Hindawi
Discrete Dynamics in Nature and Society
Volume 2019, Article ID 6903019, 11 pages
https://doi.org/10.1155/2019/6903019

Research Article

Pricing Stock Loans with the CGMY Model

Congyin Fan [1] and Chunhao Zhou [2]

[1] *Guizhou University of Commerce, Guiyang, Guizhou 550014, China*
[2] *School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China*

Correspondence should be addressed to Congyin Fan; fcy8843@163.com

Received 23 June 2019; Revised 31 August 2019; Accepted 6 September 2019; Published 27 December 2019

Academic Editor: Francisco R. Villatoro

The empirical research shows that the log-return of stock price in finance market rejects the normal distribution and admits a subclass of the asymmetric distribution. Hence, the pricing problem of stock loan is investigated under the assumption that the log-return of stock price follows the CGMY process in this work. Under this framework, the pricing model of stock loan can be described by a free boundary condition problem of space-fractional partial differential equation (FPDE). First of all, in order to change the original model defined in a fixed domain, a penalty term is introduced, and then a first order fully implicit difference schemes is developed. Secondly, based on the numerical scheme, we prove the stock loan value generated by our method does not fall below the value obtained when the contract of stock loan is exercised early. Finally, the numerical experiments are implemented and the impacts of key parameters in the CGMY model on the value and optimal redemption price of stock loan are analyzed, and some reasonable explanation should be given.

1. Introduction

A stock loan can be treated as a contract between two parties: the bank or other financial institution (the lender), and a client (the borrower). The borrower obtains a loan from the lender with their stocks as collateral, and the contract offers the borrower a right to redeem the stock at any valid time. But he/she should pay off the loan as well as the cumulative loan. As described in Ref. [1], the risk aversion investors can use stock loan to transfer the risk of holding stock to the financial institutions, and this kind of derivatives also can establish market liquidity. Therefore, the stock loan is one of the most important financial derivatives in the financial market and currently, both Florida Mortgage Corporation and Shelly Bay Capital specialize in providing stock loan services.

The pricing problem about the stock loan has been a popular topic in the academic since the first research paper [1] about the stock loan published in 2007. In this literature, the risk asset was assumed to follow the Geometric Brownian motion (GBM) and the contract of stock loan is finite maturity. Following this publication, more and more researchers paid attentions to the academical topic of stock loan valuation with other conditions. For instance, Lu and Putri [2] considered the pricing problem of stock loan with finite maturity margin. Under the framework of hyper-exponential jump diffusion model, Cai and Sun [3] studied the value and optimal redemption price of stock loan with infinite and finite maturity. The stock loan with finite maturity was also investigated in Ref. [4] under the case that risk-free interest rate follows the Rednleman–Bartter model without drift term.

It is well known that the classical Black–Scholes (B–S) model was established under a lot of strict assumptions, such as the log-return of stock price follows the nonmoral distribution, frictionless, and so on. However, these strict assumptions do not accord with the dynamic process of stock price in real financial market. The empirical research shows that discontinuities or jumps are believed to be an indispensable element of financial risk-asset price (see e.g., [5–7]) and the log-return of risk asset appear to be “asymmetric distribution” and “leptokurtic distribution” (see e.g., [8–12]). For this reason, for capturing these characters of stock price, many scholars use other complex stochastic process to drive the stock price. Prominent examples including the CEV model [13], GEV model [14], KoBoL model [15], and so on. A frequently used stochastic process is called CGMY process and it was presented by Carr et al. [16] with the aim to provide a model for the dynamic of equity log-returns. This model is rich enough to accommodate jumps of finite or infinite activity, and finite or infinite variation. In fact, this stochastic process is a particular type of pure jump Lévy process with four key parameters (e.g., C, G, M, and Y) which control its essential characteristics.The parameter C may be viewed as a measure of the overall level of activity. The aggregate activity level should be calibrated through movements under the case of keeping the other three parameters constant and integrating over all that moves exceeding a small level. Parameters G and M control the rate of exponential decay on the right and left of the Lévy density, respectively, leading to skewed distributions when they are unequal. Parameter Y is particularly useful in characterizing the fine structure of the stochastic process.The CGMY process has infinite variation and finite quadratic variation if $Y \in (1,2)$. At present, many papers discussed pricing problem of financial derivatives based on the CGMY process. Ballotta and Kyriakou [17] obtained the option price by using the Monte Carlo method in this case. Zhang et al. [18] presented a fast numerical method to double barrier option under framework of CGMY. Chen et al. [19] investigated European option and they obtained the explicit closed-form analytical based on the model.

A better refernce value to the holder of stock loan can be provided in this case of CGMY model, a better reference value to the holder of stock loan is provided; therefore, in this paper we intend to investigate the pricing of stock loans with finite-maturity under the CGMY model. The pricing model is free-boundary problem of partial differential equation with tempered fractional derivatives. The tempered fractional derivatives and nonlinearity associated with the free boundary add the difficulty of solving the pricing model. Aiming at this problem, we consider a penalty method in which the free boundary is removed by adding a small and continuous penalty term to the governing equation. Therefore, the mathematical model should be solved on a fixed domain. Even if on the fixed domain, the analytical solution is seldom available so that a full-implicit scheme is employed. Based on our numerical scheme, we prove that the stock loan value generated by the penalty method cannot fall below the value obtained when the stock loan is exercised early, and we also do simulation to verify the result.

This study has five main sections. Section 1 provides the introduction. The pricing model of stock loan is set in Section 2 and the numerical method is introduced in Section 3. The simulation and discussions are presented in Section 4, and conclusion is displayed in Section 5.

2. Mathematical Model

In this section, the governing equation of the stock loan with finite maturity, which is a partial differential equation with fractional derivative, is presented under the CGMY framework. And then, financially, the corresponding free moving boundary and terminal conditions will be given to complete the pricing model.

The CGMY process assumes the log value of stock price $(x_t = \ln(S_t))$ follows a stochastic differential equation under the risk-neutral measure $\mathbb{Q}$ [16]

$$dx_t = (r - D - w)dt + \sigma dL_t^{CGMY}, \tag{1}$$

where r, D, and t are the risk free interest rate, the dividend and the current time, respectively. dL_t^{CGMY} is a stochastic variable and its characteristic function is

$$\begin{aligned}&\phi_{CGMY}(u, dt; C, G, M, Y)\\&\quad = \exp\left\{dtC\Gamma(-Y)\left[(M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right]\right\}.\end{aligned} \tag{2}$$

In addition,

$$w = C\Gamma(Y)\left\{(M-1)^Y - M^Y + (G+1)^Y - G^Y\right\}, \tag{3}$$

where the parameters C, G, M and Y are constants, and $Y \in (1,2), C > 0, G \geq 0, M \geq 0$.

