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Research Article

A Generalization of Linear and Nonlinear Retarded Integral Inequalities in Two Independent Variables

Ying Jiang, Guojing Xing, and Chenghui Zhang

School of Control Science and Engineering, Shandong University, Jinan, Shandong 250061, China

Correspondence should be addressed to Guojing Xing; xgjsdu@sdu.edu.cn

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Integral inequalities, which provide explicit bounds on unknown functions, are used to serve as handy tools in the study of the qualitative properties of solutions to differential and integral equations. By utilizing some analysis techniques, such as amplification method, differential, and integration, several new types of linear and nonlinear retarded integral inequalities in two independent variables are provided. These results generalize and complement previous ones. An illustrative example is given to support the obtained results. The study of the numerical example shows that the new results presented in this paper work well in the analysis of retarded integral inequalities in two independent variables.

1. Introduction

With the development of science and technology, various inequalities have been paid more and more attention, and the generalization of inequalities has become one of the important research directions in modern mathematics. The integral inequality, which has integrals of unknown functions, is an important type of inequality. For nonlinear differential equations derived from the natural science and engineering technology, especially from various branches of mathematics, it is difficult or impossible to obtain explicit solutions in most cases. Therefore, it is of great significance to get the bounds of the solutions to those nonlinear differential equations. Integral inequalities just can provide the bounds of the solutions to the nonlinear differential equations and integral equations. Hence, integral inequalities are used to serve as handy tools in the study of the qualitative properties of solutions to differential and integral equations, such as existence, uniqueness, boundedness, oscillation, stability, and invariant manifold. For example, these inequalities have been widely employed to investigate the stability of switched systems which can be applied to modeling many engineering system problems in real world, such as traffic control, automobile engine control, switching power converters, and multiagent consensus [1–5]. For some related contributions on various classes of integral inequalities, we refer the reader to [1–20] and the references cited therein.

For convenience, throughout this paper, $\mathbb{R}$ represents the set of real numbers, $\mathbb{R}_+ = [0,\infty)$, and $C(A,B)$ signifies the class of all continuous functions defined on set A with range in the set B.

In what follows, we provide some background details that motivated our study. One of the most famous and widespread integral inequalities in the study of differential and integral equations is Gronwall-Bellman-type inequality [6–8], which can be described as follows.

Theorem 1. *Let u and f be nonnegative continuous functions on an interval $[a,b]$ satisfying*

$$u(t) \le c + \int_a^t f(s)\,u(s)\,\mathrm{d}s, \quad t \in [a,b] \tag{1}$$

for some constant $c \ge 0$. Then

$$u(t) \le c\exp\left(\int_a^t f(s)\,\mathrm{d}s\right), \quad t \in [a,b]. \tag{2}$$

In recent years, many scholars have done a lot of researches and generalization of the above integral inequality, which make the integral inequalities develop continually and the application fields expand gradually. Pachpatte [9, 10] investigated the inequality

$$u(t) \le u_1 + \int_0^t [f(s)u(s) + p(s)]\,ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)\,d\sigma\right)ds \tag{3}$$

and the retarded inequality

$$u(t) \le u_2 + \int_0^t f(s)u(s)\,ds + \int_0^{\alpha(t)} g(s)u(s)\,ds, \tag{4}$$

where $\alpha \in C^1(I, I)$ is nondecreasing with $\alpha(t) \le t$ on $I = [0, T)$ and u_1 and u_2 are constants. Abdeldaim and El-Deeb [11] generalized [9] and analyzed the following retarded linear and nonlinear inequalities:

$$\begin{aligned} u(t) &\le u_0 + \int_0^{\alpha(t)} [f(s)u(s) + p(s)]\,ds + \int_0^{\alpha(t)} f(s)\left(\int_0^s g(\sigma)u(\sigma)\,d\sigma\right)ds, \\ u(t) &\le u_0 + \int_0^{\alpha(t)} \varphi(u(s))[f(s)\varphi(u(s)) + p(s)]\,ds + \int_0^{\alpha(t)} \varphi(u(s))f(s)\left(\int_0^s g(\sigma)\varphi(u(\sigma))\,d\sigma\right)ds, \end{aligned} \tag{5}$$

respectively. Tian et al. [16] introduced the retarded inequalities in two independent variables as follows.

Theorem 2 (see [16, Theorem 1]). *Let u, f, and $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $a(x) > 0$, $b(y) > 0$, $a'(x) \ge 0$, $b'(y) \ge 0$, and α, $\beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(x) \le x$ and $\beta(y) \le y$ on $\mathbb{R}_+$. Moreover, let $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$ and let $\varphi(x) > 0$ on $(0, \infty)$, $\psi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\psi(x) > 0$ on $(0, \infty)$. If*

$$\varphi(u(x, y)) \le a(x) + b(y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} u(t, s) \cdot [f(t, s)\psi(u(t, s)) + g(t, s)]\,ds\,dt, \tag{6}$$

then, for $0 \le x < \xi_1$, $0 \le y < \eta_1$,

$$u(x, y) \le \varphi^{-1}\left\{\Omega^{-1}\left[G^{-1}\left(G(P(x, y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t, s)\,ds\,dt\right)\right]\right\}, \tag{7}$$

where

$$\begin{aligned} P(x, y) &= \Omega(a(0) + b(y)) + \int_0^x \frac{a'(s)}{\varphi^{-1}(a(s) + b(0))}ds + \int_0^{\alpha(x)}\int_0^{\beta(y)} g(t, s)\,ds\,dt. \\ \Omega(x) &= \int_{x_0}^x \frac{ds}{\varphi^{-1}(s)}, \quad x > x_0 > 0, \\ G(z) &= \int_{z_0}^z \frac{ds}{\psi[\varphi^{-1}(\Omega^{-1}(s))]}, \quad z > z_0 > 0, \end{aligned} \tag{8}$$

