a,b}\times\mathbb{R} \to \mathbb{R}$ *and* $y : \mathbb{N}_{a,b} \to \mathbb{R}$. *Then, system (34) has a unique solution of the form*

$$
\begin{aligned}
y(t) & = c + {}^{CF}_{\ a}\nabla^{-\alpha} f(t, y(t)) \\
& = c + \frac{2-\alpha}{B(\alpha-1)} \sum_{s=a+1}^{t} f(s, y(s)) \\
& \quad + \frac{\alpha-1}{B(\alpha-1)} \left({}_{a}\nabla^{-2} f(\cdot, y(\cdot))(t)\right).
\end{aligned}
\tag{40}
$$

Proof. If we apply ${}^{CF}_{\ a}\nabla^{-\alpha}$ to system (39) and make use of Proposition 10 with $\beta = \alpha - 1$ then we reach at the representation (40). Conversely, if we apply ${}^{CFR}_{\ \ a}\nabla^{\alpha}$, make use of Proposition 10 and by noting that

$$
{}^{CFR}_{\ \ a}\nabla^{\alpha} = {}^{CFR}_{\ \ a}\nabla^{\beta}\nabla_t c = 0,
\tag{41}
$$

we obtain system (39). Hence, $y(t)$ satisfies system (39) if and only if it satisfies (40).

Let $X = \{x : \max_{t\in\mathbb{N}_{a,b}}|x(t)| < \infty\}$ be the Banach space endowed with the norm $\|x\| = \max_{t\in\mathbb{N}_{a,b}}|x(t)|$. On X define the linear operator

$$
(Tx)(t) = c + {}^{CF}_{\ a}\nabla^{-\alpha} f(t, x(t)).
\tag{42}
$$

Then, for arbitrary $x_1, x_2 \in X$ and $t \in \mathbb{N}_{a,b}$, we have by assumption that

$$
\begin{aligned}
& \left|(Tx_1)(t) - (Tx_2)(t)\right| \\
& \quad = \left|{}^{CF}_{\ a}\nabla^{-\alpha}\left[f(t, x_1(t)) - f(t, x_2(t))\right]\right| \\
& \quad \leq \frac{A}{B(\alpha-1)}\left((2-\alpha)(b-a) + \frac{(\alpha-1)(b-a)^{\overline{2}}}{2}\right) \\
& \quad \cdot \|x_1 - x_2\|,
\end{aligned}
\tag{43}
$$

and hence T is a contraction. By Banach Contraction Principle, there exists a unique $x \in X$ such that $Tx = x$ and hence the proof is complete. □

5. The Lyapunov Inequality for the CFR Difference Boundary Value Problem

In this section, we prove a Lyapunov inequality for a CFR boundary value difference problem of order $2 < \alpha \leq 3$.

Consider the boundary value problem

$$\begin{gathered}\left({}_{a}^{\mathrm{CFR}}\nabla^{\alpha} y\right)(t)+q(t)\, y^{\rho}(t)=0, \\ 2<\alpha \leq 3,\ t \in \mathbb{N}_{a+1, b-1},\ y(a)=y(b)=0,\end{gathered} \tag{44}$$

where, $y^{\rho}(t) = y(\rho(t)) = y(t-1)$.

Lemma 17. *$y(t)$ is a solution of the boundary value problem (44) if and only if it satisfies the equation*

$$y(t)=\sum_{s=a+1}^{b} G(t, s)\, T(s, y(s)), \tag{45}$$

where

$$G(t, s) = \begin{cases}\dfrac{(t-a)(b-\rho(s))}{b-a}, & t+1 \leq s,\ t, s \in \mathbb{N}_{a, b}, \\ \left(\dfrac{(t-a)(b-\rho(s))}{b-a}-(t-\rho(s))\right), & s-1 \leq t,\ t, s \in \mathbb{N}_{a, b},\end{cases} \tag{46}$$

$$\begin{aligned}T(t, y(t)) &= {}_{a}^{\mathrm{CF}}\nabla^{\beta}\left(q(\cdot)\, y^{\rho}(\cdot)\right)(t) \\ &= \frac{1-\beta}{B(\beta)} q(t)\, y^{\rho}(t)+\frac{\beta}{B(\beta)}\left({}_{a}\nabla^{-1} q(\cdot)\, y^{\rho}(\cdot)\right)(t), \quad \beta=\alpha-2.\end{aligned}$$