2.1. Governing Equation. In fact, as described in Ref. [1] a stock loan problem can be regarded as an American call option with a time-varying strike price $Ke^{\gamma t}$, where K is the principal and γ is the continuously compounded loan interest rate and $\gamma \geq r$ in general. Therefore, the payoff function of stock loan at maturity can be written as

$$\Pi(x_T, T; \alpha) = \max\left(\exp(x_T) - Ke^{\gamma T}, 0\right). \tag{4}$$

In addition, typically as same as the American call, at each valid time there is a particular value of the underlying, which yields a boundary between two regions: in one side(called exercising region) the investor exercise the contract of stock loan and in other side (called holding region) one should hold this contract. The particular value of the asset S_f is called optimal redemption price and denoted by $x_f\left(x_f = \ln S_f\right)$. Then, based on the principle of non-arbitrage pricing, in the holding region, the value of stock loan at time t satisfies

$$V(x_t, t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\Pi(x_T, T) | \mathscr{F}_t\right], \tag{5}$$

where $\mathbb{E}_{\mathbb{Q}}$ is the conditional mean operator under the measure $\mathbb{Q}$ and $\mathscr{F}_t$ denotes the information flow at time t. Famous authors, Cartea [20] and Del-Castillo-Negrete obtained the governing equation of $V(x,t)$ through the Fourier transformation as follows:

$$\begin{aligned}&\frac{\partial V}{\partial t} + (r - D - w)\frac{\partial V}{\partial x} + C\Gamma(-Y)e^{Mx}{}_xD_{\infty}^{Y}\left(e^{-Mx}V\right)\\&\quad + C\Gamma(-Y)e^{-Gx}{}_xD_{\infty}^{Y}\left(e^{Gx}V\right) = \left[r + C\Gamma(-Y)\left(M^Y + G^Y\right)\right]V,\end{aligned} \tag{6}$$

where $x \in \left(-\infty, x_f\right]$, and

$${}_{-\infty}D_x^Y V(x,t) = \frac{1}{\Gamma(2-Y)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{x} V(y,t)(x-y)^{2-Y-1}dy, \tag{7}$$

$${}_xD_{\infty}^Y V(x,t) = \frac{(-1)^n}{\Gamma(2-Y)}\frac{\partial^2}{\partial x^2}\int_{x}^{\infty} V(y,t)(x-y)^{2-Y-1}dy, \tag{8}$$

are left and right sided Riemann–Liouville fractional derivatives.

In fact, the Y-order left and right fractional derivatives are closely related to CGMY process. The fractional differentiation is nonlocal, which relates to the stock loan value in the stopping region $\left(-\infty, x_f\right]$. This is where the stock loan value is equal to its intrinsic value. The nonlocalness of this fractional operator means that over a time step Δt, the price of risk-asset S_t should diffuse to value $S_{t+\Delta t}$ far away from S_t, provides a way in efficiently simulating the existence of jumps of the risk-asset.

2.2. The Boundary Conditions. In this subsection, to complete the pricing system, a set of appropriate boundary conditions will be given. First of all, as a rational investor, she/he does not redeem stock if the level of stock price is very low and here we assume the stock price S_t is close to zero, i.e., $x = \ln S_t \to -\infty$, then one can obtain a boundary condition as

$$\lim_{x\to-\infty} V(x,t) = 0. \tag{9}$$

On the other hand, the continuity of stock loan value $V(x,t)$ at optimal redemption price x_f should still be retained to ensure the smooth pasty of the stock loan value across the free boundary. For these reasons, we still impose the two boundary conditions as follows:

$$V\left(x_f,t\right) = e^{x_f} - Ke^{\gamma t}, \tag{10}$$

$$\frac{\partial V\left(x_f,t\right)}{\partial x} = S_f = e^{x_f}. \tag{11}$$

And the terminal condition is the payoff function

$$V(x,T) = \max\left(e^x - Ke^{\gamma T}, 0\right). \tag{12}$$

To sum up, the complete pricing model for stock loan with finite maturity under CGMY process can be written as

$$\begin{aligned}\frac{\partial V}{\partial t} &+ (r-D-w)\frac{\partial V}{\partial x} + C\Gamma(-Y)e^{Mx}\,{}_xD_{\infty}^{Y}\left(e^{-Mx}V\right)\\ &+ C\Gamma(-Y)e^{-Gx}\,{}_xD_{\infty}^{Y}\left(e^{Gx}V\right) = \left[r + C\Gamma(-Y)\left(M^Y+G^Y\right)\right]V,\end{aligned} \tag{13}$$

$$V\left(x_f,t\right) = e^{x_f} - Ke^{\gamma t}, \tag{14}$$

$$\frac{\partial V\left(x_f,t\right)}{\partial x} = S_f = e^{x_f}, \tag{15}$$

$$\lim_{x\to-\infty} V(x,t) = 0, \tag{16}$$

$$V(x,T) = \max\left(e^x - Ke^{\gamma T}, 0\right). \tag{17}$$

It must be noted that, a rational investor cannot redeem the stock if the value of redemption is less than the holding value; therefore, the value V of the stock loan should satisfy the following inequality

$$V(x,t) \geq \max\left(e^x - Ke^{\gamma t}, 0\right), \tag{18}$$

for all $x \leq x_f$ and $0 \leq t \leq T$.

