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Research Article

Existence Results for Impulsive Fractional Differential Inclusions with Two Different Caputo Fractional Derivatives

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In this paper, we study the impulsive fractional differential inclusions with two different Caputo fractional derivatives and nonlinear integral boundary value conditions. Under certain assumptions, new criteria to guarantee the impulsive fractional impulsive fractional differential inclusion has at least one solution are established by using Bohnenblust-Karlin's fixed point theorem. Also, some previous results will be significantly improved.

1. Introduction

In this paper, we consider the following fractional differential inclusions with impulsive effects:

$$\begin{aligned} &{}^{c}D_{0,t}^{\alpha}\left({}^{c}D_{0,t}^{\beta}u(t)\right)+\lambda u(t)\in F(t,u(t)), \\ &\qquad\qquad a.e.\ t\in J=[0,1],\ t\neq t_k, \\ &\Delta u(t_k)=u(t_k^+)-u(t_k^-)=I_k(u(t_k)), \\ &\qquad\qquad t=t_k,\ k=1,2,\ldots n, \\ &au(0)+bu(1)=\int_0^1 g(s,u)\,ds, \\ &\left[{}^{c}D_{0,t}^{\beta}u(t)\right]_{t=t_k}=c_k,\quad k=0,1,\ldots,n, \end{aligned}\qquad (1)$$

where $0<\alpha,\beta<1$, ${}^{c}D_{0,t}^{\alpha}$, and ${}^{c}D_{0,t}^{\beta}$ represent the different Caputo fractional derivatives of orders α and β, respectively. $F: J\times R\longrightarrow \mathscr{P}(R)$ is a multivalued map, $\mathscr{P}(R)$ is the family of all nonempty subsets of R, and $g: J\times R\longrightarrow R$ is a given continuous function. $0=t_0<t_1<\cdots<t_{n+1}=1$, $a>0$, $b\geq 0$, $0\leq c_k\leq c$, $k=0,1,\ldots,n$ are real constants and λ is a given positive parameter. $u(t_k^+)=\lim_{h\longrightarrow 0^+}u(t_k+h)$ and $u(t_k^-)=\lim_{h\longrightarrow 0^-}u(t_k+h)$ represent the right and left limits of $u(t)$ at $t=t_k$, $k=1,2,\ldots n$.

As an extension of integer-order differential equations, fractional-order differential equations have been of great interest since the equations involving fractional derivatives always have better effects in applications than the traditional differential equations of integer order. Due to these significant applications in various sciences, such as physics, engineering, chemistry, and biology, fractional differential equations have received much attention by researchers during the past two decades. Up to now, fractional boundary value problems are still heated research topics. That is why, more and more considerations by many people have been paid to study the existence of solutions for fractional boundary value problems; we refer readers to [1–12].

However, the articles of fractional boundary value problems with two different Caputo fractional derivatives are not many. More precisely, in [10], the authors have studied the following impulsive fractional Langevin equations with two different Caputo fractional derivatives:

$$\begin{aligned} &{}^{c}D_{t}^{\beta}\left({}^{c}D_{t}^{\alpha}+\lambda\right)x(t)=f(t,x(t)), \\ &\qquad t\in J'=J\setminus\{t_1,\ldots,t_m\},\ J:=[0,1], \\ &\Delta u(t_k):=u(t_k^+)-u(t_k^-)=I_k,\quad I_k\in R, \\ &x(0)=0, \end{aligned}$$

$$x(\eta_k) = 0,$$
$$x(1) = 0,$$
$$\eta_k \in (t_k, t_{k+1}), \ k = 1, 2, \ldots m-1, \quad (2)$$

where $f : J \times R \longrightarrow R$ is a given function, $0 < \alpha, \beta < 1$ and $0 < \alpha + \beta < 1$, $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$, $\lambda > 0$, $u(t_k^+) = \lim_{h \longrightarrow 0^+} u(t_k + h)$, and $u(t_k^-) = \lim_{h \longrightarrow 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$, $k = 1, 2, \ldots m$.

Then, in [11], the authors considered the following nonlinear Langevin inclusions with two different Caputo fractional derivatives:

$$\begin{aligned} &{}^c\mathrm{D}^p \left({}^c\mathrm{D}^q + \lambda\right) x(t) \in F(t, x(t)), \quad 0 < t < 1, \\ &x(0) = \sum_{i=1}^{n} \beta_i \left(I^{\mu_i} x\right)(\zeta), \\ &x(1) = \sum_{i=1}^{n} \alpha_i \left(I^{\nu_i} x\right)(\eta), \\ &0 < \zeta < \eta < 1, \end{aligned} \quad (3)$$

where $0 < p, q < 1$, λ is a real number, I^k is the Riemann-Liouville fractional integral of order $k > 0$ ($k = \nu_i, \mu_i$; $i = 1, 2, \ldots, n$), and α, β are constants.

In [12], the author investigates the following impulsive fractional differential equations with two different Caputo fractional derivatives with coefficients:

$$\begin{aligned} &{}^c\mathrm{D}_{0,t}^{\alpha}\left({}^c\mathrm{D}_{0,t}^{\beta} u(t)\right) + \lambda u(t) = f(t, u(t)), \\ &t \in J' = J \setminus \{t_1, \ldots, t_m\}, \\ &\Delta u(t_k) = u(t_k^+) - u(t_k^-) = y_k, \\ &k = 1, 2, \ldots m, \\ &au(0) + bu(1) = c, \\ &\left[{}^c\mathrm{D}_{0,t}^{\beta} u(t)\right]_{t=t_k} = d_k, \quad k = 0, 1, 2, \ldots, m, \end{aligned} \quad (4)$$

where $J = [0, 1]$, $f \in C(J \times R, R)$, $0 < \alpha, \beta < 1$, $y_k \in R$, $\lambda > 0$, $a > 0$, $b \geq 0$, $c \geq 0$, $d_k \geq 0$ are real constants.