Ω^{-1}, φ^{-1}, and G^{-1} are the inverses of Ω, φ, and G, respectively; $(\xi_1, \eta_1) \in \mathbb{R}_+ \times \mathbb{R}_+$ is chosen so that

$$\begin{aligned} &G(P(x, y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t, s)\,ds\,dt \in \operatorname{dom}(G^{-1}), \quad (t, s) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ &G^{-1}\left\{G(P(x, y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t, s)\,ds\,dt\right\} \in \operatorname{dom}(\Omega^{-1}), \quad (t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ \end{aligned} \tag{9}$$

with dom$(\cdot)$ *denoting the function domain.*

Theorem 3 (see [16, Corollary 1]). *Assume that u, f, g, a, b, α, and β are defined as in Theorem 2. Let $\varphi(u) = u^p$ and $\psi(u) = u^{q-1}$ in Theorem 2, where $p \ge q > 1$ are positive constants. If*

$$u^p(x, y) \le a(x) + b(y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} u(t, s) \cdot [f(t, s)u^{q-1}(t, s) + g(t, s)]\,ds\,dt, \tag{10}$$

then, for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$u(x, y) \le \begin{cases} \left(\left[\dfrac{p-1}{p}\lambda(x, y)\right]^{(p-q)/(p-1)} + \dfrac{p-q}{p}\displaystyle\int_0^{\alpha(x)}\int_0^{\beta(y)} f(t, s)\,ds\,dt\right)^{1/(p-q)}, & \text{when } p > q, \\ \left[\dfrac{p-1}{p}\lambda(x, y)\right]^{1/(p-1)} \exp\left(\dfrac{1}{p}\displaystyle\int_0^{\alpha(x)}\int_0^{\beta(y)} f(t, s)\,ds\,dt\right), & \text{when } p = q, \end{cases} \tag{11}$$

where

$$\lambda(x,y) = \frac{p}{p-1}[a(0)+b(y)]^{(p-1)/p} + \int_0^x \frac{a'(s)}{(a(s)+b(0))^{1/p}} ds + \int_0^{\alpha(x)}\int_0^{\beta(y)} g(t,s)\, ds\, dt. \quad (12)$$

Motivated by the recent contributions of Abdeldaim and El-Deeb [11], Zhang and Meng [14], and Tian et al. [16], our principal goal is to extend the inequalities with one variable in [11] to those with two variables which include Theorems 2 and 3 as special cases.

The rest of the work is organized as follows. A useful lemma that plays a fundamental role in the proofs of the main theorems is presented in Section 2. In Section 3, we propose our main theorems and corollary on several new types of linear and nonlinear retarded integral inequalities in two independent variables. An illustrative example is given to indicate the usefulness of these inequalities in Section 4, which is followed by a short conclusion in Section 5.

2. Lemma

The subsequent lemma is helpful in proving our main theorems.

Lemma 4. *Assume that* u, f, *and* $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *and* $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *is an increasing function with* $\varphi(\infty) = \infty$ *and* $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *is a nondecreasing function. Suppose that* c *is a nonnegative constant and* $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *are nondecreasing with* $\alpha(x) \le x$, $\beta(y) \le y$, $\alpha(0) = 0$, *and* $\beta(0) = 0$ *on* $\mathbb{R}_+$. *If*

$$\varphi(u(x,y)) \le c + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\psi(u(t,s))\, ds\, dt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s) \cdot \left(\int_0^t\int_0^s g(\tau,\omega)\psi(u(\tau,\omega))\, d\omega\, d\tau\right) ds\, dt, \quad (13)$$

then, for $0 \le x < \xi$, $0 \le y < \eta$,

$$u(x,y) \le \varphi^{-1}\left\{G^{-1}\left(G(c) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1 + \int_0^t\int_0^s g(\tau,\omega)\, d\omega\, d\tau\right) ds\, dt\right)\right\}, \quad (14)$$

where

$$G(z) = \int_{z_0}^{z} \frac{ds}{\psi[\varphi^{-1}(s)]}, \quad z > z_0 > 0; \quad (15)$$

φ^{-1} *and* G^{-1} *are the inverses of* φ *and* G, *respectively;* $(\xi, \eta) \in \mathbb{R}_+ \times \mathbb{R}_+$ *is chosen so that*

$$G(c) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1 + \int_0^t\int_0^s g(\tau,\omega)\, d\omega\, d\tau\right) ds\, dt \in \operatorname{dom}(G^{-1}), \quad 0 \le x < \xi,\ 0 \le y < \eta. \quad (16)$$

Proof. Define the nondecreasing positive function z by

$$z(x,y) = c + \varepsilon + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\psi(u(t,s))\, ds\, dt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s) \cdot \left(\int_0^t\int_0^s g(\tau,\omega)\psi(u(\tau,\omega))\, d\omega\, d\tau\right) ds\, dt, \quad (17)$$

where ε is an arbitrary small positive number. Utilizing inequality (13) and the monotonicity of φ^{-1}, we get

$$u(x,y) \le \varphi^{-1}(z(x,y)). \quad (18)$$

Differentiating (17) with respect to x and combining (18) and the monotonicities of φ^{-1}, z, and ψ, we conclude that

$$\begin{aligned} z_x(x,y) &= \alpha'(x)\int_0^{\beta(y)} f(\alpha(x),s)\psi(u(\alpha(x),s))\, ds + \alpha'(x)\int_0^{\beta(y)} f(\alpha(x),s) \cdot \left(\int_0^{\alpha(x)}\int_0^s g(\tau,\omega)\psi(u(\tau,\omega))\, d\omega\, d\tau\right) ds \\ &\le \psi\left[\varphi^{-1}(z(x,y))\right]\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),s)\, ds + \psi\left[\varphi^{-1}(z(x,y))\right]\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),s) \cdot \left(\int_0^{\alpha(x)}\int_0^s g(\tau,\omega)\, d\omega\, d\tau\right) ds. \end{aligned} \quad (19)$$