2.3. Model Normalization. It must be noted that there is a time-factor $e^{\gamma t}$ in the boundary conditions (14) and (17). And we find the time-factor should influence numerical results, hence, we introduce a new variable system to the model (14)–(17) for avoiding the effect of $e^{\gamma t}$. Take

$$z = x - \gamma t,\ \ U(z,t) = e^{\gamma t}V(x,t),\ \ z_f = x - \gamma t, \tag{19}$$

then

$$\frac{\partial V}{\partial t} = \gamma e^{\gamma t}U + e^{\gamma t}\frac{\partial U}{\partial t} - \gamma e^{\gamma t}\frac{\partial U}{\partial z}, \tag{20}$$

$$\frac{\partial U}{\partial x} = e^{\gamma t}\frac{\partial U}{\partial z}, \tag{21}$$

$$\begin{aligned}e^{Mx}\,{}_xD_{x_f}^{Y}\left(e^{-Mx}V(x,t)\right) &= \frac{(-1)^n e^{Mx}}{\Gamma(n-Y)}\frac{d^n}{dx^n}\int_x^{x_f}\frac{e^{-M\xi}V(\xi,t)}{(\xi-x)^{Y-n+1}}d\xi\\ &= \frac{(-1)^n e^{M(z+\gamma t)}}{\Gamma(n-Y)}\frac{d^n}{dx^n}\int_{x+\gamma t}^{x}\frac{e^{-M\xi}V(\xi,t)}{(\xi-z-\gamma t)^{Y-n+1}}d\xi.\end{aligned} \tag{22}$$

let $y = \xi - \gamma t$, therefore we obtain

$$\begin{aligned}&\frac{(-1)^n e^{M(z+\gamma t)}}{\Gamma(n-Y)}\frac{d^n}{dz^n}\int_z^{z_f}\frac{e^{-M(y+\gamma t)}V(\xi,t)}{(y-z)^{Y-n+1}}dy\\ &\quad= \frac{(-1)^n e^{M(z+\gamma t)}}{\Gamma(n-Y)}\frac{d^n}{dz^n}\int_z^{z_f}\frac{e^{\gamma t}e^{-M(y+\gamma t)}e^{-\gamma t}V(\xi,t)}{(y-z)^{Y-n+1}}dy\\ &\quad= \frac{(-1)^n e^{M(z+\gamma t)}}{\Gamma(n-Y)}\frac{d^n}{dz^n}\int_z^{z_f}\frac{e^{\gamma t}e^{-M(y+\gamma t)}U(y,t)}{(y-z)^{Y-n+1}}dy\\ &\quad= e^{Mz+\gamma t}\,{}_zD_{z_f}^{Y}\left(e^{-Mz}U(z,t)\right).\end{aligned} \tag{23}$$

And

$$\begin{aligned}&e^{-Gx}\,{}_{-\infty}D_{x_f}^{Y}\left(e^{Gx}V(x,t)\right)\\ &\quad= \frac{(-1)^n e^{-Gx}}{\Gamma(n-Y)}\frac{d^n}{dx^n}\int_{-\infty}^{x}\frac{e^{G\xi}V(\xi,t)}{(x-\xi)^{Y-n+1}}d\xi\\ &\quad= \frac{(-1)^n e^{-G(z+\gamma t)}}{\Gamma(n-Y)}\frac{d^n}{dx^n}\int_{-\infty}^{x}\frac{e^{G\xi}V(\xi,t)}{(z+\gamma t-\xi)^{Y-n+1}}d\xi,\end{aligned} \tag{24}$$

let $y = \xi - \gamma t$, so one obtains

$$\begin{aligned}&\frac{(-1)^n e^{-G(z+\gamma t)}}{\Gamma(n-Y)}\frac{d^n}{dz^n}\int_{-\infty}^{z}\frac{e^{G(y+\gamma t)}V(\xi,t)}{(z-y)^{Y-n+1}}d\xi\\ &\quad= \frac{(-1)^n e^{-G(z+\gamma t)}}{\Gamma(n-Y)}\frac{d^n}{dz^n}\int_{-\infty}^{z}\frac{e^{\gamma t}e^{G(y+\gamma t)}e^{-\gamma t}V(\xi,t)}{(z-y)^{Y-n+1}}dy\\ &\quad= \frac{(-1)^n e^{-G(z+\gamma t)}}{\Gamma(n-Y)}\frac{d^n}{dz^n}\int_{-\infty}^{z}\frac{e^{\gamma t}e^{G(y+\gamma t)}U(y,t)}{(z-y)^{Y-n+1}}dy\\ &\quad= e^{-Gz+\gamma t}\,{}_{-\infty}D_z^{Y}\left(e^{Gz}U(z,t)\right).\end{aligned} \tag{25}$$

Now, substituting Eq. (20), (21), (23), (25) into Eq. (13) and after transforming the boundary conditions (14)–(17) through Eq. (19), the PDE system of function $U(z,t)$ can be obtained as follows:

$$\frac{\partial U}{\partial t}+a\frac{\partial U}{\partial z}+C\Gamma(-Y)\left({}_{z}D_{z_f}^{Y,M}U+{}_{-\infty}D_{z}^{Y,G}U\right)+(\gamma-r)U=0, \quad (26)$$

$$\lim_{z\to-\infty}U(z,t)=0, \quad (27)$$

$$U\left(z_f,t\right)=e^{z_f}-K, \quad (28)$$

$$U'\left(z_f,t\right)=e^{z_f}, \quad (29)$$

$$U(z,T)=\max(e^{z}-K,0), \quad (30)$$

where $a=r-w-\gamma-D, t\in\left(-\infty,z_f\right]$,

$${}_{z}D_{z_f}^{Y,M}U={}_{z}D_{z_f}^{Y}U-M^{Y}U, \quad (31)$$

$${}_{-\infty}D_{z}^{Y,G}U={}_{-\infty}D_{z}^{Y}U-M^{G}U. \quad (32)$$

And the equation (18) should be changed as

$$U(z,t)\ge\max(e^{z}-K,0). \quad (33)$$

Mathematically, the function $U(z,t)$ in the model (26)–(30) can be viewed as an American call option with strike price K and free-boundary e^{z_f}. According to the property of American call, the values of $U(z,t)$ are not less than the payoff $\max(e^{z}-K,0)$. And the free-boundary e^{z_f} is monotonically increasing with respect to time to expiry of option contract. In the next section, a numerical scheme based on the penalty method will be proposed to solve this model.

3. Numerical Method

In this section, we first consider a penalty approach in which the free moving boundary is removed by adding a small and continuous penalty term, so that the stock loan pricing problem can be solved on a fixed domain. Then, a different scheme is proposed and an effective nonlinear Newton-type iteration strategy is employed. Furthermore, due to the coefficient matrixes of the finally linear system, which contains the full matrix with Toeplitz structure, the fast biconjugate gradient stabilized method (FBi-CGSTAB) is used to solve our system.

