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*Research Article*

# Characterization of Self-Adjoint Domains for Two-Interval Even Order Singular $C$-Symmetric Differential Operators in Direct Sum Spaces

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This paper is concerned with the characterization of all self-adjoint domains associated with two-interval even order singular $C$-symmetric differential operators in terms of boundary conditions. The previously known characterizations of Lagrange symmetric differential operators are a special case of this one.

## 1. Introduction

Self-adjoint differential operators [1–3] in Hilbert space are of interest in mathematics and physics; in Quantum Mechanics they represent observables [4–7]. These operators are generally defined by symmetric expressions and boundary conditions. Two-interval theory of differential equations was developed by W. N. Everitt and A. Zettl [8] in 1986. In 1988, A. M. Krall and A. Zettl [9, 10] generalized the method given by Coddington [11] and obtained the characterizations of self-adjoint domains for Sturm-Liouville differential operators with interior singular points. Afterwards, in [12] the two-interval theory was extended to higher order equations and any finite or infinite number of intervals. In [13] Wang et al. give an explicit characterization of all self-adjoint domains for Lagrange symmetric differential operators in terms of certain solutions for *real* $\lambda$ for the one-interval case when one endpoint is regular and the other is singular. In analogy with the celebrated Weyl limit-point, limit-circle theory in the second order case, i.e., Sturm-Liouville problems [14], they construct limit-point and limit-circle solutions and characterize the self-adjoint domains in terms of the limit-circle solutions. In [15], Hao et al. give a characterization for Lagrange symmetric differential operators by dividing one interval $(a_1, b_1)$ into two intervals $(a_1, c_1)$ and $(c_1, b_1)$ for some point $c_1 \in (a_1, b_1)$ when both endpoints $a_1$ and $b_1$ are singular. In [16], Suo et al. extend the characterization in [13] to two-interval case for one endpoint of each interval $(a_1, b_1)$, and $(a_2, b_2)$ is regular, and illustrate the interactions between the regular endpoints and singular endpoints with some examples.

As noted in survey article [17], we observe that a special type of matrix, $E_n = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n$, plays key role in the characterization of a self-adjoint differential operators, both boundary conditions and symmetric differential operators. What is more interesting is that the symbol difference of this special type matrix is equivalent to skew-diagonal matrix $\begin{pmatrix} 0 & \tilde{I} \\ -\tilde{I} & o \end{pmatrix}$, $\tilde{I} = (\delta_{r,n+1-s})_{r,s=1}^k$, which also generates self-adjoint operators. Actually these matrices can be generalized as a fixed nonsingular matrix $C$ and preserve their properties. So we can enlarge the known set of these operarors by extending the known symmetric expressions to C-symmetric expressions and charaterize the boundary conditions which determine self-adjoint extensions of these C-symmetric expressions on a single interval case. Remarkably, the same matrices C which generate the expressions also generate their self-adjoint extensions. This paper is based on all the above known works, and the complete characterization of self-adjoint domains of the two-interval case for even order $C$-symmetric differential operators is given when four endpoints $a_1, b_1, a_2, b_2$ are singular or regular. Moreover, it has shown that the previous results in [16, 17] are special cases of ours. Following this introduction, some basic notations and facts are given in Section 2, in Sections 3 and 4 we give our main theorems for characterization of all self-adjoint domains and their proofs, and at last in Section 5 we give some examples to illustrate our main results.

## 2. Notation and Basic Facts

In this section we summarize some basic facts about general $C$-symmetric quasidifferential expressions of even order ($n = 2k$, $k \geq 1$) and real or complex coefficients on one-interval and two-interval cases for the convenience of the reader.

Firstly, let $J = (a, b)$ be an interval with $-\infty \leq a < b \leq \infty$ and $M_n(S)$ denote the set of $n \times n$ complex matrices with entries from a given set $S$.

Set $C_n = (c_{r,s})_{1\leq r,s\leq n}$ as a skew-diagonal constant matrix satisfying

$$C_n^{-1} = -C_n = C_n^*, \tag{1}$$

and let

$$\begin{aligned} Z_n(J) := \Big\{ & (q_{r,s})_{r,s=1}^n \in M_n(L_{loc}(J)),\ q_{r,r+1} \\ & \neq 0 \ a.e.\ J,\ q_{r,r+1}^{-1} \in L_{loc}(J),\ 1 \leq r \leq n-1,\ q_{r,s} \\ & = 0 \ a.e.\ J,\ 2 \leq r+1 < s \leq n;\ q_{r,s} \in L_{loc}(J), s \neq r \\ & + 1,\ 1 \leq r \leq n-1 \Big\}. \end{aligned} \tag{2}$$

Let $Q \in Z_n(J)$. We define

$$V_0 := \{ y : J \longrightarrow \mathbb{C},\ y \text{ is measurable} \} \tag{3}$$

and

$$y^{[0]} := y \quad (y \in V_0). \tag{4}$$

Inductively, for $r = 1, \ldots, n$, we define

$$V_r = \left\{ y \in V_{r-1} : y^{[r-1]} \in (AC_{loc}(J)) \right\}, \tag{5}$$

$$y^{[r]} = q_{r,r+1}^{-1} \left\{ y^{[r-1]'} - \sum_{s=1}^{r} q_{r,s} y^{[s-1]} \right\} \quad (y \in V_r), \tag{6}$$

where $q_{n,n+1} := c_{n,1}$, and $AC_{loc}(J)$ denotes the set of complex-valued functions which are absolutely continuous on all compact subintervals of $J$. Finally we set

$$My = M_Q y := i^n y^{[n]} \quad (y \in V_n). \tag{7}$$

The expression $M = M_Q$ is called the quasidifferential expression associated with $Q$. For $V_n$ we also use the notations $V(M)$ and $D(Q)$.

*Definition 1.* Let $Q \in Z_n(J)$ and let $M = M_Q$ be defined as above. Assume that

$$Q = -C_n^{-1} Q^* C_n, \tag{8}$$

where

$$C_n = \begin{pmatrix} 0_{k\times k} & C_{12} \\ C_{21} & 0_{k\times k} \end{pmatrix} \tag{9}$$

satisfying

$$C_n^{-1} = -C_n = C_n^*, \quad C_{21}, C_{12} \in M_k(\mathbb{C}), \tag{10}$$

i.e.,

$$C_n = \begin{pmatrix} 0_{k\times k} & C_{12} \\ -C_{12}^* & 0_{k\times k} \end{pmatrix}, \tag{11}$$

with

$$c_{r,s}\overline{c}_{r,s} = 1, \quad r + s = n + 1. \tag{12}$$

Then $M = M_Q$ is called a $C$-symmetric differential expression.

Let $w \in L_{loc}(J)$ be positive a.e. on $J$. We consider the Hilbert space

$$H = L^2(J, w) \tag{13}$$

with its usual inner product

$$(y, z) := \int_J y\overline{z} w\, dx, \quad y, z \in H. \tag{14}$$

For the $C$-symmetry $M_Q$, the Green's formula has the form

$$\begin{aligned} \int_J \left\{ My\overline{z} - y\overline{Mz} \right\} dx = {} & [y, z](b) \\ & - [y, z](a) \quad (y, z \in D(Q)), \end{aligned} \tag{15}$$

where $[y, z](b) = \lim_{t\longrightarrow b^-} [y, z](t)$, $[y, z](a) = \lim_{t\longrightarrow a^+} [y, z](t)$ and the limits always exist and are finite. Here the skew-symmetric sesquilinear form $[\cdot, \cdot]$ maps $D(Q) \times D(Q) \longrightarrow \mathbb{C}$.

Every self-adjoint extension $T$ of the minimal operator $T_{Q,0}$ is between the minimal operator $T_{Q,0}$ and maximal operator $T_Q$; i.e., we have

$$T_{Q,0} \subset T = T^* \subset T_Q. \tag{16}$$

Thus these self-adjoint operators $T$ are distinguished from one another only by their domains.