To the best of our knowledge, integral boundary conditions appear in population dynamics and cellular systems; it has constituted a very interesting and important class of problems. However, fractional boundary value problems with integral boundary conditions have not received so much attention as periodic boundary conditions, so the main aim in this paper is intended as an attempt to establish some criteria of existence of solutions for (1). It is worth pointing out that there was no paper considering the impulsive fractional differential inclusions with two different Caputo fractional derivatives and nonlinear integral conditions by using Bohnenblust-Karlin's fixed point theorem up to now, so our results are new. Also, we improve some previous results.

The arrangement of the rest paper is as follows. In Section 2, some preliminaries and results which are applied in the later paper are presented. In Section 3, the main proof of theorems will be vividly shown. In Section 4, a corresponding example is given to illustrate the obtained results in Section 3.

2. Preliminaries

In this section, we recall some basic knowledge of definitions and lemmas that we shall use in the rest of the paper.

Let $C(J, R)$ denote a Banach space of continuous functions from J into R with the norm

$$\|u\| = \sup_{t \in J} \{|u(t)|\} \quad (5)$$

for $u \in C(J, R)$. Also, we denote the function space by

$$PC(J, R) = \{u : u \in C((t_k, t_{k+1}], R) \ \ u(t_k^+) = u(t_k) \ \ k = 1, \ldots, m\} \quad (6)$$

with the norm $\|u\|_{PC} = \sup_{t \in J} \{|u(t)|\}$. Clearly, $PC(J, R)$ is Banach spaces.

Let $L^1(J, R)$ be a Banach space of measurable functions $y : J \longrightarrow R$ which are Lebesgue integrable and normed by

$$\|y\|_{L^1} = \int_0^1 |y(t)| \, dt. \quad (7)$$

Let $(X, |\cdot|)$ be a Banach space. We give following notations for convenience: let

$$\begin{aligned} &\mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is closed}\}, \\ &\mathscr{P}_{b}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is bounded}\}, \\ &\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact}\}, \\ &\mathscr{P}_{cp,c}(X) \\ &\quad = \{Y \in \mathscr{P}(X) : Y \text{ is compact and convex}\}, \end{aligned} \quad (8)$$

and $BCC(X)$ denote the set of all nonempty bounded, closed, and convex subset of X.

A multivalued map $G : X \longrightarrow 2^X$

(i) is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$;

(ii) is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for any bounded set B of X(*i.e.* $\sup_{x \in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$);

(iii) is called upper semicontinuous (u.s.c.) on X if, for each $x_0 \in X$, the set $G(x_0)$ is nonempty closed subset of X, and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $G(\mathscr{N}_0) \subseteq N$;

(iv) is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset B of X;

(v) is completely continuous with nonempty compact values; then G is *u.s.c.* if and only if G has a closed graph; *i.e.*, $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

(vi) has a fixed point if there is $x \in X$ such that $x \in G(x)$.

Definition 1. A multivalued map $F : J \times R \longrightarrow \mathscr{P}(R)$ is *Carathéodory* if

(i) $t \longmapsto F(t,u)$ is measurable for each $u \in R$,

(ii) $u \longmapsto F(t,u)$ is upper semicontinuous for almost all $t \in J$.

Moreover, a *Carathéodory* function F is called L^1– *Carathéodory* if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,1], R^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\alpha(t) \quad (9)$$

for all $\|x\| \le \alpha$ for *a.e.* $t \in [0,1]$.

For each $y \in C(J,R)$, define that the set of selections for F by

$$S_{F,y} = \left\{v \in L^1(J,R) : v(t) \in F(t, y(t)) \ \textit{a.e. } t \in J\right\} \quad (10)$$

is nonempty.

Lemma 2 (see [13]). *Let X be a Banach space. Let $F : J\times R \longrightarrow \mathscr{P}_{cp,c}(X)$ be an L^1– Carathéodory multivalued map, and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \longrightarrow \mathscr{P}_{cp,c}(X)(C(J,X)) \quad (11)$$

and

$$x \longmapsto (\Theta \circ S_F)(x) = \Theta\left(S_{F,x,y}\right) \quad (12)$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

For more details, please refer to [13–15].

Definition 3. A function $u(t) \in PC(J,R)$ is called a solution of (1) if there exists a function $f \in L^1(J,R)$ with $f(t) \in F(t,u(t))$, *a.e.* $t \in J$ such that ${}^c\mathrm{D}^\alpha_{0,t}({}^c\mathrm{D}^\beta_{0,t}u(t)) + \lambda u(t) = f(t,u(t))$, *a.e.* $t \in J$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k))$, $t = t_k$, $k = 1,2,\ldots n$, and $au(0) + bu(1) = \int_0^1 g(s,u(s))ds$, $[{}^c\mathrm{D}^\beta_{0,t}u(t)]_{t=t_k} = c_k$, $k = 0,1,\ldots,n$.

Next, we present the following necessary basic knowledge of fractional calculus theory which is used in the later paper.

Definition 4 (see [4]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : [0,+\infty) \longrightarrow R$ is given by

$$I^\alpha_t f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\,ds, \quad a > 0, \quad (13)$$

provided that the right-hand side is pointwise defined on $[0,+\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 5 (see [4]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : [0,+\infty) \longrightarrow R$ is given by

$$ {}^l\mathrm{D}^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1} f(s)\,ds, \quad t > 0, \quad (14)$$

where $n-1 < \alpha \le n$, provided that the right-hand side is pointwise defined on $[0,+\infty)$.