On account of $\psi[\varphi^{-1}(z(x,y))] \ge \psi[\varphi^{-1}(c+\varepsilon)] > 0$, we deduce that

$$\frac{z_x(x,y)}{\psi[\varphi^{-1}(z(x,y))]} \le \alpha'(x)\int_0^{\beta(y)} f(\alpha(x),s) \cdot \left(1 + \int_0^{\alpha(x)}\int_0^s g(\tau,\omega)\, d\omega\, d\tau\right) ds. \quad (20)$$

Integrating the latter inequality on $[0, x]$ and letting $\varepsilon \to 0$, we have

$$G(z(x,y)) \le G(c) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(t,s) \cdot \left(1 + \int_0^t \int_0^s g(\tau,\omega)\, d\omega\, d\tau\right) ds\, dt \tag{21}$$

owing to (15). By virtue of (16), (18), and the last inequality, we obtain inequality (14). The proof is complete. □

Remark 5. Assume that $\int_{z_0}^{\infty} [\psi(\varphi^{-1}(s))]^{-1} ds = \infty$. Then $G(\infty) = \infty$ and (14) is valid on $\mathbb{R}_+ \times \mathbb{R}_+$; that is, one can select $\xi = \infty$ and $\eta = \infty$.

3. Main Results

The following are the main results of this paper.

Theorem 6. *Let u, a, f, g, and $h \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and let $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(x) \le x$, $\beta(y) \le y$, $\alpha(0) = 0$, and $\beta(0) = 0$ on $\mathbb{R}_+$. If the inequality*

$$u(x,y) \le a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} (f(t,s)u(t,s) + h(t,s))\, ds\, dt + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(t,s) \cdot \left(\int_0^t \int_0^s g(\tau,\omega)u(\tau,\omega)\, d\omega\, d\tau\right) ds\, dt \tag{22}$$

holds, for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, then

$$\begin{aligned} u(x,y) \le{}& a(x,y) + \exp\left(\int_0^{\alpha(x)} \int_0^{\beta(y)} f(t,s)\left(1 + \int_0^t \int_0^s g(\tau,\omega)\, d\omega\, d\tau\right) ds\, dt\right) \\ &\times \int_0^x \exp\left(-\int_0^{\alpha(l)} \int_0^{\beta(y)} f(t,s)\left(1 + \int_0^t \int_0^s g(\tau,\omega)\, d\omega\, d\tau\right) ds\, dt\right) \\ &\times \frac{\partial}{\partial l}\left[\int_0^{\alpha(l)} \int_0^{\beta(y)} (f(t,s)a(t,s) + h(t,s))\, ds\, dt + \int_0^{\alpha(l)} \int_0^{\beta(y)} f(t,s)\left(\int_0^t \int_0^s g(\tau,\omega)a(\tau,\omega)\, d\omega\, d\tau\right) ds\, dt\right] dl. \end{aligned} \tag{23}$$

Proof. Letting

$$z(x,y) = \int_0^{\alpha(x)} \int_0^{\beta(y)} (f(t,s)u(t,s) + h(t,s))\, ds\, dt + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(t,s) \cdot \left(\int_0^t \int_0^s g(\tau,\omega)u(\tau,\omega)\, d\omega\, d\tau\right) ds\, dt, \tag{24}$$

then $z(0, y) = z(x, 0) = 0$ and

$$u(x,y) \le a(x,y) + z(x,y). \tag{25}$$

Our assumptions on f, u, h, g, α, and β indicate that z is a positive function which is nondecreasing with respect to each of the two variables. Differentiating z with respect to x and using (25), we arrive at

$$\begin{aligned} z_x(x,y) ={}& \alpha'(x) \int_0^{\beta(y)} (f(\alpha(x),s)u(\alpha(x),s) + h(\alpha(x),s))\, ds + \alpha'(x) \int_0^{\beta(y)} f(\alpha(x),s) \cdot \left(\int_0^{\alpha(x)} \int_0^s g(\tau,\omega)u(\tau,\omega)\, d\omega\, d\tau\right) ds \\ \le{}& \alpha'(x) \int_0^{\beta(y)} [f(\alpha(x),s)(a(\alpha(x),s) + z(\alpha(x),s)) + h(\alpha(x),s)]\, ds + \alpha'(x) \int_0^{\beta(y)} f(\alpha(x),s) \\ &\cdot \left(\int_0^{\alpha(x)} \int_0^s g(\tau,\omega)(a(\tau,\omega) + z(\tau,\omega))\, d\omega\, d\tau\right) ds \\ ={}& \alpha'(x) \int_0^{\beta(y)} f(\alpha(x),s)z(\alpha(x),s)\, ds + \alpha'(x) \int_0^{\beta(y)} f(\alpha(x),s) \cdot \left(\int_0^{\alpha(x)} \int_0^s g(\tau,\omega)z(\tau,\omega)\, d\omega\, d\tau\right) ds \\ &+ \alpha'(x) \int_0^{\beta(y)} (f(\alpha(x),s)a(\alpha(x),s) + h(\alpha(x),s))\, ds + \alpha'(x) \int_0^{\beta(y)} f(\alpha(x),s) \cdot \left(\int_0^{\alpha(x)} \int_0^s g(\tau,\omega)a(\tau,\omega)\, d\omega\, d\tau\right) ds. \end{aligned} \tag{26}$$