Proof. Apply the integral ${}_{a}^{\mathrm{CF}}\nabla^{-\alpha}$ to (44) and make use of Definition 6 and Proposition 10 with $n = 2$ and $\beta = \alpha - 2$ to reach

$$\begin{aligned}y(t) &= c_1+c_2(t-a)-\left({}_{a}\nabla^{-2} T(\cdot, y(\cdot))\right)(t) \\ &= c_1+c_2(t-a)-\sum_{s=a+1}^{t}(t-\rho(s))\, T(s, y(s)).\end{aligned} \tag{47}$$

The condition $y(a) = 0$ implies that $c_1 = 0$ and the condition $y(b) = 0$ implies that $c_2 = (1/(b-a))\sum_{s=a+1}^{b}(b-\rho(s))T(s, y(s))$, and hence

$$\begin{aligned}y(t) = {} & \frac{t-a}{b-a} \sum_{s=a+1}^{b}(b-\rho(s))\, T(s, y(s)) \\ & -\sum_{s=a+1}^{t}(t-\rho(s))\, q(s)\, T(s, y(s))\, d s.\end{aligned} \tag{48}$$

Then, the result follows by splitting the summation

$$\begin{aligned}& \sum_{s=a+1}^{b}(b-\rho(s))\, T(s, y(s)) \\ & \quad=\sum_{s=a+1}^{t}(b-\rho(s))\, T(s, y(s)) \\ & \quad\quad+\sum_{s=t+1}^{b}(b-\rho(s))\, T(s, y(s)).\end{aligned} \tag{49}$$

□

Lemma 18. *Given that $b \equiv a \pmod 1$, the Green function $G(t, s)$ defined in Lemma 17 has the following properties:*

(1) *$G(t, s) \geq 0$ for all $t, s \in \mathbb{N}_{a,b}$.*

(2) *$\max_{t\in\mathbb{N}_{a,b}} G(t, s) = G(\rho(s), s)$ for $s \in \mathbb{N}_{a+1,b-1}$.*

(3) *$f(s) = G(\rho(s), s) = (\rho(s) - a)(b - \rho(s))/(b - a)$ has a unique maximum, given by*

$$\max_{s \in \mathbb{N}_{a, b}} f(s) = \begin{cases}f\left(\dfrac{(a+b+2)}{2}\right)=\dfrac{b-a}{4} & \text{if } b-a \text{ is even} \\ f\left(\dfrac{(a+b+3)}{2}\right)=\dfrac{(b-a)^2-1}{4(b-a)} & \text{if } b-a \text{ is odd.}\end{cases} \tag{50}$$

Hence, in either cases $\max_{s\in\mathbb{N}_{a,b}} f(s) \leq (b-a)/4$.

Proof. (1) It is clear that $g_1(t, s) = (t-a)(b-\rho(s))/(b-a) \geq 0$. Regarding the part $g_2(t, s) = ((t-a)(b-\rho(s))/(b-a)-(t-\rho(s)))$ we see that $(t-\rho(s)) = ((t-a)/(b-a))(b-(a+(\rho(s)-a)(b-a)/(t-a)))$ and that $a+(\rho(s)-a)(b-a)/(t-a) \geq \rho(s)$ if and only if $\rho(s)(b-t)+a(t-b) \geq 0$ if and only if $s \geq a+1$. Hence, we conclude that $g_2(t, s) \geq 0$ as well.

(2) Clearly, $g_1(t, s)$ is an increasing function in t. Applying ∇ to g_2 with respect to t for every fixed $s \geq a+1$ we see that $\nabla_t g(t, s) = (b-a)^{-1}(a-\rho(s))$ and hence g_2 is a decreasing function in t.

(3) Let $f(s) = G(\rho(s), s) = (\rho(s)-a)(b-\rho(s))/(b-a)$. Then,

$$(\nabla f)(s)=\frac{a+b-2 s+3}{b-a}=0, \tag{51}$$

if $s = (a+b+3)/2$ and hence for $a, b \in \mathbb{N}$, f attains its maximum at $s_1 = (a+b+3)/2$ if $a+b$ (or $b-a$) is odd and at $(a+b+2)/2$ if $a+b$ (or $b-a$) is even. More generally, if a and b are such that $b \equiv a \pmod 1$, we see that $s_1 \equiv a \pmod 1$ if $b-a$ is odd and $s_2 \equiv a \pmod 1$ if $b-a$ is even. Finally, $f(s_1) = ((b-a)^2-1)/4(b-a)$ and $f(s_2) = (b-a)/4$. □

In next lemma, we estimate $T(t, y(t))$ for a function $y \in B[\mathbb{N}_{a,b}]$, the Banach space of all Banach-valued finite sequences on $\mathbb{N}_{a,b}$ with $\|y\| = \max_{t\in\mathbb{N}_{a,b}}|y(t)|$, where $|\cdot|$ is the norm in the Banach space.