Definition 6 (see [4]). The Caputo fractional derivative of order $\alpha > 0$ of a function $f : [0,+\infty) \longrightarrow R$ is given by

$$ {}^c\mathrm{D}^\alpha_t f(t) = {}^l\mathrm{D}^\alpha_t\left[f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right], \quad t > 0, \quad (15)$$

where $n-1 < \alpha \le n$, provided that the right-hand side is pointwise defined on $[0,+\infty)$.

Definition 7 (see [10]). Functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are called classical and generalized Mittag-Leffler functions, respectively, given by

$$\begin{aligned} E_\alpha(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \\ E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}. \end{aligned} \quad (16)$$

Lemma 8 (see [10]). *Let $0 < \alpha,\beta < 1$, and then functions $E_\alpha(z), E_{\alpha,\alpha}(z)$, and $E_{\alpha,\alpha+\beta}$ are nonnegative and have the following properties.*

(i) For any $\lambda > 0$ and $t \in J$,

$$\begin{aligned} E_\alpha(-t^\alpha\lambda) &\le 1, \\ E_{\alpha,\alpha}(-t^\alpha\lambda) &\le \frac{1}{\Gamma(\alpha)}, \\ E_{\alpha,\alpha+\beta}(-t^\alpha\lambda) &\le \frac{1}{\Gamma(\alpha+\beta)}. \end{aligned} \quad (17)$$

Moreover,

$$\begin{aligned} E_\alpha(0) &= 1, \\ E_{\alpha,\alpha}(0) &= \frac{1}{\Gamma(\alpha)}, \\ E_{\alpha,\alpha+\beta}(0) &= \frac{1}{\Gamma(\alpha+\beta)}. \end{aligned} \quad (18)$$

(ii) For any $\lambda > 0$ and $t_1, t_2 \in J$, when $t_2 \longrightarrow t_1$, we have

$$\begin{aligned} E_\alpha(-t_2^\alpha\lambda) &\longrightarrow E_\alpha(-t_1^\alpha\lambda), \\ E_{\alpha,\alpha}(-t_2^\alpha\lambda) &\longrightarrow E_{\alpha,\alpha}(-t_1^\alpha\lambda), \\ E_{\alpha,\alpha+\beta}(-t_2^\alpha\lambda) &\longrightarrow E_{\alpha,\alpha+\beta}(-t_1^\alpha\lambda). \end{aligned} \quad (19)$$

(iii) For any $\lambda > 0$ and $t_1, t_2 \in J$ and $t_1 \le t_2$, we have

$$\begin{aligned} E_\alpha(-t_1^\alpha\lambda) &\ge E_\alpha(-t_2^\alpha\lambda), \\ E_{\alpha,\alpha}(-t_1^\alpha\lambda) &\ge E_{\alpha,\alpha}(-t_2^\alpha\lambda), \\ E_{\alpha,\alpha+\beta}(-t_1^\alpha\lambda) &\ge E_{\alpha,\alpha+\beta}(-t_2^\alpha\lambda). \end{aligned} \quad (20)$$

Lemma 9 (see [16]). *Let $a + bE_{\alpha+\beta}(-\lambda) \neq 0$. For a given $f \in L^1(J,R)$ with $f(t) \in F(t,u(t))$, a.e. $t \in J$, then the boundary value problem (1) has a unique solution $u(t) \in PC(J,R)$ which is defined by the following form:*

$$\begin{aligned} u(t) &= \frac{E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)}{a + bE_{\alpha+\beta}(-\lambda)} \left[a \sum_{i=1}^{n} \frac{I_i - t_i^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i - c_{i-1})}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)} \right. \\ &\quad - b \int_0^1 (1-s)^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}\left(-\lambda (1-s)^{\alpha+\beta}\right) f(s)\, ds \\ &\quad \left. - bc_n E_{\alpha+\beta,\beta+1}(-\lambda) + \int_0^1 g(s,u(s))\, ds \right] - E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right) \\ &\quad \times \sum_{j=k+1}^{n} \frac{I_j - t_j^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)(c_j - c_{j-1})}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)} \\ &\quad + \int_0^t (t-s)^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}\left(-\lambda (t-s)^{\alpha+\beta}\right) f(s)\, ds \\ &\quad + c_k t^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right) \quad \forall t \in [t_k, t_{k+1}),\ k = 0, 1, \ldots, n. \end{aligned} \tag{21}$$

Finally, we give the following lemma which is greatly important in the proof of our main results.

Lemma 10 (see [17, Bohnenblust-Karlin]). *Let X be a Banach space, D a nonempty subset of X, which is bounded, closed, and convex. Suppose $G : D \longrightarrow 2^X \setminus \{0\}$ is u.s.c. with closed, convex values, and such that $G(D) \subset D$ and $\overline{G(D)}$ are compact. Then G has a fixed point.*

3. Main Results

In order to begin our main results, we also need the following conditions:

(H1) There exists $0 < q < \alpha + \beta < 1$, and a real function $m_r(t) \in L^{1/q}(J, R_+)$ such that

$$\|F(t,u)\| = \sup\{|f| : f(t) \in F(t,u)\} \leq m_r(t), \quad \forall \|u\| \leq r \ \text{for a.e. } t \in J, \tag{22}$$

for each $r > 0$.

(H2) $g(t,0) = 0$ and there exists $L > 0$ such that

$$|g(t,u) - g(t,v)| \leq L|u - v| \tag{23}$$

for $u, v \in R$ and $t \in [0,1]$, where L satisfies $L < a$ in which a is defined in (1).