By virtue of the monotonicity of z, we get

$$z_x(x,y) - z(x,y)\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),s) \cdot \left(1+\int_0^{\alpha(x)}\int_0^{s} g(\tau,\omega)\,d\omega\,d\tau\right)ds \le \alpha'(x) \cdot \int_0^{\beta(y)} \left(f(\alpha(x),s)\,a(\alpha(x),s) + h(\alpha(x),s)\right)ds + \alpha'(x)\int_0^{\beta(y)} f(\alpha(x),s) \cdot \left(\int_0^{\alpha(x)}\int_0^{s} g(\tau,\omega)\,a(\tau,\omega)\,d\omega\,d\tau\right)ds. \tag{27}$$

Multiplying the latter inequality by $e^{-\int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)(1+\int_0^t\int_0^s g(\tau,\omega)d\omega\,d\tau)ds\,dt}$ yields

$$\frac{\partial}{\partial x}\left(z(x,y)\,e^{-\int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)(1+\int_0^t\int_0^s g(\tau,\omega)d\omega\,d\tau)ds\,dt}\right) \le \exp\left(-\int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right)ds\,dt\right) \times \alpha'(x)\left(\int_0^{\beta(y)}\left(f(\alpha(x),s)\,a(\alpha(x),s)+h(\alpha(x),s)\right)ds + \int_0^{\beta(y)} f(\alpha(x),s)\left(\int_0^{\alpha(x)}\int_0^{s} g(\tau,\omega)\,a(\tau,\omega)\,d\omega\,d\tau\right)ds\right). \tag{28}$$

Integrating this inequality on $[0,x]$, we deduce that

$$z(x,y) \le \exp\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right)ds\,dt\right) \times \int_0^x \left[\exp\left(-\int_0^{\alpha(l)}\int_0^{\beta(y)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right)ds\,dt\right)\times\alpha'(l) \cdot \left(\int_0^{\beta(y)}\left(f(\alpha(l),s)\,a(\alpha(l),s)+h(\alpha(l),s)\right)ds + \int_0^{\beta(y)} f(\alpha(l),s)\left(\int_0^{\alpha(l)}\int_0^{s} g(\tau,\omega)\,a(\tau,\omega)\,d\omega\,d\tau\right)ds\right)\right]dl. \tag{29}$$

Combining (25) with (29), we get inequality (23). This completes the proof. □

Theorem 7. *Let u, a, f, g, and $h \in C(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)$, $a(x,y)>0$, $a_x \ge 0$, $a_y \ge 0$, and $\alpha,\beta \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ be nondecreasing with $\alpha(x)\le x$, $\beta(y)\le y$, $\alpha(0)=0$, and $\beta(0)=0$ on $\mathbb{R}_+$. Moreover, let γ and $\psi \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ be nondecreasing function with $\gamma>0$ and $\psi>0$ on $(0,\infty)$. If*

$$u(x,y) \le a(x,y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \gamma(u(t,s)) \cdot \left(f(t,s)\,\psi(u(t,s)) + h(t,s)\right)ds\,dt + \int_0^{\alpha(x)}\int_0^{\beta(y)} \gamma(u(t,s))\,f(t,s) \cdot \left(\int_0^t\int_0^s g(\tau,\omega)\,\psi(u(\tau,\omega))\,d\omega\,d\tau\right)ds\,dt, \tag{30}$$

then, for $0\le x<\xi$, $0\le y<\eta$,

$$u(x,y) \le \Omega^{-1}\left\{G^{-1}\left(G(P(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right)ds\,dt\right)\right\}, \tag{31}$$

where

$$P(x,y) = \Omega(a(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)\,ds\,dt, \tag{32}$$

$$\Omega(x) = \int_{x_0}^{x} \frac{ds}{\gamma(s)}, \quad x > x_0 > 0, \tag{33}$$

$$G(z) = \int_{z_0}^{z} \frac{ds}{\psi[\Omega^{-1}(s)]}, \quad z > z_0 > 0. \tag{34}$$

Ω^{-1} and G^{-1} are the inverses of Ω and G, respectively; $(\xi,\eta)\in\mathbb{R}_+\times\mathbb{R}_+$ is chosen so that

$$G(P(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right)ds\,dt \in \operatorname{dom}(G^{-1}),$$
$$G^{-1}\left\{G(P(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s) \cdot \left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right)ds\,dt\right\} \in \operatorname{dom}(\Omega^{-1}) \tag{35}$$

for $0\le x<\xi$, $0\le y<\eta$.

Proof. Define the nondecreasing function z by

$$z(x,y) = \int_0^{\alpha(x)}\int_0^{\beta(y)} \gamma(u(t,s)) \cdot \left(f(t,s)\,\psi(u(t,s)) + h(t,s)\right)ds\,dt$$

$$+\int_0^{\alpha(x)}\int_0^{\beta(y)}\gamma(u(t,s))f(t,s)\cdot\left(\int_0^t\int_0^s g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\right)ds\,dt. \tag{36}$$