3.1. Model Transformation. In this paper the penalty function is defined as

$$\frac{\varepsilon H}{U_{\varepsilon}(z,t)+\varepsilon-q(z)}, \quad (34)$$

where ε is a regularization constant and $0<\varepsilon\ll 1$, H is a constant,

$$q(z)=e^{z}-K. \quad (35)$$

If adding the penalty term (34) to the FPDE in (26), one can obtain a nonlinear FPDE system defined on a fixed domain as follows:

$$\begin{aligned}\frac{\partial U}{\partial t}+a\frac{\partial U}{\partial z}&+C\Gamma(-Y)\left({}_{z}D_{z_f}^{Y,M}U+{}_{-\infty}D_{z}^{Y,G}U\right)\\&+(\gamma-r)U+\frac{\Delta t\varepsilon H}{U_{\varepsilon}+\varepsilon-q}=0,\end{aligned} \quad (36)$$

$$\lim_{z\to-\infty}U_{\varepsilon}(z,t)=0, \quad (37)$$

$$U_{\varepsilon}(z_{\max},t)=e^{z_{\max}}-K, \quad (38)$$

$$U_{\varepsilon}=\max(e^{z}-K,0), \quad (39)$$

where $\exp(z_{\max})$ denotes the maximum stock price, $z\in\left(-\infty,z_{\max}\right]$. And according to the results in Ref. [21], the maximum value of stock is equal to 3 or 4 times of strike price, hence we take $\exp(z_{\max})=3K$ in this paper. In addition, we will omit the subscript ε of function $U_{\varepsilon}(z,t)$ for convenience.

3.2. Difference Scheme. For the FPDE of (37) is nonlinear, it results in the analytical solution of the mathematical model (37)–(39) that is laborious and even impossible to achieve even if the model is defined on the fixed domain $\left(-\infty,z_{\max}\right]\times[0,T]$. Therefore, an effective numerical algorithm should be preferred.

Specifically, taking $\Delta z>0$ as spatial step such that $N_1\Delta z=z_{\max}$, where N_1 is a positive integer, and placing N_2+1 uniform grids in the time t direction, namely $\Delta t=T/N_2$, that is

$$z_j=(j-1)\Delta z,\quad t_i=(i-1)\Delta t, \quad (40)$$

and $j=\cdots,-2,-1,0,1,2,N_1+1; i=1,2,\ldots,N_2+1$. For the first order spatial and time, we use the following difference scheme

$$\frac{\partial U\left(z_j,t_i\right)}{\partial z}=\frac{U\left(z_j,t_i\right)-U\left(z_{j-1},t_i\right)}{\Delta z}+o(\Delta z), \quad (41)$$

$$\frac{\partial U\left(z_j,t_i\right)}{\partial t}=\frac{U\left(z_j,t_{i+1}\right)-U\left(z_j,t_i\right)}{\Delta t}+o(\Delta t). \quad (42)$$

And the fractional derivative can be approximated by the first-order Grnwald–Letnikov formula as [22]

$$\begin{aligned}{}_{-\infty}D_{z}^{Y,G}U_j^i&=\frac{1}{(\Delta z)^Y}\sum_{k=0}^{j+1}g_kU_{j-k+1}^i-\frac{1}{(\Delta z)^Y}\left(\gamma_1e^{\Delta zG}+\gamma_2+\gamma_3e^{-\Delta zG}\right)\\&\cdot\left(1-e^{-\Delta zM}\right)^YU_j^i+o(\Delta z),\end{aligned} \quad (43)$$

$$\begin{aligned}{}_{z}D_{z_{\max}}^{Y,M}U_j^i&=\frac{1}{(\Delta z)^Y}\sum_{k=0}^{N_1-j+1}g_kU_{j+k-1}^i-\frac{1}{(\Delta z)^Y}\left(\gamma_1e^{\Delta zM}+\gamma_2+\gamma_3e^{-\Delta zM}\right)\\&\cdot\left(1-e^{-\Delta zM}\right)^YU_j^i+o(\Delta z),\end{aligned} \quad (44)$$

where U_j^i is the value of function $U(z,t)$ at grid point (z_j, t_i), $g_k (k = 0, 1, 2, \ldots)$ are fractional difference coefficients

$$g_k = (-1)^k \binom{\alpha}{k}, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}, \tag{45}$$

and the parameters $\gamma_i (i = 1, 2, 3)$ satisfy the following system

$$\gamma_1 + \gamma_2 + \gamma_3 = 1, \quad \gamma_1 - \gamma_3 = \frac{Y}{2}. \tag{46}$$

If taking γ_3 as the free variable, the solution of system (46) can be obtained as

$$\gamma_1 = \frac{Y}{2} + \gamma_3, \quad \gamma_2 = \frac{2-Y}{2} - 2\gamma_3. \tag{47}$$

So, the fully implicit difference scheme without the truncation error for (36) is got as follows:

$$\frac{U_j^{i+1} - U_j^i}{\Delta t} + a\frac{U_j^i - U_{j-1}^i}{\Delta z} + (\gamma - r)U_j^i + \frac{\varepsilon H}{U_j^i - \varepsilon + q_j} + \frac{C\Gamma(-Y)}{(\Delta z)^Y}\left(\sum_{k=0}^{N_1-j+2} g_k U_{j+k-1}^i - f_1 U_j^i + \sum_{k=0}^{\infty} g_k U_{j-k+1}^i - f_2 U_j^i\right) = 0, \tag{48}$$

with the boundary and terminal conditions

$$\lim_{j\to-\infty} U_j^i = 0, \quad U_{N_1+1}^i = e^{x_{\max}} - K, \quad U_j^{N_2+1} = \max(e^{x_j} - K, 0), \tag{49}$$

where

$$f_1 = \left(\gamma_1 e^{\Delta z G} + \gamma_2 + \gamma_3 e^{-\Delta z G}\right)\left(1 - e^{-\Delta z G}\right)^Y, \quad f_2 = \left(\gamma_1 e^{\Delta z M} + \gamma_2 + \gamma_3 e^{-\Delta z M}\right)\left(1 - e^{-\Delta z M}\right)^Y. \tag{50}$$

For our proposed difference scheme, we have a discrete analogue form of the important property inherited by the stock loan model. Prior to presenting the proof, we need the following two lemmas.