For convenience, we denote

$$\Omega = \sum_{i=1}^{n} \frac{|I_i| + \left|t_i^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i - c_{i-1})\right|}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)}. \tag{24}$$

Theorem 11. *Suppose that (H1) and (H2) hold; then system (1) has at least one solution on J.*

Proof. We transform problem (1) into a fixed point problem. Consider the operator $N : C(J,R) \longrightarrow PC(J,R)$ defined by

$$\begin{aligned} N(u) = \Bigg\{ & h(t) \in PC(J,R) : h(t) \\ &= \frac{E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)}{a + bE_{\alpha+\beta}(-\lambda)} \left[a \sum_{i=1}^{n} \frac{I_i - t_i^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i - c_{i-1})}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)} \right. \\ &\quad - b \int_0^1 (1-s)^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}\left(-\lambda (1-s)^{\alpha+\beta}\right) f(s)\, ds \\ &\quad \left. - bc_n E_{\alpha+\beta,\beta+1}(-\lambda) + \int_0^1 g(s,u(s))\, ds \right] - E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right) \\ &\quad \times \sum_{j=k+1}^{n} \frac{I_j - t_j^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)(c_j - c_{j-1})}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)} \\ &\quad + \int_0^t (t-s)^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}\left(-\lambda (t-s)^{\alpha+\beta}\right) f(s)\, ds \\ &\quad + c_k t^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right) \ \forall t \in [t_k, t_{k+1}),\ k = 0, 1, \ldots, n. \Bigg\} \end{aligned} \tag{25}$$

for $f \in S_{F,u}$.

Next we shall show that N satisfies all the assumptions of Lemma 10; that is to say, N has a fixed point which is a solution of problem (1). For the sake of convenience, we subdivide the proof into several steps.

Step 1 ($N(u)$ is convex for each $u \in PC(J,R)$). In fact, assume $h_1, h_2 \in N(u)$, then there exist $f_1, f_2 \in S_{F,u}$ such that, for each $t \in J$, we have

$$\begin{aligned} h_i(t) &= \frac{E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)}{a + bE_{\alpha+\beta}(-\lambda)} \left[a \sum_{i=1}^{n} \frac{I_i - t_i^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i - c_{i-1})}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)} \right. \\ &\quad - b \int_0^1 (1-s)^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}\left(-\lambda (1-s)^{\alpha+\beta}\right) f_i(s)\, ds \\ &\quad \left. - bc_n E_{\alpha+\beta,\beta+1}(-\lambda) + \int_0^1 g(s,u(s))\, ds \right] - E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right) \\ &\quad \times \sum_{j=k+1}^{n} \frac{I_j - t_j^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)(c_j - c_{j-1})}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)} \\ &\quad + \int_0^t (t-s)^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}\left(-\lambda (t-s)^{\alpha+\beta}\right) f_i(s)\, ds \\ &\quad + c_k t^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right), \quad i = 1, 2. \end{aligned} \tag{26}$$

Let $0 \leq \chi \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned} [\chi h_1 + (1-\chi) h_2](t) &= \frac{E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)}{a + bE_{\alpha+\beta}(-\lambda)} \left[a \sum_{i=1}^{n} \frac{I_i - t_i^{\beta} E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i - c_{i-1})}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)} \right. \\ &\quad - b \int_0^1 (1-s)^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}\left(-\lambda (1-s)^{\alpha+\beta}\right) \end{aligned}$$

$$\cdot\left[\chi f_1(s)+(1-\chi)f_2(s)\right]ds-bc_nE_{\alpha+\beta,\beta+1}(-\lambda)+\int_0^1 g(s,u(s))\,ds\Bigg]-E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)\times\sum_{j=k+1}^{n}\frac{I_j-t_j^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)(c_j-c_{j-1})}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)}+\int_0^t(t-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(t-s)^{\alpha+\beta}\right)\left[\chi f_1(s)+(1-\chi)\cdot f_2(s)\right]ds+c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right). \tag{27}$$

Since $S_{F,u}$ is convex (F has convex values), so it follows that $\chi h_1+(1-\chi)h_2\in N(u)$.

Step 2. Let $B_r=\{u\in PC(J,R):\|u\|\le r\}$, where

$$\frac{a}{a-L}\left(\frac{(a+b)\Omega}{a}+\frac{bc}{a\Gamma(1+\beta)}+\frac{c}{\Gamma(1+\beta)}+\frac{(a+b)\|m_r\|_{L^{1/q}}}{a\Gamma(\alpha+\beta)}\left(\frac{1-q}{\alpha+\beta-q}\right)^{1-q}\right)\le r. \tag{28}$$

Then B_r is a bounded closed convex set in $PC(J,R)$. Thus we need to verify $N(B_r)\subseteq B_r$. In fact, from Lemma 8, (H1), and (H2), for each $u\in B_r$, $t\in J_k$, $k=0,1,\ldots,n$, we have