Then

$$u(x,y)\le a(x,y)+z(x,y). \tag{37}$$

Differentiating (36) and using (37) and the monotonicity of γ, we obtain

$$\begin{aligned}z_{xy}(x,y) &= \alpha'(x)\beta'(y)\gamma(u(\alpha(x),\beta(y)))\cdot(f(\alpha(x),\beta(y))\psi(u(\alpha(x),\beta(y)))\\&\quad+h(\alpha(x),\beta(y)))+\alpha'(x)\beta'(y)\cdot\gamma(u(\alpha(x),\beta(y)))f(\alpha(x),\beta(y))\\&\quad\cdot\int_0^{\alpha(x)}\int_0^{\beta(y)}g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\le\alpha'(x)\\&\quad\cdot\beta'(y)\gamma[a(\alpha(x),\beta(y))+z(\alpha(x),\beta(y))]\\&\quad\cdot\Bigg(f(\alpha(x),\beta(y))\psi(u(\alpha(x),\beta(y)))\\&\quad+h(\alpha(x),\beta(y))+f(\alpha(x),\beta(y))\\&\quad\cdot\int_0^{\alpha(x)}\int_0^{\beta(y)}g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\Bigg).\end{aligned} \tag{38}$$

Let $T_1\le\xi$ and $T_2\le\eta$ be arbitrary numbers. Utilizing (38) and the monotonicities of a, z, and γ, we get that, for $0\le x<T_1$ and $0\le y<T_2$,

$$\begin{aligned}z_{xy}(x,y)&\le\alpha'(x)\beta'(y)\gamma[a(T_1,T_2)+z(x,y)]\\&\quad\cdot\Bigg(f(\alpha(x),\beta(y))\psi(u(\alpha(x),\beta(y)))\\&\quad+h(\alpha(x),\beta(y))+f(\alpha(x),\beta(y))\\&\quad\cdot\int_0^{\alpha(x)}\int_0^{\beta(y)}g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\Bigg).\end{aligned} \tag{39}$$

For $a(T_1,T_2)>0$ and $\gamma[a(T_1,T_2)+z(x,y)]>0$,

$$\begin{aligned}\frac{z_{xy}(x,y)}{\gamma[a(T_1,T_2)+z(x,y)]}&\le\alpha'(x)\beta'(y)\\&\quad\cdot\Bigg(f(\alpha(x),\beta(y))\psi(u(\alpha(x),\beta(y)))\\&\quad+h(\alpha(x),\beta(y))+f(\alpha(x),\beta(y))\\&\quad\cdot\int_0^{\alpha(x)}\int_0^{\beta(y)}g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\Bigg).\end{aligned} \tag{40}$$

From another point of view,

$$\frac{\partial}{\partial y}\left(\frac{z_x}{\gamma[a(T_1,T_2)+z(x,y)]}\right)\le\frac{z_{xy}}{\gamma[a(T_1,T_2)+z(x,y)]}. \tag{41}$$

It follows from (40) and (41) that

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{z_x}{\gamma[a(T_1,T_2)+z(x,y)]}\right)&\le\alpha'(x)\beta'(y)\\&\quad\cdot\Bigg(f(\alpha(x),\beta(y))\psi(u(\alpha(x),\beta(y)))\\&\quad+h(\alpha(x),\beta(y))+f(\alpha(x),\beta(y))\\&\quad\cdot\int_0^{\alpha(x)}\int_0^{\beta(y)}g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\Bigg).\end{aligned} \tag{42}$$

Integrating the above inequality on $[0,y]$ with respect to the second variable and taking $z_x(x,0)=0$ into account, we have

$$\begin{aligned}\frac{z_x(x,y)}{\gamma[a(T_1,T_2)+z(x,y)]}&\le\frac{z_x(x,0)}{\gamma[a(T_1,T_2)+z(x,0)]}\\&\quad+\alpha'(x)\int_0^{\beta(y)}\Bigg(f(\alpha(x),s)\psi(u(\alpha(x),s))\\&\quad+h(\alpha(x),s)+f(\alpha(x),s)\\&\quad\cdot\int_0^{\alpha(x)}\int_0^{s}g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\Bigg)ds\\&=\alpha'(x)\int_0^{\beta(y)}\Bigg(f(\alpha(x),s)\psi(u(\alpha(x),s))\\&\quad+h(\alpha(x),s)+f(\alpha(x),s)\\&\quad\cdot\int_0^{\alpha(x)}\int_0^{s}g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\Bigg)ds.\end{aligned} \tag{43}$$

From (33), the latter relation gives

$$\begin{aligned}\frac{\partial}{\partial x}(\Omega[a(T_1,T_2)+z(x,y)])&\le\alpha'(x)\\&\quad\cdot\int_0^{\beta(y)}\Bigg(f(\alpha(x),s)\psi(u(\alpha(x),s))+h(\alpha(x),s)\\&\quad+f(\alpha(x),s)\\&\quad\cdot\int_0^{\alpha(x)}\int_0^{s}g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\Bigg)ds.\end{aligned} \tag{44}$$

Integrating the last inequality over $[0, x]$, we get

$$\begin{aligned}
&\Omega(a(T_1,T_2)+z(x,y)) \le \Omega(a(T_1,T_2)) \\
&\quad + \int_0^{\alpha(x)}\int_0^{\beta(y)} \Big(f(t,s)\psi(u(t,s)) + h(t,s) + f(t,s) \\
&\quad \cdot \int_0^t\int_0^s g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau \Big)\,ds\,dt = P(x, \\
&\quad y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\psi(u(t,s))\,ds\,dt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s) \\
&\quad \cdot \left(\int_0^t\int_0^s g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\right) ds\,dt \\
&\quad \le P(T_1,T_2) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\psi(a(T_1,T_2) \\
&\quad + z(t,s))\,ds\,dt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s) \\
&\quad \cdot \left(\int_0^t\int_0^s g(\tau,\omega)\psi(a(T_1,T_2)+z(\tau,\omega))\,d\omega\,d\tau\right) ds\,dt,
\end{aligned} \tag{45}$$