**Lemma 2** (Lagrange identity). *Assume $Q \in Z_n(J)$ satisfies (8) and let $M = M_Q$ be the corresponding $C$-symmetric differential expression. Then for any $y, z \in D(Q)$ we have*

$$\overline{z} My - y\overline{Mz} = [y, z]', \tag{17}$$

*and*

$$\begin{aligned} [y, z] &= (-1)^{k+1} \sum_{r=0}^{n-1} c_{n-r,r+1} \overline{z^{[n-r-1]}} y^{[r]} = (-1)^{k+1} \\ & \cdot \sum_{r=1}^{k} \left\{ c_{r,n-r+1} \overline{z^{[r-1]}} y^{[n-r]} - \overline{c}_{r,n-r+1} \overline{z^{[n-r]}} y^{[r-1]} \right\} \\ & = (-1)^{k+1} (Z^* C_n Y), \end{aligned} \tag{18}$$

*where*

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix}, \qquad Z = \begin{pmatrix} z^{[0]} \\ z^{[1]} \\ \vdots \\ z^{[n-1]} \end{pmatrix}, \tag{19}$$

*and*

$$C_n = \begin{pmatrix} 0_{k\times k} & C_{12} \\ C_{21} & 0_{k\times k} \end{pmatrix} \tag{20}$$

*is defined by (11).*

*In fact,*

$$C_{12} = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_{1,n} \\ 0 & 0 & \cdots & c_{2,n-1} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & c_{k-1,k+2} & \cdots & 0 & 0 \\ c_{k,k+1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{21}$$

*and by (11) we have* $C_{21} = -C_{12}^*$ *and* $c_{r,s} = -\overline{c}_{s,r}$, $r + s = n + 1$.

*Proof.* Set $Q = (q_{r,s})_{r,s=1}^n$, and $Q^+ := -C_n^{-1}Q^*C_n = (p_{r,s})_{r,s=1}^n$. Then we infer that

$$p_{r,s} = \sum_{l=1}^n c_{l,s}\left(\sum_{j=1}^n c_{r,j}\overline{q}_{l,j}\right) = c_{r,n-r+1}\overline{q}_{n-s+1,n-r+1}c_{n-s+1,s}, \quad r,s = 1,2,\ldots,n. \tag{22}$$

So for $1 \le r \le n-1$,

$$p_{r,r+1} = c_{r,n-r+1}\overline{q}_{n-r,n-r+1}c_{n-r,r+1} \tag{23}$$

is invertible a.e. on $J$.

Since for $2 \le r+1 < s \le n$, $r+1-s = (n-s+1)+1-(n-r+1) < 0$, $q_{n-s+1,n-r+1} = 0$, then

$$p_{r,s} = c_{r,n-r+1}\overline{q}_{n-s+1,n-r+1}c_{n-s+1,s} = 0. \tag{24}$$

This concludes that $Q^+ \in Z_n(J)$.

Since $Q \in Z_n(J)$ satisfies (8), i.e., $Q = Q^+$.

Now, let $f = -\overline{c}_{1,n}y_Q^{[n]}$, $g = -\overline{c}_{1,n}z_Q^{[n]}$, $y, z \in V_n$; then, from (4) and (6) we have

$$Y' = QY + F, \qquad Z' = QZ + G, \tag{25}$$

where

$$F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g \end{pmatrix}. \tag{26}$$

So from $Q^*C_n = -C_nQ$, we have

$$\begin{aligned}(Z^*C_nY)' &= (Z^*)'C_nY + Z^*C_n'Y + Z^*C_nY' \\ &= (Z')^*C_nY + Z^*C_nY' \\ &= (QZ+G)^*C_nY + Z^*C_n(QY+F) \\ &= (Z^*Q^* + G^*)C_nY + Z^*C_n(QY+F) \\ &= Z^*(Q^*C_n + C_nQ)Y + G^*C_nY + Z^*C_nF \\ &= Z^*C_nF + G^*C_nY = c_{1,n}\overline{z^{[0]}}f - \overline{c}_{1,n}\overline{g}y^{[0]} \\ &= -\overline{z^{[0]}}y^{[n]} + \overline{z^{[n]}}y^{[0]} \\ &= -(-i)^n\left\{\overline{z^{[0]}}My - \overline{Mz}y^{[0]}\right\}.\end{aligned} \tag{27}$$

After integrating both sides of the above equation, we get

$$\int_a^b \overline{z}M_Q y\,dx - \int_a^b y\overline{M_Q z}\,dx = (-1)^{k+1}Z^*C_nY\Big|_a^b. \tag{28}$$

Hence from (15) we have

$$\overline{z}M_Q y - y\overline{M_Q z} = [y,z]', \tag{29}$$

and

$$[y,z] = (-1)^{k+1}Z^*C_nY. \tag{30}$$

Together with some caculations we have

$$Z^*C_nY = \sum_{r=0}^{n-1} c_{n-r,r+1}\overline{z^{[n-r-1]}}y^{[r]}, \tag{31}$$

and $C_n$ has the form (11) and $c_{r,s} = -\overline{c}_{s,r}$, $r+s = n+1$.

Then we also have

$$Z^*C_nY = \sum_{r=1}^k\left\{c_{r,n-r+1}\overline{z^{[r-1]}}y^{[n-r]} - \overline{c}_{r,n-r+1}\overline{z^{[n-r]}}y^{[r-1]}\right\}. \tag{32}$$

This completes the proof. □

Following this we consider direct sum Hilbert space

$$H = H_1 \oplus H_2,$$
$$H_j = L^2(J_j, w_j), \quad w_j > 0, \tag{33}$$

where $J_j = (a_j, b_j), -\infty \le a_j < b_j \le \infty, j = 1, 2$.

The inner product in space $H$ is defined by

$$(y, z) = \sum_{j=1}^{2} (y_j, z_j)_j, \quad y = \{y_1, y_2\}, \ z = \{z_1, z_2\}, \tag{34}$$

and $(\cdot, \cdot)_j$ is the usual inner product in $H_j$:

$$(y_j, z_j)_j = \int_{J_j} y_j \bar{z}_j w_j dx, \quad y_1, z_1 \in H_1, \ y_2, z_2 \in H_2. \tag{35}$$

Define two differential expressions with complex-valued coefficients by

$$M_j y = M_{Q_j} \quad y := i^n y_{Q_j}^{[n]} \text{ on } J_j. \tag{36}$$

Let $M = \{M_1, M_2\}$; i.e., $My = \{M_1 y_1, M_2 y_2\}$.

*Definition 3* (see [1, 8, 16]). The two-interval maximal and minimal domains and operators are simply the direct sums of the corresponding one-interval domains and operators, i.e.,

$$T_Q = T_{Q_1} \oplus T_{Q_2},$$
$$T_{Q,0} = T_{Q_1,0} \oplus T_{Q_2,0}, \tag{37}$$

and

$$D_Q = D(T_Q) = D(T_{Q_1}) \oplus D(T_{Q_2}),$$
$$D_{Q,0} = D(T_{Q,0}) = D(T_{Q_1,0}) \oplus D(T_{Q_2,0}). \tag{38}$$

We also have the following lemma.