$$\begin{aligned}
|N(u)|&\le\left|E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)\right|\cdot\Bigg|\frac{1}{a+bE_{\alpha+\beta}(-\lambda)}\Bigg[a\sum_{i=1}^{n}\frac{I_i-t_i^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i-c_{i-1})}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)}\\
&\quad-b\int_0^1(1-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(1-s)^{\alpha+\beta}\right)f(s)\,ds-bc_nE_{\alpha+\beta,\beta+1}(-\lambda)+\int_0^1 g(s,u(s))\,ds\Bigg]\\
&\quad-\sum_{j=k+1}^{n}\frac{I_j-t_j^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)(c_j-c_{j-1})}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)}\Bigg|+\Bigg|\int_0^t(t-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(t-s)^{\alpha+\beta}\right)f(s)\,ds\\
&\quad+c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right)\Bigg|\le\Bigg|\frac{1}{a+bE_{\alpha+\beta}(-\lambda)}\cdot\Bigg[a\sum_{i=1}^{n}\frac{I_i-t_i^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i-c_{i-1})}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)}\\
&\quad-b\int_0^1(1-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(1-s)^{\alpha+\beta}\right)f(s)\,ds-bc_nE_{\alpha+\beta,\beta+1}(-\lambda)+\int_0^1\left[g(s,u(s))-g(s,0)\right]ds\\
&\quad-\left(a+bE_{\alpha+\beta}(-\lambda)\right)\cdot\sum_{j=k+1}^{n}\frac{I_j-t_j^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)(c_j-c_{j-1})}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)}\Bigg]\Bigg|\\
&\quad+\Bigg|\int_0^t(t-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(t-s)^{\alpha+\beta}\right)f(s)\,ds+c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right)\Bigg|\\
&\le\Bigg|\frac{1}{a}\Bigg[a\sum_{i=1}^{k}\frac{I_i-t_i^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i-c_{i-1})}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)}-b\int_0^1(1-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(1-s)^{\alpha+\beta}\right)m_r(s)\,ds\\
&\quad-bc_nE_{\alpha+\beta,\beta+1}(-\lambda)+L\int_0^1u(s)\,ds-bE_{\alpha+\beta}(-\lambda)\cdot\sum_{j=k+1}^{n}\frac{I_j-t_j^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)(c_j-c_{j-1})}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)}\Bigg]\Bigg|\\
&\quad+\Bigg|\int_0^t(t-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(t-s)^{\alpha+\beta}\right)m_r(s)\,ds+c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right)\Bigg|\\
&\le\sum_{i=1}^{k}\frac{|I_i|\left|t_i^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i-c_{i-1})\right|}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)}+\frac{bc}{a\Gamma(1+\beta)}+\frac{b}{a\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\left|m_r(s)\right|ds+\frac{L}{a}\int_0^1|u(s)|\,ds\\
&\quad+\frac{b}{a}\sum_{j=k+1}^{n}\frac{|I_j|\left|t_j^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)(c_j-c_{j-1})\right|}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)}+\frac{1}{\Gamma(\alpha+\beta)}\cdot\int_0^t(t-s)^{\alpha+\beta-1}\left|m_r(s)\right|ds+\frac{c}{\Gamma(1+\beta)}=\frac{(a+b)\Omega}{a}\\
&\quad+\frac{bc}{a\Gamma(1+\beta)}+\frac{c}{\Gamma(1+\beta)}+\frac{L}{a}\int_0^1|u(s)|\,ds+\frac{b}{a\Gamma(\alpha+\beta)}\cdot\int_0^1(1-s)^{\alpha+\beta-1}\left|m_r(s)\right|ds+\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\\
&\quad\cdot\left|m_r(s)\right|ds\le\frac{(a+b)\Omega}{a}+\frac{bc}{a\Gamma(1+\beta)}+\frac{c}{\Gamma(1+\beta)}+\frac{L\|u\|_{L^1}}{a}+\frac{b}{a\Gamma(\alpha+\beta)}\left(\int_0^1\left[(1-s)^{\alpha+\beta-1}\right]^{1/(1-q)}ds\right)^{1-q}
\end{aligned}$$

$$\cdot\left(\int_0^1 \left|m_r(s)\right|^{1/q} ds\right)^q + \frac{1}{\Gamma(\alpha+\beta)}\left(\int_0^t \left[(t-s)^{\alpha+\beta-1}\right]^{1/(1-q)} ds\right)^{1-q} \cdot\left(\int_0^t \left|m_r(s)\right|^{1/q} ds\right)^q \le \frac{(a+b)\Omega}{a} + \frac{bc}{a\Gamma(1+\beta)} + \frac{c}{\Gamma(1+\beta)} + \frac{Lr}{a} + \frac{b}{a\Gamma(\alpha+\beta)}\left(\frac{1-q}{\alpha+\beta-q}\right)^{1-q} + \frac{1}{\Gamma(\alpha+\beta)}\left(\frac{1-q}{\alpha+\beta-q}\right)^{1-q}\cdot t^{\alpha+\beta-q} \le \frac{(a+b)\Omega}{a} + \frac{bc}{a\Gamma(1+\beta)} + \frac{c}{\Gamma(1+\beta)} + \frac{Lr}{a} + \frac{(a+b)\left\|m_r\right\|_{L^{1/q}}}{a\Gamma(\alpha+\beta)}\left(\frac{1-q}{\alpha+\beta-q}\right)^{1-q}. \tag{29}$$

From (28), we have $N(B_r) \subseteq B_r$.

Step 3 ($N(B_r)$ is equicontinuous). Let $\Delta = J \times B_r, \delta_1, \delta_2 \in J, \delta_1 < \delta_2$. For convenience, we also let $\sup_{(t,u)\in\Delta}|f(t,u)| := \Upsilon$,

$$\Phi = \frac{1}{a+bE_{\alpha+\beta}}\left[a\sum_{i=1}^{n}\frac{I_i - t_i^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)(c_i - c_{i-1})}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)} - bc_nE_{\alpha+\beta,\beta+1}(-\lambda) + \int_0^1 g(s,u(s))\,ds\right] \tag{30}$$

and

$$\Psi = \sum_{j=k+1}^{n}\frac{I_j - t_j^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)\left(c_j - c_{j-1}\right)}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)}, \tag{31}$$