Lemma 1. *If the parameter $\gamma_3 > 0$, then $f_i (i = 1.2)$ in the Eq. (50) are more than zero.*

The proof process of Lemma 1 is provided in Appendix A.1

Lemma 2 (see [23]). *For $1 < Y < 2$, the coefficients g_k satisfy*

$$g_0 = 1, \quad g_1 = -Y, \quad 0 \leqslant \cdots \leq g_3 \leq g_2 \leq 1, \quad \sum_{k=0}^{\infty} g_k = 0, \quad \sum_{k=0}^{m} g_k < 0, \tag{51}$$

where $m \geq 1$.

In fact, according to Lemma 2, we obtain the following equation:

$$\sum_{k=0,\, k\neq 1}^{N} g_k < Y, \tag{52}$$

where $N \geq 1$.

Theorem 3. *If the time step $\Delta t \leq 1/(\gamma - r)$, $\gamma_3 \geq 0$ and*

$$H \geq 2Y\lambda\left(e^{z_{\max}} + K\right) + |a| e^{z_{\max}} \frac{e^{z_{\max}} - 1}{e^{z_{\max}}} + R(G, M) + (\gamma - r)e^{z_{\max}}, \tag{53}$$

then the approximate stock loan values $\{U_j^i\}$ generated by the scheme Eq. (38) satisfy

$$U_j^i \geq \max(e^{z_j} - K, 0) = \max(q_j, 0), \tag{54}$$

for all i, j, where

$$\lambda = \frac{C\Gamma(-Y)}{(\Delta z)^Y}, \tag{55}$$

$$R(G, M) = C\Gamma(-Y)\left[\left(\gamma_1 e^{z_{\max} G} + \gamma_2 + \gamma_3\right)G^Y + \left(\gamma_1 e^{z_{\max} M} + \gamma_2 + \gamma_3\right)M^Y\right]. \tag{56}$$

For details of proof process, please refer to Appendix A.2.

4. Numerical Examples and Discussions

In this part, the numerical simulation should be executed for verifying the theoretical consideration of our proposed difference scheme. The system (48) is nonlinear, hence a practical Newton-type iteration strategy is given to realize the nonlinear difference scheme. The special structure preserved by the coefficient matrix resulted by difference scheme is carefully exploited. Furthermore, the fractional derivatives usually result in a dense coefficient matrix in the system. It has significant computational and storage requirements. In terms of computational cost; it is very important to use an effective and efficient method to solve our linear system. Therefore, the fast biconjugate gradient stabilized method (FBi-CGSTAB) [18] is employed. Finally, the impacts of important parameters to the optimal redemption price should be analyzed.

Practically, to implement the numerical scheme (48) in computer, the semi-infinite domain $(-\infty, x_{\max}]$ must be truncated into a finite domain:

$$\{[z_{\min}, z_{\max}] \times [0, T]\}, \tag{57}$$

here we take $z_{\min} = \ln(0.0001)$, and now $U(z_{\min}, t) = 0$. In this case, we should redefine the space step $\Delta z = (z_{\max} - z_{\min})/N_2$, and

$$z_j = z_{\min} + (j-1)\Delta z, \quad j = 1, 2, \ldots, N_2 + 1. \tag{58}$$

If we let

$$\beta = 1 - \Delta t(\gamma - r) - \frac{a\Delta t}{\Delta z}, \quad \xi = \frac{a\Delta t}{\Delta z}, \quad \eta = -\frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}, \tag{59}$$

then the matrix of numerical scheme (48) can be written as:

$$\left\{[\beta - \eta(f_1 + f_2)]\mathbf{I} + \xi\mathbf{B} + \eta(\mathbf{A} + \mathbf{A}^{\tau})\mathbf{U}^i\right\} - F\left(\mathbf{U}^i\right) = \mathbf{U}^{i+1} - \mathbf{E}^i, \tag{60}$$

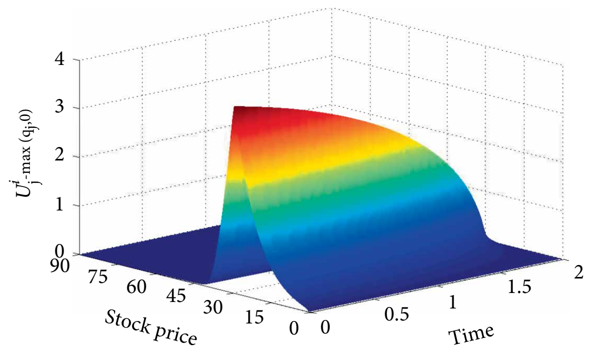

FIGURE 1: Mesh surface of $U(z,t) - q(z)$ with $r = 0.05$, $\gamma = 0.06$, $D = 0.1$, $C = 0.03$, $G = 1.2$, $M = 1$, $Y = 1.5$, $T = 2$, $K = 50$.

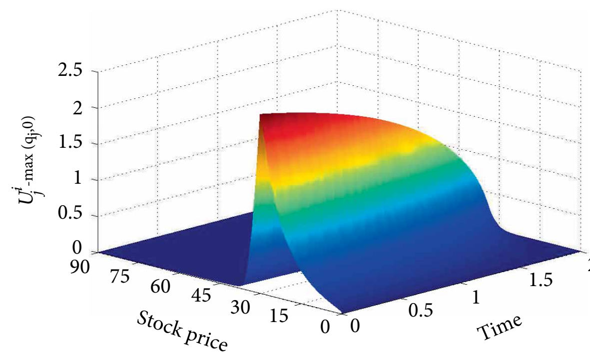

FIGURE 2: Mesh surface of $U(z,t) - q(z)$ with $r = 0.05$, $\gamma = 0.06$, $D = 0.1$, $C = 0.25$, $G = 1.1$, $M = 1.2$, $Y = 1.2$, $T = 2$, $K = 50$.