**Lemma 4** (see [8, 16]). *In the direct sum spaces, we have*

$$T_{Q,0}^* = T_{Q_1,0}^* \oplus T_{Q_2,0}^* = T_{Q_1} \oplus T_{Q_2} = T_Q,$$
$$T_Q^* = T_{Q_1}^* \oplus T_{Q_2}^* = T_{Q_1,0} \oplus T_{Q_2,0} = T_{Q,0}. \tag{39}$$

*The minimal operator $T_{Q,0}$ is a closed, symmetric, densely defined operator in the Hilbert space H with deficiency index d given by $d = d_1 + d_2$.*

It is interesting to note that Lemma 2 extends to the two-interval case:

$$\bar{z}My - y\overline{Mz} = [y, z]',$$
$$[y, z] = \sum_{j=1}^{2} [y_j, z_j]_j (b_j) - [y_j, z_j]_j (a_j), \tag{40}$$
$$y, z \in H,$$

where

$$[y_j, z_j]_j = (-1)^{k+1} (Z_j^* C_n Y_j),$$
$$Y_j = \begin{pmatrix} y_j^{[0]} \\ y_j^{[1]} \\ \vdots \\ y_j^{[n-1]} \end{pmatrix},$$
$$Z_j = \begin{pmatrix} z_j^{[0]} \\ z_j^{[1]} \\ \vdots \\ z_j^{[n-1]} \end{pmatrix}, \tag{41}$$
$$j = 1, 2,$$

and $C_n$ has the form (11).

**Lemma 5.** *Let $a_j \le \alpha_j < \beta_j \le b_j$. The number $d_j$ of linearly independent solutions of*

$$M_j y = \lambda_j w_j y \quad on \ (\alpha_j, \beta_j) \tag{42}$$

*lying in $L_2((\alpha_j, \beta_j), w_j)$ is independent of $\lambda_j \in \mathbb{C}$, provided* $\mathrm{Im}(\lambda_j) \neq 0$. *If one endpoint of $(\alpha_j, \beta_j)$ is regular and the other is singular, then the inequalities*

$$k \le d_j \le 2k = n \tag{43}$$

*hold. For $\lambda = \lambda_j \in \mathbb{R}$, the number of linearly independent solutions of $(42)_{j=1}$ lying in $L_2((\alpha_1, \beta_1), w_1)$ is less than or equal to $d_1$ and of $(42)_{j=2}$ lying in $L_2((\alpha_2, \beta_2), w_2)$ is less than or equal to $d_2$.*

*Let $c_j \in J_j = (a_j, b_j)$. If $d_{j1}$ is the deficiency index on $(a_j, c_j)$, $d_{j2}$ is the deficiency index on $(c_j, b_j)$ and $d_j$ is the deficiency index on $(a_j, b_j)$, then*

$$d_j = d_{j1} + d_{j2} - n, \quad j = 1, 2. \tag{44}$$

*Proof.* See [15, 16]. □

W. N. Everitt and A. Zettl extend the well-known single interval GKN characterization of all self-adjoint extensions to the two-interval case for Lagrange symmetric differential expressions in [12], and it is obvious that this extended GKN theorem also can be established for two-interval $C$-symmetric differential expression. It is expressed as follows.

**Lemma 6** (GKN). *Let $T_{Q,0}$ be the two-interval minimal operator in H and let d be the deficiency index of $T_{Q,0}$. Then a linear submanifold $D(T)$ of $D_Q$ is the domain of a self-adjoint extension $T$ of $T_{Q,0}$ if and only if there exist vectors $w_1, w_2, \cdots, w_d$ in $D_Q$ satisfying the following conditions:*

(i) $w_1, w_2, \cdots, w_d$ are linearly independent modulo $D_{Q,0}$;

(ii) $[w_i, w_l] = [w_{i1}, w_{l1}]_1(b_1) - [w_{i1}, w_{l1}]_1(a_1) + [w_{i2}, w_{l2}]_2(b_2) - [w_{i2}, w_{l2}]_2(a_2) = 0, \ i, l = 1, \cdots, d$;

(iii) $D(T) = \{y = \{y_1, y_2\} \in D_Q : [y, w_l] = [y_1, w_{l1}]_1(b_1) - [y_1, w_{l1}]_1(a_1) + [y_2, w_{l2}]_2(b_2) - [y_2, w_{l2}]_2(a_2) = 0, l = 1, \cdots, d\}$.

## 3. Characterization of All Self-Adjoint Domains for Singular Two-Interval Problems

In this section we assume that $M = \{M_{Q_1}, M_{Q_2}\}$ are generated by $Q_j \in Z_{n(j)}(J_j)$, $j = 1, 2$ satisfying (8), $n = 2k$, $k \geq 1$, the endpoints $a_j$ and $b_j$ are singular. We give the decomposition of the maximal domain and the characterization of all self-adjoint extensions of the two-interval minimal operator.

First we have the following theorem.

**Theorem 7.** *Let $M_j$ be a C-symmetric differential expression on $(a_j, b_j)$ and let $c_j \in (a_j, b_j)$. Consider the equations*

$$M_j y = \lambda_j w_j y, \quad j = 1, 2. \tag{45}$$

*Assume that for some $\lambda = \lambda_{j1} \in \mathbb{R}$ (45) has $d_{j1}$ linearly independent solutions $u_{j1}, u_{j2}, \cdots, u_{jd_{j1}}$ on $(a_j, c_j)$ which lie in $L^2((a_j, c_j), w_j)$ and that for some $\lambda = \lambda_{j2} \in \mathbb{R}$ (45) has $d_{j2}$ linearly independent solutions $v_{j1}, v_{j2}, \cdots, v_{jd_{j2}}$ on $(c_j, b_j)$ which lie in $L^2((c_j, b_j), w_j)$. Then, we have the following:*

(1) *The solutions $u_{j1}, u_{j2}, \cdots, u_{jd_{j1}}$ can be extended to $J_j = (a_j, b_j)$ such that the extended functions, also denoted by $u_{j1}, u_{j2}, \cdots, u_{jd_{j1}}$, satisfy $u_{jl} \in D_{Q_j}(a_j, b_j)$ and $u_{jl}$ is identically zero in a left neighborhood of $b_j$, $l = 1, \cdots, d_{j1}$. The solutions $v_{j1}, v_{j2}, \cdots, v_{jd_{j2}}$ can be extended to $(a_j, b_j)$ such that the extended functions, also denoted by $v_{j1}, v_{j2}, \cdots, v_{jd_{j2}}$, satisfy $v_{jl} \in D_{Q_j}(a_j, b_j)$ and $v_{jl}$ is identically zero in a right neighborhood of $a_j$, $l = 1, \cdots, d_{j2}$.*

(2) *For $m_j = 2d_{j1} - 2k$ the solutions $u_{j1}, u_{j2}, \cdots, u_{jd_{j1}}$ on $(a_j, c_j)$ can be ordered such that the $m_j \times m_j$ matrix $U_j = ([u_{jl_1}, u_{jl_2}]_j(c_j))$, $1 \leq l_1, l_2 \leq m_j$, is given by*

$$U_j = (-1)^{k+1} C_{m_j}^T, \quad j = 1, 2. \tag{46}$$

*For $n_j = 2d_{j2} - 2k$ the solutions $v_{j1}, v_{j2}, \cdots, v_{jd_{j2}}$ on $(c_j, b_j)$ can be ordered such that the $n_j \times n_j$ matrix $V_j = ([v_{jl_1}, v_{jl_2}]_j(c_j))$, $1 \leq l_1, l_2 \leq n_j$, is given by*

$$V_j = (-1)^{k+1} C_{n_j}^T, \quad j = 1, 2. \tag{47}$$

(3) *For every $y = \{y_1, y_2\} \in D_Q$ we have*

$$[y_j, u_{jl}]_j (a_j) = 0, \quad \text{for } l = m_j + 1, \cdots, d_{j1}, \tag{48}$$

$$[y_j, v_{jl}]_j (b_j) = 0, \quad \text{for } l = n_j + 1, \cdots, d_{j2}. \tag{49}$$