and then we have

$$\begin{aligned}
&\left|(Nu)(\delta_1) - (Nu)(\delta_2)\right| = \Bigg|\left[E_{\alpha+\beta}\left(-\lambda\delta_1^{\alpha+\beta}\right) - E_{\alpha+\beta}\left(-\lambda\delta_2^{\alpha+\beta}\right)\right]\Phi - \left[E_{\alpha+\beta}\left(-\lambda\delta_1^{\alpha+\beta}\right) - E_{\alpha+\beta}\left(-\lambda\delta_2^{\alpha+\beta}\right)\right]\Psi + \int_0^{\delta_1}(\delta_1 - s)^{\alpha+\beta-1} \\
&\cdot E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_1 - s)^{\alpha+\beta}\right)f(s)\,ds - \int_0^{\delta_2}(\delta_2 - s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_2 - s)^{\alpha+\beta}\right)f(s)\,ds \\
&+ c_k\delta_1^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_1^{\alpha+\beta}\right) - c_k\delta_2^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_2^{\alpha+\beta}\right)\Bigg| \le \Bigg|\left[E_{\alpha+\beta}\left(-\lambda\delta_1^{\alpha+\beta}\right) - E_{\alpha+\beta}\left(-\lambda\delta_2^{\alpha+\beta}\right)\right](\Phi - \Psi) \\
&+ \int_0^{\delta_1}\left[(\delta_1 - s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_1 - s)^{\alpha+\beta}\right) - (\delta_2 - s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_2 - s)^{\alpha+\beta}\right)\right] \\
&\cdot f(s)\,ds - \int_{\delta_1}^{\delta_2}(\delta_2 - s)^{\alpha+\beta-1}\cdot E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_2 - s)^{\alpha+\beta}\right)f(s)\,ds + c_k\delta_1^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_1^{\alpha+\beta}\right) \\
&- c_k\delta_2^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_2^{\alpha+\beta}\right)\Bigg| \le \left|\left[E_{\alpha+\beta}\left(-\lambda\delta_1^{\alpha+\beta}\right) - E_{\alpha+\beta}\left(-\lambda\delta_2^{\alpha+\beta}\right)\right](\Phi - \Psi)\right| + \left|\int_0^{\delta_1}\left[(\delta_1 - s)^{\alpha+\beta-1} - (\delta_2 - s)^{\alpha+\beta-1}\right]E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_1 - s)^{\alpha+\beta}\right)\cdot f(s)\,ds\right| \\
&+ \left|\int_0^{\delta_1}\left[(\delta_2 - s)^{\alpha+\beta-1}\right]\cdot\left[E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_1 - s)^{\alpha+\beta}\right) - E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_2 - s)^{\alpha+\beta}\right)\right]f(s)\,ds\right| + \left|\int_{\delta_1}^{\delta_2}(\delta_2 - s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_2 - s)^{\alpha+\beta}\right)f(s)\,ds\right| \\
&+ \left|c_k\left(\delta_1^{\beta} - \delta_2^{\beta}\right)E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_1^{\alpha+\beta}\right)\right| + \left|c_k\delta_2^{\beta}\left(E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_1^{\alpha+\beta}\right) - E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_2^{\alpha+\beta}\right)\right)\right|, \\
&\le \left|\left[E_{\alpha+\beta}\left(-\lambda\delta_1^{\alpha+\beta}\right) - E_{\alpha+\beta}\left(-\lambda\delta_2^{\alpha+\beta}\right)\right](\Phi - \Psi)\right| + \frac{\Upsilon}{\Gamma(\alpha+\beta)}\left|\int_0^{\delta_1}\left[(\delta_1 - s)^{\alpha+\beta-1} - (\delta_2 - s)^{\alpha+\beta-1}\right]ds\right| + \left|\int_0^{\delta_1}\left[(\delta_2 - s)^{\alpha+\beta-1}\right]\right. \\
&\left.\cdot\left[E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_1 - s)^{\alpha+\beta}\right) - E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(\delta_2 - s)^{\alpha+\beta}\right)\right]f(s)\,ds\right| + \frac{\Upsilon}{\Gamma(\alpha+\beta)}\left|\int_{\delta_1}^{\delta_2}(\delta_2 - s)^{\alpha+\beta-1}ds\right| + \left|c_k\left(\delta_1^{\beta} - \delta_2^{\beta}\right)\right. \\
&\left.\cdot E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_1^{\alpha+\beta}\right)\right| + \left|c_k\delta_2^{\beta}\left(E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_1^{\alpha+\beta}\right) - E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_2^{\alpha+\beta}\right)\right)\right| \le \left|\left[E_{\alpha+\beta}\left(-\lambda\delta_1^{\alpha+\beta}\right)\right.\right.
\end{aligned}$$

$$- E_{\alpha+\beta}\left(-\lambda\delta_2^{\alpha+\beta}\right)\Big](\Phi-\Psi)\Big| + \frac{\Upsilon\left[\left(\delta_2-\delta_1\right)^{\alpha+\beta}+\delta_2^{\alpha+\beta}-\delta_1^{\alpha+\beta}\right]}{(\alpha+\beta)\Gamma(\alpha+\beta)} + \frac{\Upsilon\left(\delta_2-\delta_1\right)^{\alpha+\beta}}{(\alpha+\beta)\Gamma(\alpha+\beta)} + \left|\int_0^{\delta_1}\left[\left(\delta_2-s\right)^{\alpha+\beta-1}\right]\right. \cdot\left[E_{\alpha+\beta,\alpha+\beta}\left(-\lambda\left(\delta_1-s\right)^{\alpha+\beta}\right) - E_{\alpha+\beta,\alpha+\beta}\left(-\lambda\left(\delta_2-s\right)^{\alpha+\beta}\right)\right]f(s)\,ds\Big| + \Big|c_k\left(\delta_1^{\beta} - \delta_2^{\beta}\right)E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_1^{\alpha+\beta}\right)\Big| + \left|c_k\delta_2^{\beta}\left(E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_1^{\alpha+\beta}\right)-E_{\alpha+\beta,\beta+1}\left(-\lambda\delta_2^{\alpha+\beta}\right)\right)\right|, \tag{32}$$

and from Lemma 8, we clearly see the right hand side of the above inequality tends to zero as $\delta_1 \longrightarrow \delta_2$. This implies that N is equicontinuous on J. As a consequence of Steps 1–3 together with the Ascoli-Arzela theorem, we can conclude that N is a compact valued map.