where

$$\mathbf{U}^i = \left(U_2^i, U_3^i, \ldots, U_{N_2}^i\right), \quad F\left(\mathbf{U}^i\right) = \left(f\left(U_2^i\right), f\left(U_3^i\right), \ldots, f\left(U_{N_2}^i\right)\right), \tag{61}$$

$$f\left(U_j^i\right) = \frac{\Delta t \varepsilon H}{U_j^i + \varepsilon - q_j}, \quad \mathbf{E}^i = \left(\eta g_{N_1}, \eta g_{N_1}, \ldots, \eta g_{N_1} + \beta g_0\right)^{\tau}, \tag{62}$$

and $\mathbf{I}$ denotes the $(N_1 - 1) \times (N_1 - 1)$ identity matrix, $\mathbf{A}^{\tau}$ means the transposition of matrix $\mathbf{A}$. Both $\mathbf{A}$ and $\mathbf{B}$ are Toeplitz matrix

$$\mathbf{A} = \begin{bmatrix} g_1 & g_0 & 0 & \cdots & 0 \\ g_2 & g_1 & 0 & \cdots & 0 \\ g_3 & g_2 & g_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ g_{M-2} & g_{M-3} & \cdots & g_1 & g_0 \\ g_{M-1} & g_{M-2} & \cdots & g_2 & g_1 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{63}$$

In fact, the system (60) is not solved directly for the penalty function $F(U)$ is nonlinear with respect to U, so this nonlinear system can be solved through Newton-type iteration approach, which is provided in Appendix A.3.

4.1. Numerical Testing. To ensure that both the theoretical model and method are feasible, many results in our work must be verified before quantitative analysis; therefore, in this part we will carry out some numerical testing. First of all, the fact that our proposed numerical algorithm satisfies the discrete analogue of the positive constraint, $U(x,t) \geq \max(e^z - K, 0)$ in Theorem 3 will be verified. As shown in Figures 1 and 2, the mesh surface of $U(x,t) \geq \max(e^z - K, 0)$ for the different time and stock price under various parameter setting is presented. And the mesh surface shows that the present difference scheme conserves the inequality $U_j^i \geq \max\left(q_j, 0\right)$ for all i, j.

As described in Section 2.3, the function $U(z,t)$ in the model (36)–(39) can be recognized as an American call option

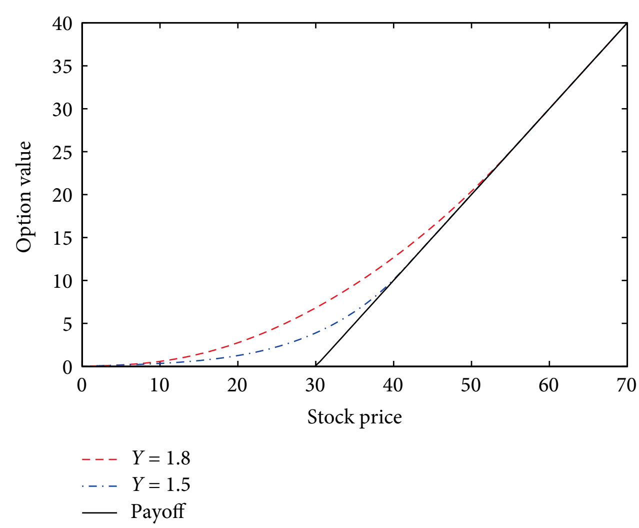

FIGURE 3: American call option prices $U(z,t)$ vs payoff with $r = 0.05$, $\gamma = 0.06$, $C = 1.13$, $G = 1.1$, $M = 1$, $Y = 1.5$, $T = 2$, $K = 50$.

with strike K and optimal exercise boundary e^{z_f}. Then, mathematically, the values of $U(z,t)$ are not less than payoff function $\max(e^z - K, 0)$ and e^{z_f} is increasing with respect to time to expiry. Figures 3 and 4 show the two facts, respectively.

4.2. Impact of Parameters. As described in Section 2, the essential characteristics of CGMY model are controlled by four key parameters, namely, C, G, M, and Y and. In other words these parameters should affect the value and optimal redemption price of stock loan. Hence, in this subsection we employ the proposed numerical method to investigate the impacts of the four parameters on the stock loan value and capture the optimal redemption price for different parameters setting with some reasonable explanation.

Figures 5 and 6 display the behaviours of stock loan valuation and optimal redemption price for different C, respectively. As shown in Figure 5, the mesh surf of $U(x,t)$ is higher with the increase of C. Financially, this phenomenon could be explained from the fact that except keeping the other parameters constant, the aggregate activity level of CGMY process may be calibrated through movements in C and the level is increasing with respect to C. In addition, the stock loan and

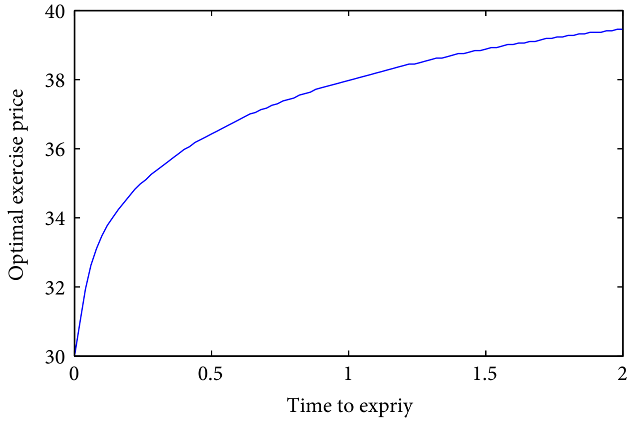

FIGURE 4: Optimal exercise price $\exp(z_f)$ with $r = 0.05$, $\gamma = 0.06$, $C = 1.13$, $G = 1.1$, $M = 1$, $Y = 1.5$, $T = 2$, $K = 50$.

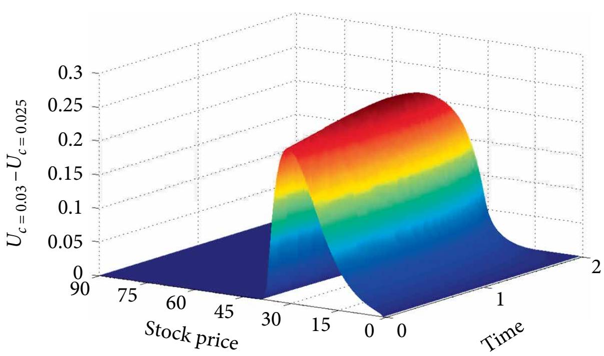

FIGURE 5: $U_{C=0.03} - U_{C=0.025}$ with $r = 0.05$, $\gamma = 0.06$, $G = 1.2$, $M = 1$, $Y = 1.5$, $T = 2$, $K = 50$.