where P is defined as in (32). Combining (37) and the monotonicity of a and employing Lemma 4, we obtain

$$\begin{aligned}
&u(x,y) \le a(T_1,T_2) + z(x,y) \le \Omega^{-1}\Big\{G^{-1}\Big(G(P(T_1,T_2)) \\
&\quad + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right) ds\,dt\Big)\Big\},
\end{aligned} \tag{46}$$

where G is defined as in (34). Taking $x = T_1$ and $y = T_2$, we conclude that

$$\begin{aligned}
&u(T_1,T_2) \le \Omega^{-1}\Big\{G^{-1}\Big(G(P(T_1,T_2)) \\
&\quad + \int_0^{\alpha(T_1)}\int_0^{\beta(T_2)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right) ds\,dt\Big)\Big\}.
\end{aligned} \tag{47}$$

As $T_1 \le \xi$ and $T_2 \le \eta$ are arbitrary, we get the desired inequality (31). The proof is complete. □

Theorem 8. *Assume that $u, a, f, g, h, \alpha, \beta, \gamma$, and ψ are defined as in Theorem 7. Moreover, let $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be increasing function with $\varphi(\infty) = \infty$ and $\varphi(x) > 0$ on $(0,\infty)$. If*

$$\begin{aligned}
&\varphi(u(x,y)) \le a(x,y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \gamma(u(t,s)) \\
&\quad \cdot (f(t,s)\psi(u(t,s)) + h(t,s))\,ds\,dt \\
&\quad + \int_0^{\alpha(x)}\int_0^{\beta(y)} \gamma(u(t,s)) f(t,s) \\
&\quad \cdot \left(\int_0^t\int_0^s g(\tau,\omega)\psi(u(\tau,\omega))\,d\omega\,d\tau\right) ds\,dt,
\end{aligned} \tag{48}$$

then, for $0 \le x < \xi$, $0 \le y < \eta$,

$$\begin{aligned}
&u(x,y) \le \varphi^{-1}\Big\{\Omega^{-1}\Big[G^{-1}\Big(G(P(x,y)) \\
&\quad + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right) ds\,dt\Big)\Big]\Big\},
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
P(x,y) &= \Omega(a(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)\,ds\,dt, \\
\Omega(x) &= \int_{x_0}^{x} \frac{ds}{\gamma[\varphi^{-1}(s)]}, \quad x > x_0 > 0, \\
G(z) &= \int_{z_0}^{z} \frac{ds}{\psi[\varphi^{-1}(\Omega^{-1}(s))]}, \quad z > z_0 > 0;
\end{aligned} \tag{50}$$

φ^{-1}, Ω^{-1}, and G^{-1} are the inverses of φ, Ω, and G, respectively; $(\xi,\eta) \in \mathbb{R}_+ \times \mathbb{R}_+$ is chosen so that

$$\begin{aligned}
&G(P(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\Big(1 \\
&\quad + \int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\Big) ds\,dt \\
&\quad \in \operatorname{dom}(G^{-1}), \\
&G^{-1}\Big\{G(P(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s) \\
&\quad \cdot \left(1+\int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right) ds\,dt\Big\} \\
&\quad \in \operatorname{dom}(\Omega^{-1})
\end{aligned} \tag{51}$$

for $0 \le x < \xi$, $0 \le y < \eta$.

Proof. Define function z by (36). Then

$$u(x,y) \le \varphi^{-1}[a(x,y) + z(x,y)]. \tag{52}$$

The rest of the proof is similar to that of Theorem 7 and hence is omitted. □

Remark 9. Letting $a(x,y) = a(x) + b(y)$, $\gamma(u(x,y)) = u(x,y)$, and $g(x,y) = 0$ in Theorem 8, Theorem 8 turns out to be Theorem 2. Therefore, the inequality established in Theorem 8 generalizes that of [16, Theorem 1].

If $\varphi(u) = u^p$, $\gamma(u) = u^q$, and $\psi(u) = u^n$ in Theorem 8, where $p \ge q + n > 1$, and p, q, and n are positive constants, then we have the following corollary.

Corollary 10. *Assume that u, a, f, g, h, α, and β are defined as in Theorem 8. If*

$$u^p(x,y) \le a(x,y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} u^q(t,s) \cdot \left(f(t,s)u^n(t,s) + h(t,s)\right) \mathrm{d}s\,\mathrm{d}t + \int_0^{\alpha(x)}\int_0^{\beta(y)} u^q(t,s) f(t,s) \cdot \left(\int_0^t\int_0^s g(\tau,\omega)u^n(\tau,\omega)\,\mathrm{d}\omega\,\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}t, \quad (53)$$

then, for all $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$,

$$u(x,y) \le \begin{cases} \left(\left[\dfrac{p-q}{p}\lambda(x,y)\right]^{(p-q-n)/(p-q)} + \dfrac{p-q-n}{p}\theta(x,y)\right)^{1/(p-q-n)}, & \text{when } p > q+n, \\ \left(\dfrac{n}{p}\lambda(x,y)\right)^{1/n}\exp\left(\dfrac{1}{p}\theta(x,y)\right), & \text{when } p = q+n, \end{cases} \quad (54)$$

where

$$\lambda(x,y) = \frac{p}{p-q}\left[a(x,y)\right]^{(p-q)/p} + \int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$\theta(x,y) = \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s) \cdot \left(1 + \int_0^t\int_0^s g(\tau,\omega)\,\mathrm{d}\omega\,\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}t. \quad (55)$$