(4) *For $1 \leq l_1, l_2 \leq d_{j1}$, we have*

$$[u_{jl_1}, u_{jl_2}]_j (a_j) = [u_{jl_1}, u_{jl_2}]_j (c_j). \tag{50}$$

*For $1 \leq l_1, l_2 \leq d_{j2}$, we have*

$$[v_{jl_1}, v_{jl_2}]_j (b_j) = [v_{jl_1}, v_{jl_2}]_j (c_j). \tag{51}$$

*Proof.* By Naimark Patching Lemma the solutions $u_{j1}, u_{j2}, \cdots, u_{jd_{j1}}$ can be "patched" at $c_j$ to obtain maximal domain functions in $D_{Q_j}(a_j, b_j)$. By another application of Naimark Patching Lemma these extended functions can be modified to be identically zero in a left neighborhood of $b_j$, $j = 1, 2$. By using the similar method, we can proof the latter part of (1). Parts (2) and (3) are established by Corollary 6 in [13] for complex case. Part (4) follows from Corollary 3.8 in [15]. □

*Remark 8.* We call that the solutions $u_{jm_j+1}, \cdots, u_{jd_{j1}}$ and $v_{jn_j+1}, \cdots, v_{jd_{j2}}$ are of LP (limit-point) type at $a_j$ and $b_j$, respectively, which satisfy conditions (3) of Theorem 7. The LP solutions play an important role in studies on distribution of continuous spectrum (see [15]). These solutions play no role in the formulation of the self-adjoint boundary conditions. But the LC (limit-circle) case requires boundary conditions to determine self-adjoint extensions. For this reason we call $u_{j1}, u_{j2}, \cdots, u_{jm_j}$ LC solutions at $a_j$, $v_{j1}, v_{j2}, \cdots, v_{jn_j}$ LC solutions at $b_j$.

Next we give the decomposition of the maximal domain and the characterization of all self-adjoint domains.

**Theorem 9.** *Let the hypotheses and notations of Theorem 7 hold. Then*

$$\begin{aligned} D_{Q_j}(a_j, b_j) = {} & D_{Q_j,0}(a_j, b_j) \\ & \oplus \operatorname{span}\{u_{j1}, u_{j2}, \cdots, u_{jm_j}\} \\ & \oplus \operatorname{span}\{v_{j1}, v_{j2}, \cdots, v_{jn_j}\}. \end{aligned} \tag{52}$$

*Proof.* The method of this proof is similar to the citation [16]. □

According to Theorems 7 and 9 we have our main result as follows.

**Theorem 10.** *Let the hypotheses and notations of Theorem 7 hold. Then a linear submanifold $D(T) \subset D_Q$ is the domain of a self-adjoint extension $T$ of two-interval minimal operator $T_{Q,0}$ if and only if there exist complex $d \times m_j$ matrices $A_j$ and complex $d \times n_j$ matrices $B_j$ such that the following three conditions hold:*

(1) $\operatorname{rank}(A_1, B_1, A_2, B_2) = d$;

(2) $\sum_{j=1}^{2} \{A_j C_{m_j} A_j^* - B_j C_{n_j} B_j^*\} = 0$;

(3) $D(T) = \{y = \{y_1, y_2\} \in D_Q :$

$$\sum_{j=1}^{2} \left\{ A_j \begin{pmatrix} [y_j, u_{j1}]_j (a_j) \\ \vdots \\ [y_j, u_{jm_j}]_j (a_j) \end{pmatrix} + B_j \begin{pmatrix} [y_j, v_{j1}]_j (b_j) \\ \vdots \\ [y_j, v_{jn_j}]_j (b_j) \end{pmatrix} \right\} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Bigg\}, \tag{53}$$

*where $(A_1, B_1, A_2, B_2)$ denotes the $d$ by $4d$ matrix whose first $d$ columns are those of $A_1$, the second $d$ columns are those of $B_1$, etc. And $C_{m_j}, C_{n_j}$ are complex matrices of the form (11).*

*Proof (necessity).* Let $D(T)$ be the domain of a self-adjoint extension $T$ of $T_{Q,0}$. By Lemma 6 there exist $w_1 = \{w_{11}, w_{12}\}, \cdots, w_d = \{w_{d1}, w_{d2}\} \in D_Q$ satisfying conditions (i), (ii), (iii) of Lemma 6. By Theorem 9, each $w_{i1}$ and $w_{i2}$ can be uniquely written as

$$w_{i1} = \widehat{y}_{i1} + \sum_{l=1}^{m_1} a_{il}u_{1l} + \sum_{l=1}^{n_1} b_{il}v_{1l},$$
$$w_{i2} = \widehat{y}_{i2} + \sum_{l=1}^{m_2} c_{il}u_{2l} + \sum_{l=1}^{n_2} d_{il}v_{2l}, \tag{54}$$

where $\widehat{y}_{i1} \in D_{Q_1,0}$, $\widehat{y}_{i2} \in D_{Q_2,0}$, $a_{il}, b_{il}, c_{il}, d_{il} \in \mathbb{C}$, $i = 1, 2, \ldots, d$.

Let

$$A_1 = -\left(\overline{a}_{il}\right)_{d\times m_1},$$
$$B_1 = \left(\overline{b}_{il}\right)_{d\times n_1},$$
$$A_2 = -\left(\overline{c}_{il}\right)_{d\times m_2},$$
$$B_2 = \left(\overline{d}_{il}\right)_{d\times n_2}, \tag{55}$$

Then

$$\begin{pmatrix} [y_1, w_{11}]_1(a_1) \\ \vdots \\ [y_1, w_{d1}]_1(a_1) \end{pmatrix} = \begin{pmatrix} \left[y_1, \sum_{l=1}^{m_1} a_{1l}u_{1l}\right]_1 (a_1) \\ \vdots \\ \left[y_1, \sum_{l=1}^{m_1} a_{dl}u_{1l}\right]_1 (a_1) \end{pmatrix} = -A_1 \begin{pmatrix} [y_1, u_{11}]_1(a_1) \\ \vdots \\ [y_1, u_{1m_1}]_1(a_1) \end{pmatrix},$$
$$\begin{pmatrix} [y_1, w_{11}]_1(b_1) \\ \vdots \\ [y_1, w_{d1}]_1(b_1) \end{pmatrix} = \begin{pmatrix} \left[y_1, \sum_{l=1}^{n_1} b_{1l}v_{1l}\right]_1 (b_1) \\ \vdots \\ \left[y_1, \sum_{l=1}^{n_1} b_{dl}v_{1l}\right]_1 (b_1) \end{pmatrix} = B_1 \begin{pmatrix} [y_1, v_{11}]_1(b_1) \\ \vdots \\ [y_1, v_{1n_1}]_1(b_1) \end{pmatrix}. \tag{56}$$

Similarly,

$$\begin{pmatrix} [y_2, w_{12}]_2(a_2) \\ \vdots \\ [y_2, w_{d2}]_2(a_2) \end{pmatrix} = -A_2 \begin{pmatrix} [y_2, u_{21}]_2(a_2) \\ \vdots \\ [y_2, u_{2m_2}]_2(a_2) \end{pmatrix},$$
$$\begin{pmatrix} [y_2, w_{12}]_2(b_2) \\ \vdots \\ [y_2, w_{d2}]_2(b_2) \end{pmatrix} = B_2 \begin{pmatrix} [y_2, v_{21}]_2(b_2) \\ \vdots \\ [y_2, v_{2n_2}]_2(b_2) \end{pmatrix}. \tag{57}$$

Hence the boundary condition (iii) of Lemma 6 is equivalent to part (3) of Theorem 10.

Next we prove that $A_1$, $B_1$, $A_2$, and $B_2$ satisfy conditions (1) and (2) of Theorem 10.