Step 4 (N has a closed graph). Let $u_n \longrightarrow u_*, h_n \in N(u_n)$ and $h_n \longrightarrow h_*$. Then we need to verify $h_* \in N(u_*)$. $h_n \in N(u_n)$ implies that there exists $f_n \in S_{F,u_n}$ such that for each $t \in J$ we have

$$h_n(t) = \frac{E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)}{a+bE_{\alpha+\beta}(-\lambda)}\left[a\sum_{i=1}^{n}\frac{I_i-t_i^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)\left(c_i-c_{i-1}\right)}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)} - b\int_0^1(1-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(1-s)^{\alpha+\beta}\right)f_n(s)\,ds - bc_nE_{\alpha+\beta,\beta+1}(-\lambda)+\int_0^1 g(s,u(s))\,ds\right] - E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right) \times\sum_{j=k+1}^{n}\frac{I_j-t_j^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)\left(c_j-c_{j-1}\right)}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)} + \int_0^t(t-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(t-s)^{\alpha+\beta}\right)f_n(s)\,ds + c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right)\Big\}, \tag{33}$$

and thus we must verify that there exists $f_* \in S_{F,u_*}$ such that for each $t \in J$ we have

$$h_*(t) = \frac{E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)}{a+bE_{\alpha+\beta}(-\lambda)}\left[a\sum_{i=1}^{n}\frac{I_i-t_i^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_i^{\alpha+\beta}\right)\left(c_i-c_{i-1}\right)}{E_{\alpha+\beta}\left(-\lambda t_i^{\alpha+\beta}\right)} - b\int_0^1(1-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(1-s)^{\alpha+\beta}\right)f_*(s)\,ds - bc_nE_{\alpha+\beta,\beta+1}(-\lambda)+\int_0^1 g(s,u(s))\,ds\right] - E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right) \times\sum_{j=k+1}^{n}\frac{I_j-t_j^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t_j^{\alpha+\beta}\right)\left(c_j-c_{j-1}\right)}{E_{\alpha+\beta}\left(-\lambda t_j^{\alpha+\beta}\right)} + \int_0^t(t-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(t-s)^{\alpha+\beta}\right)f_*(s)\,ds + c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right)\Big\}. \tag{34}$$

Consider the continuous linear operator

$$\Theta : L^1(J,R) \longrightarrow C(J,R), \tag{35}$$

$$f \longmapsto \Theta(f)(t) = \int_0^t (t-s)^{\alpha+\beta-1} \cdot E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(t-s)^{\alpha+\beta}\right)f(s)\,ds - \frac{bE_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)}{a+bE_{\alpha+\beta}(-\lambda)}\int_0^1(1-s)^{\alpha+\beta-1} \cdot E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(1-s)^{\alpha+\beta}\right)f(s)\,ds; \tag{36}$$

then,

$$\Big\|\Big\{h_n(t)-\left[E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)(\Phi-\Psi)\right] - c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right)\Big\} - \Big\{h_*(t) - \left[E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)(\Phi-\Psi)\right] - c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right)\Big\}\Big\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{37}$$

By Lemma 2, we know $\Theta \circ S_F$ is a closed graph operator.

Also from the definition of Θ we have

$$h_n(t)-\left[E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)(\Phi-\Psi)\right] - c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right) \in \Theta\left(S_{F,u_n}\right). \tag{38}$$

Since $u_n \longrightarrow u_*$, Lemma 2 implies that

$$h_*(t)-\left[E_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)(\Phi-\Psi)\right] - c_kt^{\beta}E_{\alpha+\beta,\beta+1}\left(-\lambda t^{\alpha+\beta}\right) = \int_0^t(t-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(t-s)^{\alpha+\beta}\right)f_*(s)\,ds - \frac{bE_{\alpha+\beta}\left(-\lambda t^{\alpha+\beta}\right)}{a+bE_{\alpha+\beta}(-\lambda)} \cdot\int_0^1(1-s)^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(-\lambda(1-s)^{\alpha+\beta}\right)f_*(s)\,ds \tag{39}$$

for some $f_* \in S_{F,u_*}$.

Therefore, N is a compact multivalued map, *u.s.c.* with convex closed values. By Lemma 10, we have that N has a fixed point $u(t)$ which is a solution of problem (1). □

Corollary 12. *Assume that (H2) and (H3) hold.*

(H3) There exist continuous and bounded functions $\tau_1(t), \tau_2(t) \in L^1(J, R_+), \sigma \in [0,1]$ *such that*

$$|F(t,u)| \leq \tau_1(t) + \tau_2(t)|u|^{\sigma}; \quad (40)$$

then problem (1) has at least a solution on J.

Proof. The proof is the same as Theorem 11 which we can take as $m(t) = \tau_1(t) + \tau_2(t)|u|^{\sigma}$. □

Remark 13. If we let $f(t,u) \in \{F(t,u)\}$ and $g(t,u)$ be a constant function, then the above Corollary 12 improves Theorem 3.1 in [12].

Remark 14. Note that if $\gamma = 0$ and $\gamma = 1$, we have ${}^c{}_0D_t^{\gamma}u(t) = u(t)$ and ${}^c{}_0D_t^{\gamma}u(t) = u'(t)$, respectively. Thus, in this paper, let $\alpha = 1$, $\beta = 0$, $\lambda = 0$, $c_k = 0$; our system (1) reduces to [18], so our problem (1) gives generalization of [18].