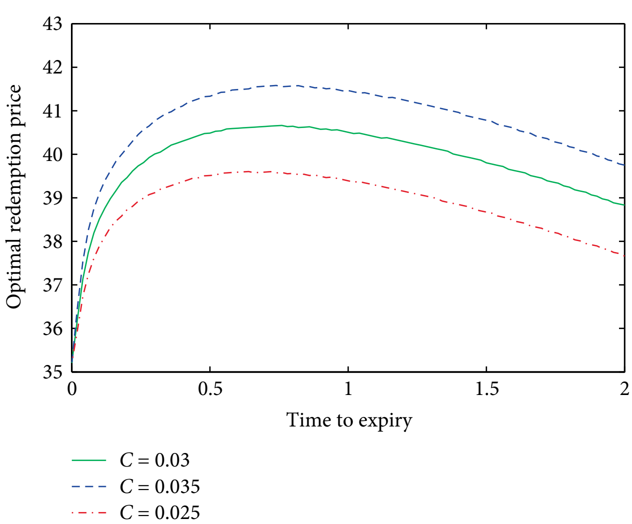

FIGURE 6: Optimal redemption prices with $r = 0.05$, $\gamma = 0.08$, $G = 1.2$, $M = 1$, $Y = 1.5$, $T = 2$, $K = 50$.

American call have a similar property that the higher activity level of underlying asset results in the higher value. Hence, it is reasonable to suggest that the stock loan value mesh surface $U(x,t)$ should move upwards as C becomes larger. Mathematically, in the exercising region where the payoff

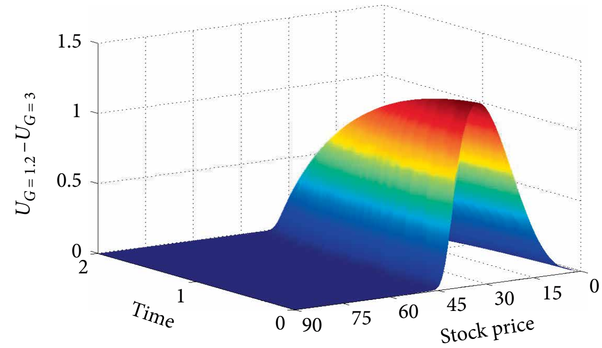

FIGURE 7: $U_{G=1.2} - U_{G=3}$ with $r = 0.05$, $\gamma = 0.06$, $C = 0.03$, $M = 1$, $Y = 1.5$, $T = 2$, $K = 50$.

$\max(S - K\exp(\gamma t), 0)$ must be equal to the stock loan value and the optimal redemption price is the boundary of exercising region, the higher stock loan values should yield the bigger optimal redemption price, namely, a bigger C shall yield a higher optimal redemption boundary shown in Figure 6.

Theoretically, the two parameters G and M control the rate of exponential decay on the right and left of the density of CGMY process,respectively. As for $G < M (G > M)$, the left (right) tail of the distribution for this process is heavier than the right (left) tail, which is consistent with the neutral distribution typically implied from stock loan values [16]. Also a smaller value of $G(M)$ results in a heavier left (right) tail. Moreover, according to the results in Ref. [24], it is suggested that the stock loan should have a positive value only if there is a large decrease in the risk-asset. Hence, the stock loan value relies on the left (right) tail of the risk-neutral distribution of stock. In other words, the fatter the left (right) tail is, the more valuable the stock loan value. Therefore, as shown in Figures 7 and 8, the mesh surface of $U(x,t)$ is decreasing with respect to G and M during with other parameters given, as depicted in the complete time dimension.

According to the analysis in explained earlier, it is not difficult to obtain that a smaller value of G or M would yield a higher stock loan value, hence, as a rational investor she/he should raise the redemption price at any valid time when there are smaller value of G or M. And in other words, the higher optimal redemption boundary is increasing with respect to G and M at any valid time, as shown in Figures 9 and 10.

Finally, we should focus on the impact of parameter Y. Carr [16] claims that parameter Y decides whether the up jumps and down jumps of CGMY process have a completely monotone Lévy density. Moreover, both activity and variation of this process will become larger as Y becomes bigger. So, the investors must be willing to obtain a higher value of contract under the case of "out-the-money", then the mesh-surface is increasing with respect to Y with other parameters are kept fixed as shown in Figure 11. However, when stock prices are less than principal, our numerical results are consistent with the conclusion of call option calculated by the fat tails distribution under the "in-the-money" case [14]. As for the stock loan contract, the stock prices must be larger than principal in the exercising region where the value of stock loan decreases monotonically with respect to Y. This leads to a fact that the

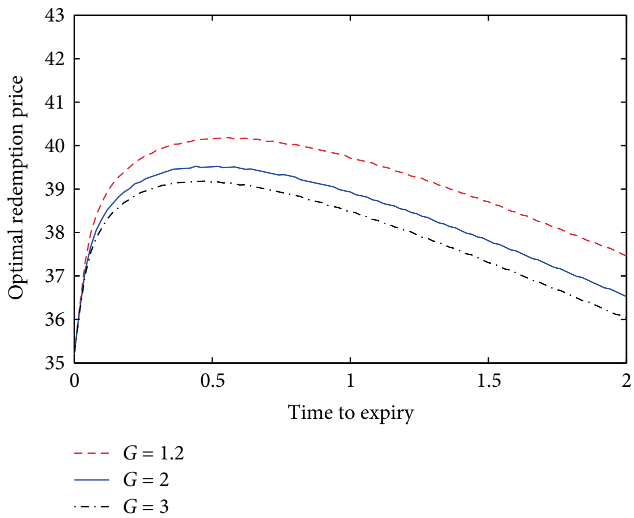

FIGURE 8: Optimal redemption prices with $r = 0.05, \gamma = 0.08$, $C = 0.03, M = 1, Y = 1.5, T = 2, K = 50$.

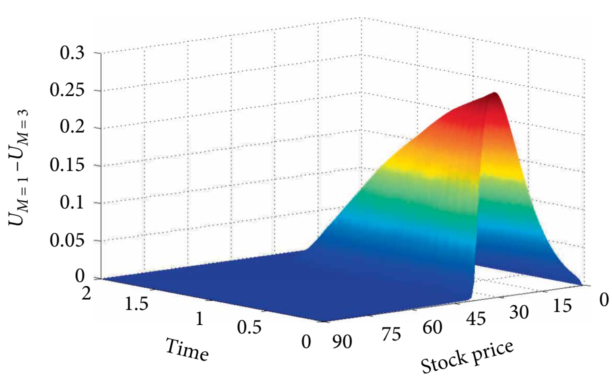

FIGURE 9: $U_{M=1} - U_{M=3}$ with $r = 0.05, \gamma = 0.08, C = 0.03$, $G = 1.2, Y = 1.5, T = 2, K = 50$.