Clearly $\operatorname{rank}(A_1, B_1, A_2, B_2) \leq d$. If $\operatorname{rank}(A_1, B_1, A_2, B_2) < d$, then there exist constants $h_1, \cdots, h_d$, not all zero, such that

$$(h_1, \ldots, h_d)(A_1, B_1, A_2, B_2) = 0. \tag{58}$$

Let $f = \{f_1, f_2\} = \sum_{i=1}^d \overline{h}_i w_i$, so $f_1 = \sum_{i=1}^d \overline{h}_i w_{i1}$, $f_2 = \sum_{i=1}^d \overline{h}_i w_{i2}$; from (54), we obtain

$$f_1 = \sum_{i=1}^d \overline{h}_i \widehat{y}_{i1} + \sum_{i=1}^d \sum_{l=1}^{m_1} \overline{h}_i a_{il} u_{1l} + \sum_{i=1}^d \sum_{l=1}^{n_1} \overline{h}_i b_{il} v_{1l},$$
$$f_2 = \sum_{i=1}^d \overline{h}_i \widehat{y}_{i2} + \sum_{i=1}^d \sum_{l=1}^{m_2} \overline{h}_i c_{il} u_{2l} + \sum_{i=1}^d \sum_{l=1}^{n_2} \overline{h}_i d_{il} v_{2l}. \tag{59}$$

By (58), we have $(h_1 \cdots h_d)A_1 = (h_1 \cdots h_d)B_1 = (h_1 \cdots h_d)A_2 = (h_1 \cdots h_d)B_2 = 0$. Hence

$$f_1 = \sum_{i=1}^d \overline{h}_i \widehat{y}_{i1},$$
$$f_2 = \sum_{i=1}^d \overline{h}_i \widehat{y}_{i2}. \tag{60}$$

So we have $f_1 \in D_{Q_1,0}$ and $f_2 \in D_{Q_2,0}$; thus, $f = \{f_1, f_2\} \in D_{Q,0}$. This contradicts the fact that the functions $w_1, w_2, \ldots, w_d$ are linearly independent modulo $D_{Q,0}$. Therefore $\operatorname{rank}(A_1, B_1, A_2, B_2) = d$.

Now we verify part (2). By (54), we have

$$[w_{i1}, w_{l1}]_1(a_1) = \left[\sum_{k=1}^{m_1} a_{ik}u_{1k}, \sum_{s=1}^{m_1} a_{ls}u_{1s}\right]_1 (a_1)$$
$$= \sum_{k=1}^{m_1} \sum_{s=1}^{m_1} a_{ik}\overline{a}_{ls} [u_{1k}, u_{1s}]_1(a_1),$$
$$(i, l = 1, \cdots, d). \tag{61}$$

So

$$\left([w_{i1}, w_{l1}]_1(a_1)\right)^T_{d\times d} = A_1 U_1^T A_1^*$$
$$= (-1)^{k+1} A_1 C_{m_1} A_1^*, \tag{62}$$

where the matrix $U_1$ is defined in Theorem 7.

Similarly, we have

$$\begin{aligned}([w_{i1},w_{l1}]_1(b_1))^T_{d\times d} &= B_1V_1^TB_1^* = (-1)^{k+1}B_1C_{n_1}B_1^*,\\ ([w_{i2},w_{l2}]_2(a_2))^T_{d\times d} &= A_2U_2^TA_2^*\\ &= (-1)^{k+1}A_2C_{m_2}A_2^*,\\ ([w_{i2},w_{l2}]_2(b_2))^T_{d\times d} &= B_2V_2^TB_2^* = (-1)^{k+1}B_2C_{n_2}B_2^*.\end{aligned} \tag{63}$$

Hence condition (ii) of Lemma 6 becomes

$$A_1C_{m_1}A_1^* - B_1C_{n_1}B_1^* + A_2C_{m_2}A_2^* - B_2C_{n_2}B_2^* = 0. \tag{64}$$

*(sufficiency)*. Let the matrices $A_1$, $B_1$, $A_2$, and $B_2$ satisfy conditions (1) and (2) of Theorem 10. We need to prove that $D(T)$ defined by (3) is the domain of a self-adjoint extension $T$ of $T_{Q,0}$.

Let

$$\begin{aligned}A_1 &= -(\overline{a}_{il})_{d\times m_1},\\ B_1 &= (\overline{b}_{il})_{d\times n_1},\\ A_2 &= -(\overline{c}_{il})_{d\times m_2},\\ B_2 &= (\overline{d}_{il})_{d\times n_2},\\ w_{i1} &= \sum_{l=1}^{m_1}a_{il}u_{1l} + \sum_{l=1}^{n_1}b_{il}v_{1l},\\ w_{i2} &= \sum_{l=1}^{m_2}c_{il}u_{2l} + \sum_{l=1}^{n_2}d_{il}v_{2l}.\end{aligned} \tag{65}$$

Then for $y = \{y_1, y_2\} \in D_Q$ we have

$$\begin{aligned}-A_1\begin{pmatrix}[y_1,u_{11}]_1(a_1)\\ \vdots\\ [y_1,u_{1m_1}]_1(a_1)\end{pmatrix} &= \begin{pmatrix}\left[y_1,\sum_{l=1}^{m_1}a_{1l}u_{1l}\right]_1(a_1)\\ \vdots\\ \left[y_1,\sum_{l=1}^{m_1}a_{dl}u_{1l}\right]_1(a_1)\end{pmatrix} = \begin{pmatrix}[y_1,w_{11}]_1(a_1)\\ \vdots\\ [y_1,w_{d1}]_1(a_1)\end{pmatrix},\\ B_1\begin{pmatrix}[y_1,v_{11}]_1(b_1)\\ \vdots\\ [y_1,v_{1n_1}]_1(b_1)\end{pmatrix} &= \begin{pmatrix}\left[y_1,\sum_{l=1}^{n_1}b_{1l}v_{1l}\right]_1(b_1)\\ \vdots\\ \left[y_1,\sum_{l=1}^{n_1}b_{dl}v_{1l}\right]_1(b_1)\end{pmatrix} = \begin{pmatrix}[y_1,w_{11}]_1(b_1)\\ \vdots\\ [y_1,w_{d1}]_1(b_1)\end{pmatrix}.\end{aligned} \tag{66}$$

Similarly, we have

$$\begin{aligned}-A_2\begin{pmatrix}[y_2,u_{21}]_2(a_2)\\ \vdots\\ [y_2,u_{2m_2}]_2(a_2)\end{pmatrix} &= \begin{pmatrix}[y_2,w_{12}]_2(a_2)\\ \vdots\\ [y_2,w_{d2}]_2(a_2)\end{pmatrix},\\ B_2\begin{pmatrix}[y_2,v_{21}]_2(b_2)\\ \vdots\\ [y_2,z_{2n_2}]_2(b_2)\end{pmatrix} &= \begin{pmatrix}[y_2,w_{12}]_2(b_2)\\ \vdots\\ [y_2,w_{d2}]_2(b_2)\end{pmatrix}.\end{aligned} \tag{67}$$

Therefore the boundary condition (3) in Theorem 10 becomes the boundary condition (iii) in Lemma 6; i.e.,

$$\begin{aligned}&[y_1,w_{i1}]_1(b_1) - [y_1,w_{i1}]_1(a_1) + [y_2,w_{i2}]_2(b_2)\\ &\quad - [y_2,w_{i2}]_2(a_2) = 0, \quad i = 1,\cdots,d.\end{aligned} \tag{68}$$

It remains to show that $w_i, i = 1,\cdots,d$ satisfy conditions (i) and (ii) of Lemma 6.