Lemma 19. *For $y \in B[\mathbb{N}_{a,b}]$ and $2 < \alpha \leq 3$, $\beta = \alpha - 2$, we have, for any $t \in \mathbb{N}_{a,b}$,*

$$|T(t, y(t))| \leq R(t)\|y\|, \tag{52}$$

where

$$R(t)=\left[\frac{3-\alpha}{B(\alpha-2)}|q(t)|+\frac{\alpha-2}{B(\alpha-2)} \sum_{s=a+1}^{t}|q(s)|\right]. \tag{53}$$

Theorem 20 (the CFR fractional difference Lyapunov inequality). *If the boundary value problem (44) has a nontrivial solution, where $q(t)$ is a real-valued bounded function on $\mathbb{N}_{a,b}$, then*

$$\sum_{s=a+1}^{b} R(s)>\frac{4}{b-a}. \tag{54}$$

Proof. Assume $y \in Y = B[\mathbb{N}_{a,b}]$ is a nontrivial solution of the boundary value problem (44). By Lemma 17, y must satisfy

$$y(t) = \sum_{s=a+1}^{b} G(t,s)\, T(s, y(s)). \tag{55}$$

Then, by using the properties of the Green function $G(t,s)$ proved in Lemmas 18 and 19, we come to the conclusion that

$$\|y\| < \frac{b-a}{4} \sum_{s=a+1}^{b} R(s)\, \|y\| \tag{56}$$

from which (54) follows. □

Remark 21. Note that if $\alpha \to 2^+$, then $R(t)$ tends to $|q(t)|$ and hence we obtain the classical nabla discrete version of the Lyapunov inequality (2). For the sake of more comparisons of Lyapunov inequalities on time scales we refer to [40].

Example 22. Consider the following CFR Sturm-Liouville difference eigenvalue problem (SLDEP) of order $2 < \alpha \le 3$

$$\begin{aligned} &\left({}_0^{\mathrm{CFR}}\nabla^{\alpha} y\right)(t) + \lambda y^{\rho}(t) = 0, \\ &\qquad t \in \mathbb{N}_{1,b-1},\ b \in \mathbb{N},\ y(0) = y(b) = 0. \end{aligned} \tag{57}$$

If λ is an eigenvalue of (57), then by Theorem 20 with $q(t) = \lambda$, we have

$$\begin{aligned} T(t) &= \left[\frac{3-\alpha}{B(\alpha-2)} |\lambda| + \frac{\alpha-2}{B(\alpha-2)} \left({}_0\nabla^{-1} |\lambda|\right)(t)\right] \\ &= |\lambda| \left[\frac{3-\alpha}{B(\alpha-2)} + \frac{\alpha-2}{B(\alpha-2)} t\right]. \end{aligned} \tag{58}$$

Hence, we must have

$$\sum_{s=1}^{b} T(s) = |\lambda| \left[\frac{b(3-\alpha)}{B(\alpha-2)} + \frac{\overline{b^2}(\alpha-2)}{2B(\alpha-2)}\right] > \frac{4}{b}. \tag{59}$$

Notice that the limiting case $\alpha \to 2^+$ implies that $|\lambda| > 4/b^2$ which is the lower bound for the eigenvalues of the ordinary difference eigenvalue problem:

$$\begin{aligned} &\left(\nabla^2 y\right)(t) + \lambda y^{\rho}(t) = 0, \\ &\qquad t \in \mathbb{N}_{1,b-1},\ y(0) = y(b) = 0. \end{aligned} \tag{60}$$