Proof. Assume that $p > q+n$ and let $\varphi(u) = u^p$, $\gamma(u) = u^q$, and $\psi(u) = u^n$. Then we have $\varphi^{-1}(u) = u^{1/p}$, and so

$$\Omega(x) = \int_{x_0}^x \frac{\mathrm{d}s}{\gamma\left[\varphi^{-1}(s)\right]} = \int_{x_0}^x s^{-q/p}\,\mathrm{d}s = \frac{p}{p-q}\left(x^{(p-q)/p} - x_0^{(p-q)/p}\right),$$
$$\Omega^{-1}(x) = \left(\frac{p-q}{p}x + x_0^{(p-q)/p}\right)^{p/(p-q)},$$
$$\psi\left[\varphi^{-1}\left(\Omega^{-1}(x)\right)\right] = \left(\frac{p-q}{p}x + x_0^{(p-q)/p}\right)^{n/(p-q)},$$
$$G(z) = \int_{z_0}^z \frac{\mathrm{d}s}{\psi\left[\varphi^{-1}\left(\Omega^{-1}(s)\right)\right]} = \int_{z_0}^z \left(\frac{p-q}{p}s + x_0^{(p-q)/p}\right)^{-n/(p-q)}\mathrm{d}s = \frac{p}{p-q-n}\left(\frac{p-q}{p}z + x_0^{(p-q)/p}\right)^{(p-q-n)/(p-q)} - \frac{p}{p-q-n}\left(\frac{p-q}{p}z_0 + x_0^{(p-q)/p}\right)^{(p-q-n)/(p-q)},$$
$$G^{-1}(z) = \frac{p}{p-q}\left\{\left[\frac{p-q-n}{p}z + \left(\frac{p-q}{p}z_0 + x_0^{(p-q)/p}\right)^{(p-q-n)/(p-q)}\right]^{(p-q)/(p-q-n)} - x_0^{(p-q)/p}\right\},$$
$$\Omega^{-1}\left[G^{-1}\left(G(P(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,\mathrm{d}\omega\,\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}t\right)\right] = \left\{\left[\left(\frac{p-q}{p}\lambda(x,y)\right)^{(p-q-n)/(p-q)} + \frac{p-q-n}{p}\theta(x,y)\right]^{(p-q)/(p-q-n)} - x_0^{(p-q)/p} + x_0^{(p-q)/p}\right\}^{p/(p-q)} = \left[\left(\frac{p-q}{p}\lambda(x,y)\right)^{(p-q-n)/(p-q)} + \frac{p-q-n}{p}\theta(x,y)\right]^{p/(p-q-n)}, \quad (56)$$

where λ and θ are defined in (55). Using Theorem 8, one can easily obtain

$$u(x,y) \le \varphi^{-1}\left\{\Omega^{-1}\left[G^{-1}\left(G(P(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1+\int_0^t\int_0^s g(\tau,\omega)\,\mathrm{d}\omega\,\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}t\right)\right]\right\} = \left(\left[\frac{p-q}{p}\lambda(x,y)\right]^{(p-q-n)/(p-q)} + \frac{p-q-n}{p}\theta(x,y)\right)^{1/(p-q-n)}. \quad (57)$$

When $p = q + n$,

$$\psi\left[\varphi^{-1}\left(\Omega^{-1}(x)\right)\right] = \frac{p-q}{p}x + x_0^{(p-q)/p},$$
$$G(z) = \int_{z_0}^{z} \frac{ds}{\psi\left[\varphi^{-1}\left(\Omega^{-1}(s)\right)\right]} = \int_{z_0}^{z}\left(\frac{p-q}{p}s + x_0^{(p-q)/p}\right)^{-1} ds = \frac{p}{p-q}\left[\ln\left(z + \frac{p}{p-q}x_0^{(p-q)/p}\right) - \ln\left(z_0 + \frac{p}{p-q}x_0^{(p-q)/p}\right)\right],$$
$$G^{-1}(z) = \left(z_0 + \frac{p}{p-q}x_0^{(p-q)/p}\right)\exp\left(\frac{p-q}{p}z\right) - \frac{p}{p-q}\cdot x_0^{(p-q)/p},$$
$$\Omega^{-1}\left[G^{-1}\left(G(P(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)\left(1 + \int_0^t\int_0^s g(\tau,\omega)\,d\omega\,d\tau\right)ds\,dt\right)\right] = \left[\frac{p-q}{p}\lambda(x,y)\exp\left(\frac{p-q}{p}\theta(x,y)\right) - x_0^{(p-q)/p} + x_0^{(p-q)/p}\right]^{p/(p-q)} = \left[\frac{n}{p}\lambda(x,y)\exp\left(\frac{n}{p}\theta(x,y)\right)\right]^{p/n}, \tag{58}$$

where λ and θ are the same as in (55). By Theorem 8, similar discussions can give

$$u(x,y) \le \left(\frac{n}{p}\lambda(x,y)\right)^{1/n}\exp\left(\frac{1}{p}\theta(x,y)\right). \tag{59}$$

This completes the proof. □

Remark 11. Letting $a(x, y) = a(x) + b(y)$, $q = 1$, $n = q - 1$, and $g(x, y) = 0$, Corollary 10 reduces to Theorem 3. Hence, the inequality established in Corollary 10 includes the result of [16, Corollary 1].