Condition (i) holds. If not, then there exist constants $c_1,\cdots,c_d$, not all zero, such that

$$\gamma = \sum_{i=1}^{d}c_iw_i \in D_{Q,0}, \tag{69}$$

i.e.,

$$\begin{aligned}\gamma_1 &= \sum_{i=1}^{d}c_iw_{i1} \in D_{Q_1,0},\\ \gamma_2 &= \sum_{i=1}^{d}c_iw_{i2} \in D_{Q_2,0}.\end{aligned} \tag{70}$$

Hence we have $[\gamma_1,y_1]_1(a_1) = [\gamma_1,y_1]_1(b_1) = [\gamma_2,y_2]_2(a_2) = [\gamma_2,y_2]_2(b_2) = 0$, for any $y = \{y_1,y_2\} \in D_Q$. Using the notation $U_1$ from Theorem 7,

$$\begin{aligned}&(0\cdots 0)\\ &= \left(\left[\sum_{l=1}^{d}c_lw_{l1},u_{11}\right]_1(a_1)\cdots\left[\sum_{l=1}^{d}c_lw_{l1},u_{1m_1}\right]_1(a_1)\right)\\ &= (c_1\cdots c_d)(a_{il})_{d\times m_1}U_1.\end{aligned} \tag{71}$$

Since $U_1$ is nonsingular, we have $(\overline{c}_1\cdots\overline{c}_d)A_1 = 0$.

Similarly, we have $(\overline{c}_1 \cdots \overline{c}_d)B_1 = 0$, $(\overline{c}_1 \cdots \overline{c}_d)A_2 = 0$, and $(\overline{c}_1 \cdots \overline{c}_d)B_2 = 0$. Hence

$$(\overline{c}_1 \cdots \overline{c}_d)(\mathrm{A}_1, B_1, A_2, B_2) = 0. \tag{72}$$

This contradicts the fact that $\operatorname{rank}(A_1, B_1, A_2, B_2) = d$.

Next we show that (ii) holds. We have

$$\begin{aligned} [w_{i1}, w_{l1}]_1(a_1) &= \left[\sum_{s=1}^{m_1} a_{is}u_{1s}, \sum_{k=1}^{m_1} a_{lk}u_{1k}\right]_1 (a_1) \\ &= \sum_{s=1}^{m_1}\sum_{k=1}^{m_1} a_{is}\overline{a}_{lk}[u_{1s}, u_{1k}]_1(a_1). \end{aligned} \tag{73}$$

From Theorem 7 we get

$$\begin{aligned} ([w_{i1}, w_{l1}]_1(a_1))^T_{d\times d} &= A_1U_1^TA_1^* \\ &= (-1)^{k+1}A_1C_{m_1}A_1^*. \end{aligned} \tag{74}$$

Similarly,

$$\begin{aligned} ([w_{i1}, w_{l1}]_1(b_1))^T_{d\times d} &= (-1)^{k+1}B_1C_{n_1}B_1^*, \\ ([w_{i2}, w_{l2}]_2(a_2))^T_{d\times d} &= (-1)^{k+1}A_2C_{m_2}A_2^*, \\ ([w_{i2}, w_{l2}]_2(b_2))^T_{d\times d} &= (-1)^{k+1}B_2C_{n_2}B_2^*. \end{aligned} \tag{75}$$

Therefore

$$\begin{aligned} [w_i, w_l]^T_{d\times d} &= ([w_{i1}, w_{l1}]_1(b_1) - [w_{i1}, w_{l1}]_1(a_1) \\ &\quad + [w_{i2}, w_{l2}]_2(b_2) - [w_{i2}, w_{l2}]_2(a_2))^T = (-1)^{k+1} \\ &\quad \cdot B_1C_{n_1}B_1^* - (-1)^{k+1}A_1C_{m_1}A_1^* + (-1)^{k+1}B_2C_{n_2}B_2^* \\ &\quad - (-1)^{k+1}A_2C_{m_2}A_2^* = 0. \end{aligned} \tag{76}$$

By Lemma 6, we conclude that $D(T)$ is a self-adjoint domain. □

## 4. Special Case

In Theorem 10 it is assumed that all four endpoints $a_1, b_1, a_2, b_2$ are singular. It can be specialized to the results when at least one endpoint is regular. We state several cases here for the convenience of the reader.

**Theorem 11.** *Let the hypotheses and notations of Theorem 7 hold and assume that the endpoints $b_1$ and $b_2$ are regular. Then $n_1 = n_2 = n$ and $d = d_{11} + d_{21}$. Then a linear submanifold $D(T) \subset D_Q$ is the domain of a self-adjoint extension $T$ of $T_{Q,0}$ if and only if there exist a complex $d \times m_1$ matrix $A_1$ and a complex $d \times n$ matrix $B_1$ and a complex $d \times m_2$ matrix $A_2$ and a complex $d \times n$ matrix $B_2$ such that the following three conditions hold:*

(1) $\operatorname{rank}(A_1, B_1, A_2, B_2) = d$;

(2) $A_1C_{m_1}A_1^* + A_2C_{m_2}A_2^* = B_1C_nB_1^* + B_2C_nB_2^*$;

(3) $D(T) = \{y = \{y_1, y_2\} \in D_Q$:

$$\begin{aligned} & A_1\begin{pmatrix} [y_1, u_{11}]_1(a_1) \\ \vdots \\ [y_1, u_{1m_1}]_1(a_1) \end{pmatrix} + B_1\begin{pmatrix} y_1(b_1) \\ \vdots \\ y_1^{[n-1]}(b_1) \end{pmatrix} \\ & + A_2\begin{pmatrix} [y_2, u_{21}]_2(a_2) \\ \vdots \\ [y_2, u_{2m_2}]_2(a_2) \end{pmatrix} \\ & \left. + B_2\begin{pmatrix} y_2(b_2) \\ \vdots \\ y_2^{[n-1]}(b_2) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}\right\}. \end{aligned} \tag{77}$$

*Proof.* Since $M_j$ are regular at $b_j$, for any $y = \{y_1, y_2\} \in D_Q$ the limits

$$y_j^{[s]}(b_j) = \lim_{t\longrightarrow b_j^-} y_j^{[s]}(t), \quad j = 1, 2, \tag{78}$$

exist and are finite for $s = 0, 1, \ldots, n-1$.

When $d_{12} = d_{22} = n$, for matrices $B_1, B_2$ determined by Theorem 10, we let

$$\begin{aligned} B_1 &= (-1)^k R_1^*C_n, \\ B_2 &= (-1)^k R_2^*C_n, \end{aligned} \tag{79}$$

where

$$R_j^* = \begin{pmatrix} \overline{w}_{1j} & \cdots & \overline{w}_{1j}^{[n-1]} \\ \cdots & \cdots & \cdots \\ \overline{w}_{dj} & \cdots & \overline{w}_{dj}^{[n-1]} \end{pmatrix}, \quad j = 1, 2. \tag{80}$$

Then we have

$$([y_j, w_{lj}]_j(b_j))_{d\times 1} = (-1)^{k+1}R_1^*C_n\begin{pmatrix} y_j(b_j) \\ \vdots \\ y_j^{[n-1]}(b_j) \end{pmatrix}, \quad j = 1, 2, \tag{81}$$

$$\begin{aligned} ([w_{i1}, w_{l1}]_1(b_1))^T_{d\times d} &= (-1)^{k+1}R_1^*C_nR_1 \\ &= (-1)^{k+1}R_1^*C_nC_n^{-1}C_nR_1 \\ &= (-1)^{k+1}(R_1^*C_n)(-C_n)C_nR_1 = (-1)^{k+1}B_1C_nB_1^*, \end{aligned}$$

and

$$([w_{i2}, w_{l2}]_2(b_2))^T_{d\times d} = (-1)^{k+1}B_2C_nB_2^*. \tag{82}$$

So by Theorem 10, we may complete the proof. □

*Remark 12.* In the minimal deficiency case $d_{11} = k, m_1 = 0, d_{21} = k, m_2 = 0$ the terms involving $A_1$ and $A_2$ in (77) drop out and Theorem 11 reduces to the self-adjoint boundary conditions at the regular endpoints $b_1$ and $b_2$:

$$B_1 \begin{pmatrix} y_1(b_1) \\ \vdots \\ y_1^{[n-1]}(b_1) \end{pmatrix} + B_2 \begin{pmatrix} y_2(b_2) \\ \vdots \\ y_2^{[n-1]}(b_2) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (83)$$

where the $n \times n$ complex matrices $B_1$ and $B_2$ satisfy $\operatorname{rank}(B_1, B_2) = n$ and $B_1 C_n B_1^* + B_2 C_n B_2^* = 0$. In this case there are no conditions required or allowed at the endpoints $a_1$ and $a_2$.