6. Conclusions

Fractional differences and their correspondent fractional sum operators are of importance in discrete modeling of various problems in science. We extended the fractional difference calculus whose difference operators depend on a discrete exponential function kernel to arbitrary positive order. The correspondent arbitrary order fractional sum operators have been defined as well and applied to solve fractional initial and boundary value difference problems. The extension for right fractional differences and sums is also achieved. To set up the basic concepts, we proved existence and uniqueness theorems by means of Banach fixed point theorem for initial value problems in the frame of CFC and CFR fractional differences. We have come to the conclusion that the condition $f(a, y(a)) = 0$ is necessary to guarantee the existence of solution and hence fractional linear difference initial value problem with constant coefficients results in the trivial solution unless the order is positive integer. We used our extension to arbitrary order to prove a Lyapunov type inequality for a CFR boundary value problem of order $2 < \alpha \le 3$ and then obtain the classical ordinary case when α tends to 2 from right. This proves different behavior from the classical fractional difference case, where the Lyapunov inequality was proved for a fractional difference boundary problem of order $1 < \alpha \le 2$ and the classical ordinary case was then recovered when α tends to 2 from left.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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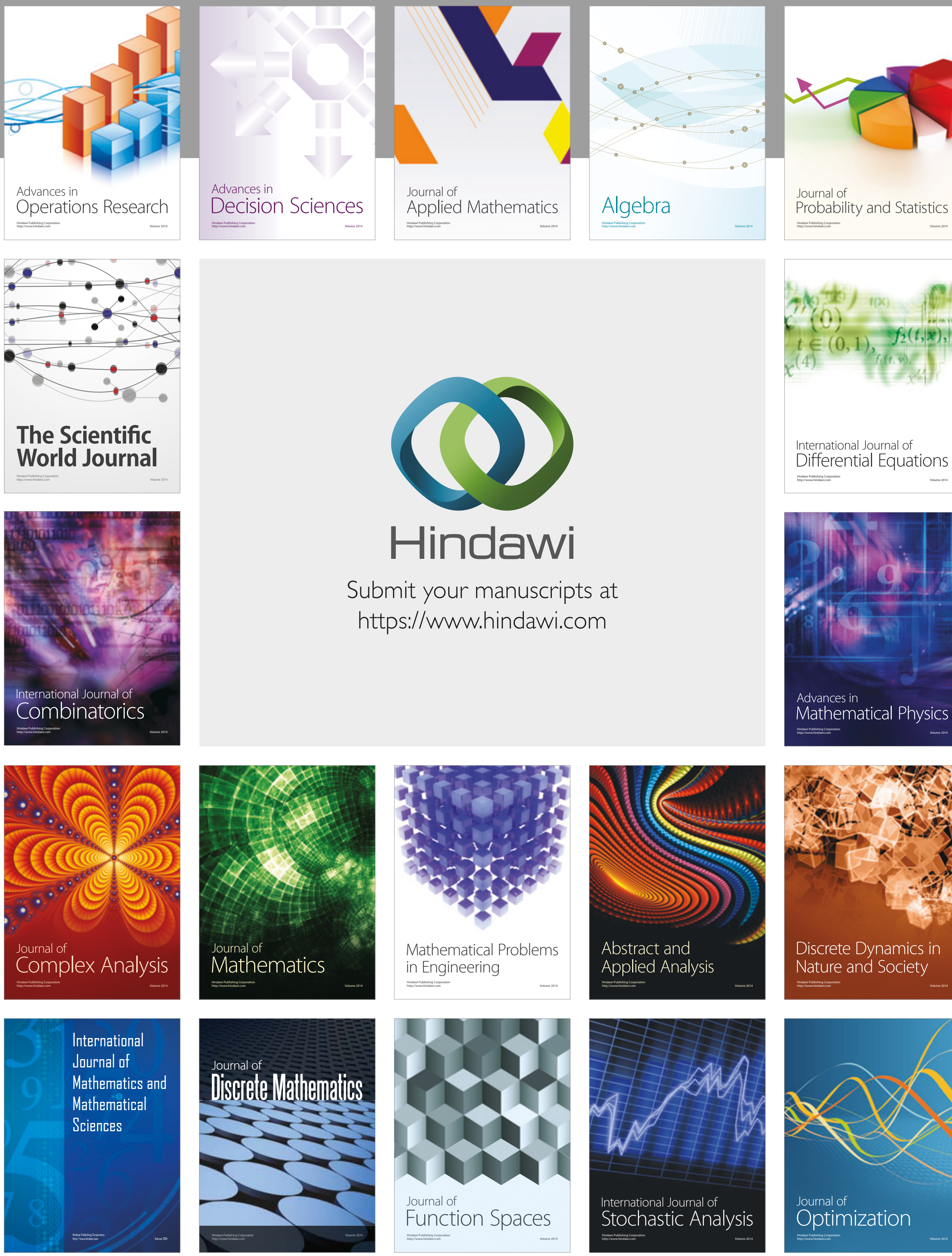
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