Remark 15. If $\beta = 0$, $\lambda = 0$, the boundary value condition becomes $u(0) = u_0$, and our system (1) reduces to [16, 19]. If $\alpha = 1$, $\beta = 0$, the boundary value condition becomes $u(0) - u(T) = \mu$, and our system (1) reduces to [20]. Thus, our problem (1) gives generalizations of [16, 19, 20].

4. An Example

In this part, we will give corresponding example to illustrate the main results in our paper.

Example 1. Consider the following system:

$$\begin{aligned}
&{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}u(t)\right) + \lambda u(t) \in F(t,u(t)), \\
&\qquad\qquad a.e.\ t \in J = [0,1] \setminus \left\{\frac{1}{5}\right\} \\
&\Delta u\left(\frac{1}{5}\right) = I_1\left(u\left(\frac{1}{5}\right)\right) \\
&au(0) + bu(1) = \int_0^1 g(s,u)\,ds, \\
&\left[{}^cD_{0,t}^{\beta}u(t)\right]_{t=0} = c_0, \\
&\left[{}^cD_{0,t}^{\beta}u(t)\right]_{t=1/5} = c_1,
\end{aligned} \quad (41)$$

where $0 < \alpha + \beta < 1, \lambda > 0, a = 4, b = 1$, and let $F(t,u(t)) = [(\sin t/e^t)(\cos u(t) + 1), ((\sin t)/e^t)(\cos u(t) + 3)]$ and $g(t,u(t)) = (\cos t/e^t)u(t)/(1 + u(t)), (t,u) \in [0,1] \times [0,+\infty)$. Then we let $m_r(t) = 4\sin t/e^t$ and $L = 1$, and we have

$$|g(t,u) - g(t,v)| \leq |u - v|; \quad (42)$$

then (H1) and (H2) of Theorem 11 all hold. Hence, system (41) has at least one solution on J.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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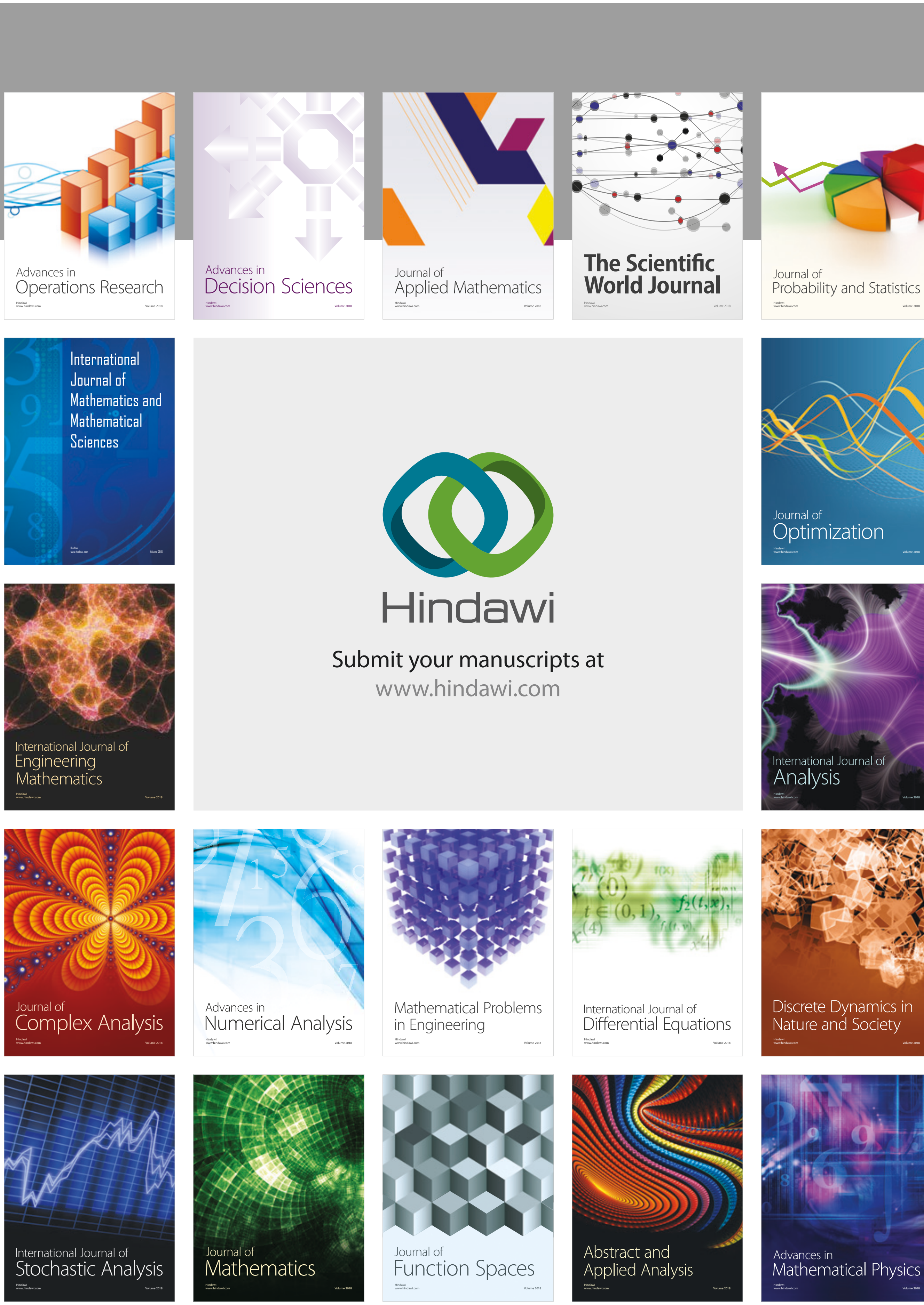
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