**Theorem 13.** *Let $M_j$ be two C-symmetric differential expressions of order $n = 2k$ on $(a_j, b_j)$, $j = 1, 2$ and $w_j$ a positive function in $L(a_j, b_j)$ and assume that each endpoint is regular. Then a linear submanifold $D(T)$ of $D_Q$ is the domain of a self-adjoint extension $T$ of $T_{Q,0}$ if and only if there exist a complex $2n \times n$ matrix $A_1$ and a complex $2n \times n$ matrix $B_1$ and a complex $2n \times n$ matrix $A_2$ and a complex $2n \times n$ matrix $B_2$ such that the following three conditions hold:*

(1) $\operatorname{rank}(A_1, B_1, A_2, B_2) = 2n$;

(2) $A_1 C_n A_1^* - B_1 C_n B_1^* + A_2 C_n A_2^* - B_2 C_n B_2^* = 0$;

(3) $D(T) = \{y = \{y_1, y_2\} \in D_Q$:

$$A_1 \begin{pmatrix} y_1(a_1) \\ \vdots \\ y_1^{[n-1]}(a_1) \end{pmatrix} + B_1 \begin{pmatrix} y_1(b_1) \\ \vdots \\ y_1^{[n-1]}(b_1) \end{pmatrix} + A_2 \begin{pmatrix} y_2(a_2) \\ \vdots \\ y_2^{[n-1]}(a_2) \end{pmatrix} + B_2 \begin{pmatrix} y_2(b_2) \\ \vdots \\ y_2^{[n-1]}(b_2) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Bigg\}. \quad (84)$$

*Proof.* In this case $d = d_1 + d_2 = d_{11} + d_{12} - n + d_{21} + d_{22} - n = n + n - n + n + n - n = 2n$. And for any $y = \{y_1, y_2\} \in D_Q$ the limits

$$y_j^{[s]}(a_j) = \lim_{t \to a_j^+} y_j^{[s]}(t); \quad y_j^{[s]}(b_j) = \lim_{t \to b_j^-} y_j^{[s]}(t), \quad j = 1, 2, \quad (85)$$

exist and are finite for $s = 0, 1, \ldots, n-1$.

From Lagrange identity in two-interval case (40) we have

$$\int_a^b \overline{z} My\,dx - \int_a^b \overline{Mz}\, y\,dx = [y, z]_a^b = 0, \quad (86)$$

where

$$[y, z]_a^b = \sum_{j=1}^{2} [y_j, z_j]_j (b_j) - [y_j, z_j]_j (a_j) = \sum_{j=1}^{2} \{Z_j^*(b_j) C_n Y_j(b_j) - Z_j^*(a_j) C_n Y_j(a_j)\}. \quad (87)$$

Then

$$D(T) = \{y = \{y_1, y_2\} \in D_Q : A_1 Y_1(a_1) + B_1 Y_1(b_1) + A_2 Y_2(a_2) + B_2 Y_2(b_2) = 0\} \quad (88)$$

is a self-adjoint domain if and only if

$$A_1 C_n A_1^* + A_2 C_n A_2^* = B_1 C_n B_1^* + B_2 C_n B_2^*. \quad (89)$$

□

It is worthy noting that if we set $C_n = E_n$, where

$$E_n = ((-1)^r \delta_{r, n+1-s})_{r,s=1}^{n}, \quad (90)$$

then our $C$-symmetric condition can be reduced to Lagrange symmetric case; therefore, we have the following well-known chracterization.

**Corollary 14** (see [16]). *Let $M_j$ be two Lagrange symmetric differential expressions of order $n = 2k$ on $(a_j, b_j)$, $j = 1, 2$, and assume that each endpoint is regular. Then a linear submanifold $D(T)$ of $D_Q$ is the domain of a self-adjoint extension $T$ of $T_{Q,0}$ if and only if there exist a complex $2n \times n$ matrix $\widetilde{A}_1$ and a complex $2n \times n$ matrix $\widetilde{A}_2$ and a complex $2n \times n$ matrix $\widetilde{A}_3$ and a complex $2n \times n$ matrix $\widetilde{A}_4$ such that the following three conditions hold:*

(1) $\operatorname{rank}(\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4) = 2n$;

(2) $\sum_{k=1}^{4} (-1)^{k+1} \widetilde{A}_k E_n \widetilde{A}_k^* = 0$;

(3) $D(T) = \{y = \{y_1, y_2\} \in D_Q$:

$$\widetilde{A}_1 \begin{pmatrix} y_1(a_1) \\ \vdots \\ y_1^{[n-1]}(a_1) \end{pmatrix} + \widetilde{A}_2 \begin{pmatrix} y_1(b_1) \\ \vdots \\ y_1^{[n-1]}(b_1) \end{pmatrix} + \widetilde{A}_3 \begin{pmatrix} y_2(a_2) \\ \vdots \\ y_2^{[n-1]}(a_2) \end{pmatrix} + \widetilde{A}_4 \begin{pmatrix} y_2(b_2) \\ \vdots \\ y_2^{[n-1]}(b_2) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Bigg\}. \quad (91)$$

This corollary is the part IV of theorem 4.12 in paper [16].

## 5. Examples

A number of examples are given here to account for the main theorems. These examples include the interactions between the singular endpoints: interactions which generate self-adjoint operators. The self-adjoint interactions involve jump discontinuous of the Lagrange bracket of solutions at singular interior points. Here, let us take the case of $n = 4$ as an example. Since the conditions when $0 \le d \le 2$ are the same as in the one-interval case, we give examples for $3 \le d \le 8$ in the following.

*Example 1.* Assume $d_{11} = d_{12} = 3, d_{21} = 2, d_{22} = 3$. Then $d_1 = d_{11} + d_{12} - 4 = 2, d_2 = d_{21} + d_{22} - 4 = 1, d = d_1 + d_2 = 3$ and $m_1 = 2d_{11} - 4 = 2, m_2 = 2d_{21} - 4 = 0, n_1 = 2d_{12} - 4 = 2, n_2 = 2d_{22} - 4 = 2$. If $C_2 = \left(\begin{smallmatrix} 0 & c \\ -\bar{c} & 0 \end{smallmatrix}\right)$ satisfy $C_2^{-1} = -C_2 = C_2^*$, i.e., $c\bar{c} = 1$, let

$$\begin{gathered} A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & c \end{pmatrix}, \\ A_2 = O, \\ B_2 = \begin{pmatrix} 1 & 0 \\ \bar{c} & 1 \\ 0 & 0 \end{pmatrix}, \\ c \in \mathbb{C}. \end{gathered} \tag{92}$$

Then

$$\begin{aligned} &[y_1, u_{11}]_1 (a_1) + [y_2, v_{21}]_2 (b_2) = 0, \\ &[y_1, u_{12}]_1 (a_1) + [y_2, v_{22}]_2 (b_2) = -\bar{c} [y_2, v_{21}]_2 (b_2), \\ &[y_1, v_{11}]_1 (b_1) + c [y_1, v_{12}]_1 (b_1) = 0, \end{aligned} \tag{93}$$

and (93) is a self-adjoint boundary condition. Furthermore we notice that there is one separated singular boundary condition at $b_1$, one singular 'continuity' boundary condition and one singular jump boundary condition; these singular conditions involve the Lagrange bracket.