4. Example

Example 1. Consider the integral equation

$$u^p(x,y) = k(x,y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} F\left(t,s,u(t,s), \int_0^t\int_0^s H(\tau,\omega,u(\tau,\omega))\,d\omega\,d\tau\right)ds\,dt, \tag{60}$$

where $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $F : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing with $\alpha(x) \le x$, $\beta(y) \le y$, $\alpha(0) = 0$, and $\beta(0) = 0$ on $\mathbb{R}_+$, and $p \ge 4$ is a constant. Suppose that

$$|k(x,y)| \le a(x,y),$$
$$|F(t,s,u,v)| \le f(t,s)|u|^4 + h(t,s)|u|^2 + f(t,s)|u|^2 v,$$
$$|H(t,s,u)| \le g(t,s)|u|^2, \tag{61}$$

where a, f, h, and g are defined as in Corollary 10. Combining (60)-(61) yields

$$|u(x,y)|^p \le a(x,y) + \int_0^{\alpha(x)}\int_0^{\beta(y)}\left(f(t,s)|u(t,s)|^4 + h(t,s)|u(t,s)|^2\right)ds\,dt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(t,s)|u(t,s)|^2\cdot\left(\int_0^t\int_0^s g(\tau,\omega)|u(\tau,\omega)|^2\,d\omega\,d\tau\right)ds\,dt. \tag{62}$$

Exploiting Corollary 10, we obtain an explicit bound to the solutions of (60):

$$|u(x,y)| \le \begin{cases}\left(\left[\frac{p-2}{p}\lambda(x,y)\right]^{(p-4)/(p-2)} + \frac{p-4}{p}\theta(x,y)\right)^{1/(p-4)}, & \text{when } p > 4,\\ \left(\frac{1}{2}\lambda(x,y)\right)^{1/2}\exp\left(\frac{1}{4}\theta(x,y)\right), & \text{when } p = 4,\end{cases} \tag{63}$$

where λ and θ are defined as in Corollary 10.

5. Conclusions

This paper investigates some new types of linear and nonlinear retarded integral inequalities in two independent variables. Several theorems and a corollary of these inequalities are obtained based on some analysis techniques, such as amplification method, differential, and integration. An illustrative example is studied to demonstrate the effectiveness of the new results.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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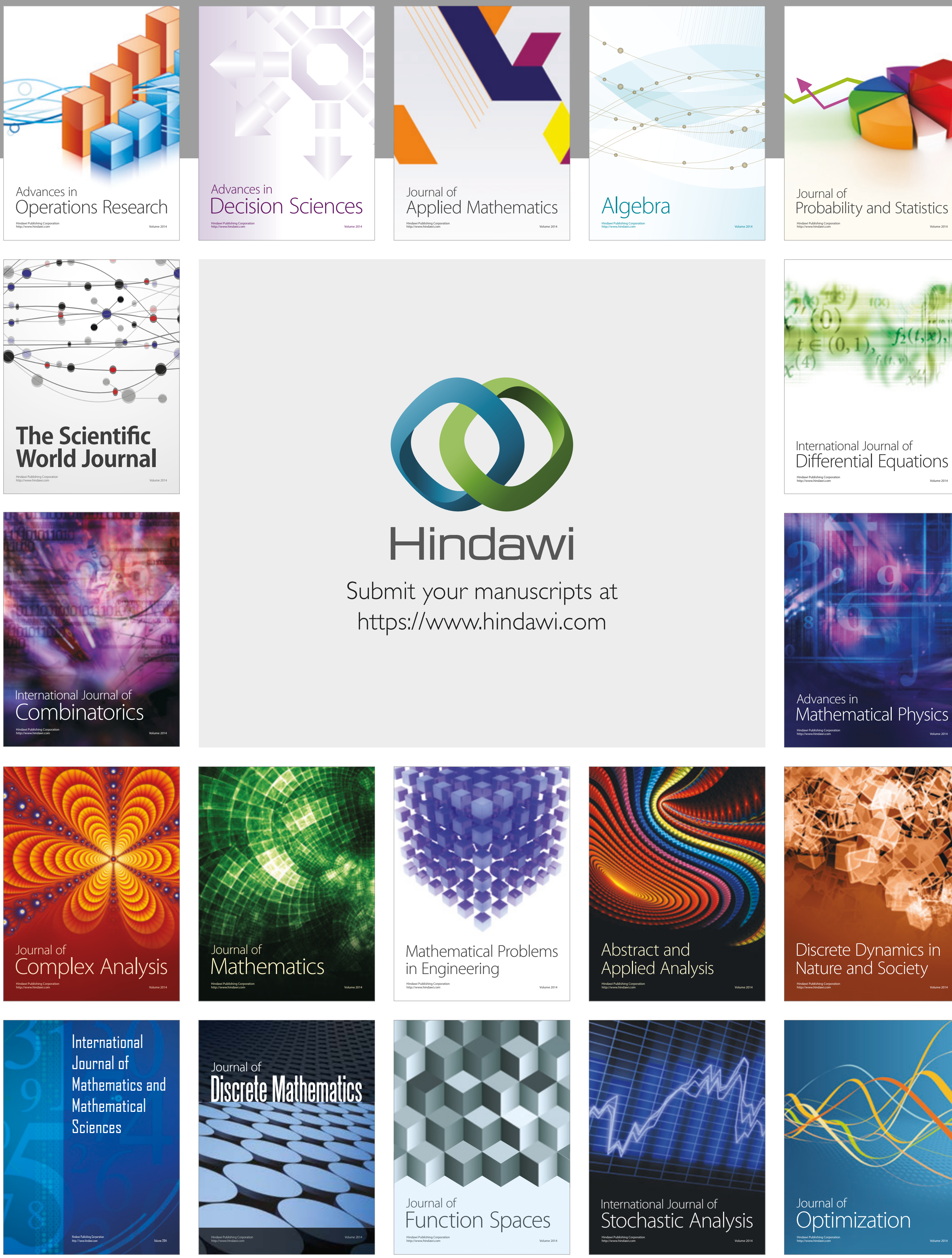
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