*Example 2.* Set $d_{11} = d_{12} = d_{21} = d_{22} = 3$. Then $d_1 = d_2 = 2$, $d = 4$ and $m_1 = m_2 = n_1 = n_2 = 2$. Let

$$(A_1, B_1, A_2, B_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \bar{c} & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \bar{c} & -1 & 0 & 0 \end{pmatrix}, \tag{94}$$

then (94) satisfies condition (1) and (2) in Theorem 10. Therefore the following conditions are self-adjoint boundary conditions and all of them involve interactions between singular endpoints, i.e., interactions between Lagrange brackets.

$$\begin{aligned} [y_1, u_{11}]_1 (a_1) &= [y_2, v_{21}]_2 (b_2), \\ [y_1, u_{12}]_1 (a_1) - [y_2, v_{22}]_2 (b_2) &= -\bar{c} [y_2, v_{21}]_2 (b_2), \\ [y_1, v_{11}]_1 (b_1) &= [y_2, u_{21}]_2 (a_2), \\ [y_1, v_{12}]_1 (b_1) - [y_2, u_{22}]_2 (a_2) &= -\bar{c} [y_2, u_{21}]_2 (a_2), \end{aligned} \tag{95}$$

$$c \in \mathbb{C}. \tag{96}$$

Here $C_2$ has the form as Example 1.

*Example 3.* Assume $d_{11} = 2, d_{12} = 4, d_{21} = 4, d_{22} = 3$. Then $d_1 = 2, d_2 = 3, d = 5$ and $m_1 = 0, m_2 = 4, n_1 = 4, n_2 = 2$. Let $C_2, C_4$ satisfy $C^{-1} = -C = C^*$; i.e., $C_2$ has the form as Example 2 and

$$\begin{gathered} C_4 = \begin{pmatrix} 0 & 0 & 0 & c_1 \\ 0 & 0 & c_2 & 0 \\ 0 & -\bar{c}_2 & 0 & 0 \\ -\bar{c}_1 & 0 & 0 & 0 \end{pmatrix}, \\ c\bar{c} = 1, \\ c_1\bar{c}_1 = c_2\bar{c}_2 = 1. \end{gathered} \tag{97}$$

Then

$$\begin{aligned} &[y_2, v_{21}]_2 (b_2) + c [y_2, v_{22}]_2 (b_2) = 0, \\ &[y_1, v_{11}]_1 (b_1) = [y_2, u_{21}]_2 (a_2), \\ &[y_1, v_{12}]_1 (b_1) - [y_2, u_{22}]_2 (a_2) = \bar{c}_1 [y_2, u_{21}]_2 (a_2), \\ &[y_1, v_{13}]_1 (b_1) = [y_2, u_{23}]_2 (a_2), \\ &[y_1, v_{14}]_1 (b_1) - [y_2, u_{24}]_2 (a_2) = -c_2 [y_2, u_{23}]_2 (a_2). \end{aligned} \tag{98}$$

are self-adjoint boundary conditions and there are two singular "continuity" conditions, one separated singular boundary condition at $b_2$ and two interior coupled singular jump conditions.

*Example 4.* Assume $d_{11} = d_{22} = 3, d_{12} = d_{21} = 4$. Then $d_1 = d_2 = 3, d = 6$ and $m_1 = 2, m_2 = 4, n_1 = 4, n_2 = 2$. Then we have two-interval self-adjoint boundary conditions below:

$$\begin{aligned} &[y_2, u_{21}]_2 (a_2) - [y_2, u_{22}]_2 (a_2) = 0, \\ &c_2 [y_2, u_{23}]_2 (a_2) + c_1 [y_2, u_{24}]_2 (a_2) = 0, \\ &[y_1, v_{11}]_1 (b_1) - [y_1, v_{12}]_1 (b_1) = 0, \\ &c_2 [y_1, v_{13}]_1 (b_1) + c_1 [y_1, v_{14}]_1 (b_1) = 0, \\ &[y_1, u_{11}]_1 (a_1) = [y_2, v_{21}]_2 (b_2), \\ &[y_1, u_{12}]_1 (a_1) - [y_2, v_{22}]_2 (b_2) = -\bar{c} [y_2, v_{21}]_2 (b_2). \end{aligned} \tag{99}$$

Here the complex numbers $c, c_1, c_2$ are shown as Example 3.

*Example 5.* In this example we consider the case: $d = 7$. Assume $d_{11} = 3, d_{12} = 5, d_{21} = 5, d_{22} = 2$. Then $d_1 = 4, d_2 = 3, d = 7$ and $m_1 = 2, m_2 = 6, n_1 = 6, n_2 = 0$. Let $C_2, C_4, C_6$ satisfy $C^{-1} = -C = C^*$; i.e., $C_2, C_4$ has the form as Example 3 and set $C_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & -\bar{c}_3 & 0 & 0 & 0 \\ 0 & -\bar{c}_2 & 0 & 0 & 0 & 0 \\ -\bar{c}_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The following self-adjoint boundary conditions consist of one separated singular condition at $a_1$, three singular "continuity" conditions, and three singular jump conditions.

$$\begin{aligned} &\bar{c}\,[y_1, u_{11}]_1(a_1) + [y_1, u_{12}]_1(a_1) = 0, \\ &[y_2, u_{21}]_2(a_2) = [y_1, v_{11}]_1(b_1), \\ &[y_1, v_{12}]_1(b_1) - [y_2, u_{22}]_2(a_2) = \bar{c}_1\,[y_2, u_{21}]_2(a_2), \\ &[y_2, u_{23}]_2(a_2) = [y_1, v_{13}]_1(b_1), \\ &[y_1, v_{14}]_1(b_1) - [y_2, u_{24}]_2(a_2) = -\bar{c}_3\,[y_2, u_{23}]_2(a_2), \\ &[y_2, u_{25}]_2(a_2) = [y_1, v_{15}]_1(b_1), \\ &[y_1, v_{16}]_1(b_1) - [y_2, u_{26}]_2(a_2) = -c_2\,[y_2, u_{25}]_2(a_2). \end{aligned} \tag{100}$$

*Example 6.* In this example we set $C_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$. Assume $d_{11} = d_{12} = 4, d_{21} = d_{22} = 4$. Then $d_1 = d_2 = 4, d = 8$ and $m_1 = m_2 = 4, n_1 = n_2 = 4$. The following boundary conditions feature separated self-adjoint boundary conditions at all four endpoints:

$$\begin{aligned} &[y_1, u_{11}]_1(a_1) - i\,[y_1, u_{12}]_1(a_1) = 0, \\ &\qquad i\,[y_1, u_{13}]_1(a_1) - [y_1, u_{14}]_1(a_1) = 0, \\ &[y_1, v_{11}]_1(b_1) - i\,[y_1, v_{12}]_1(b_1) = 0, \\ &\qquad i\,[y_1, v_{13}]_1(b_1) - [y_1, v_{14}]_1(b_1) = 0, \\ &[y_2, u_{21}]_2(a_2) - i\,[y_2, u_{22}]_2(a_2) = 0, \\ &\qquad i\,[y_2, u_{23}]_2(a_2) - [y_2, u_{24}]_2(a_2) = 0, \\ &[y_2, v_{21}]_2(b_2) - i\,[y_2, v_{22}]_2(b_2) = 0, \\ &\qquad i\,[y_2, v_{23}]_2(b_2) - [y_2, v_{24}]_2(b_2) = 0. \end{aligned} \tag{101}$$

## 6. Conclusion

This paper characterize all self-adjoint domains for two-interval even order $C$-symmetric differential operators in direct sum spaces, where both endpoints in each interval are singular, and there is not any singular point in each interval. And this characterization can be reduced to the regular case. Moreover the characterization in this paper is generalization of previous results for Lagrange symmetric case. So our work is valued.

## Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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