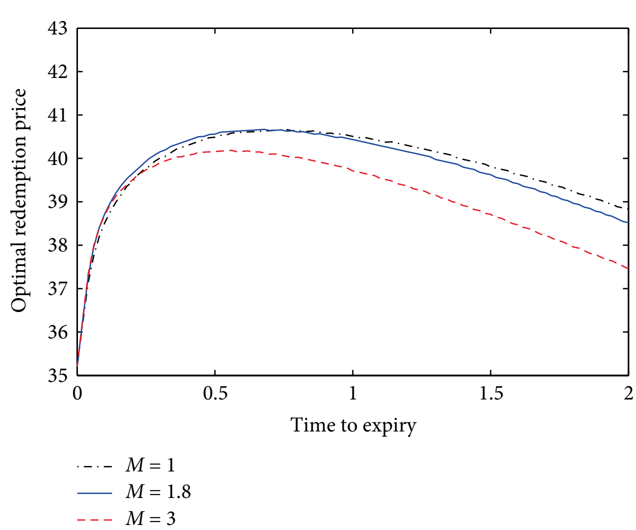

FIGURE 10: Optimal redemption prices with $r = 0.05, \gamma = 0.08$, $C = 0.03, G = 1.2, Y = 1.5, T = 2, K = 50$.

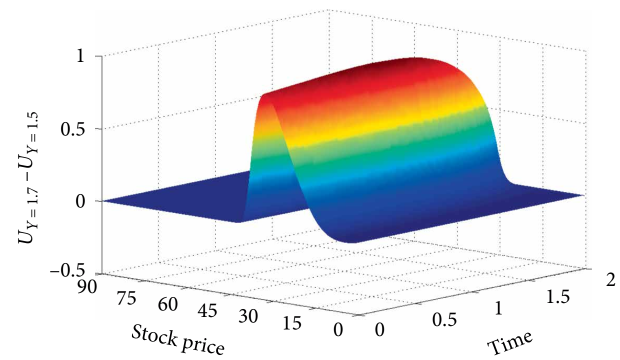

FIGURE 11: $U_{Y=1.7} - U_{Y=1.5}$ with $r = 0.05$, $\gamma = 0.08$, $C = 0.03$, $G = 1.2$, $M = 3$, $T = 2$, $K = 50$.

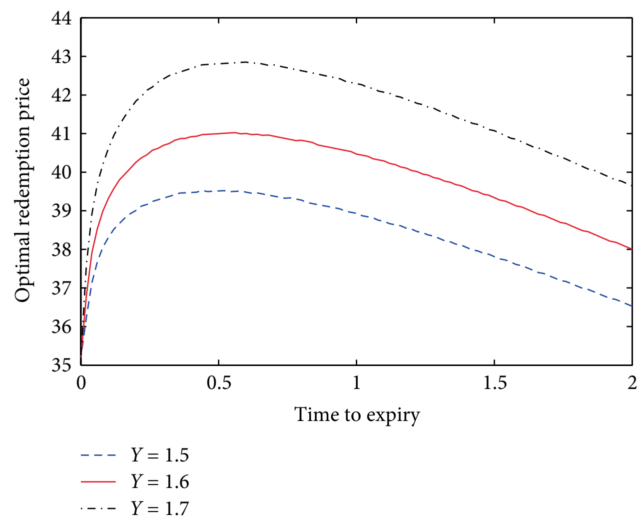

FIGURE 12: Optimal redemption prices with $r = 0.05$, $\gamma = 0.08$, $C = 0.03$, $G = 1.2$, $M = 3$, $T = 2$, $K = 50$.

bigger the value of Y is, the higher the optimal redemption boundary is, as depicted in Figure 12.

5. Conclusions

In this paper, the stock loan valuation under the CGMY framework is investigated. In this case, the pricing mathematical model is a FPDF free-boundary problem. A nonlinear penalty term is introduced to transform the free-boundary model into one with fixed domain. Then, a full nonlinear implicit numerical scheme is proposed and we also proved that the numerical solution of transformed model satisfied the inequality $U(z,t) \geq \max(\exp(z) - K, 0)$ In addition, to improve efficiency of computation, the FBi-CGSTAB method was employed.

In numerical simulation, the impacts of key parameters C, G, M, and Y on the stock loan value and optimal redemption price are analyzed. Parameter C decides activity level of underlying asset so that both stock loan value and optimal redemption price are increasing with respect to C. For the two parameters G and M, they control the rate of exponential decay on the right and left of the Lévy density, respectively; in other words, the bigger value of $G(M)$ results in the fatter left (right) tail. For this reason, both mesh surface of $U(x,t)$ and optimal redemption boundary decrease with respect to G and M. For parameter Y, in the case of "out-of-the-moneyness", the bigger Y can yield the higher mesh surface and optimal redemption price. However, this phenomenon would disappear if stock price became less than the strike price.

Appendix

A.1. Proof Process of Lemma 1

Proof. We prove $f_1 > 0$. Substituting (47) into the formula of f_1, one can obtain

$$f_1 = \left[\left(\frac{Y}{2} + \gamma_3\right)e^{\Delta zG} + \frac{2-Y}{2} - 2\gamma_4 + \gamma_3\right]\left(1 - e^{-\Delta zG}\right)^Y = \left[\left(e^{\Delta zG} + e^{-\Delta zG} - 2\right)\gamma_3 + \frac{Y}{2} + \frac{2-Y}{2}\right]\left(1 - e^{-\Delta zG}\right)^Y. \quad (A.1)$$

Due to $Y \in (1,2]$, so $(1/2)Y + (1/2)(2-Y) < 0$. Moreover,

$$e^{\Delta zG} + e^{-\Delta zG} - 2 \geq 2\sqrt{e^{\Delta zG}e^{-\Delta zG}} - 2 = 0, \quad (A.2)$$

and $\left(1 - e^{-\Delta zG}\right)^Y > 0$, therefore if $\gamma_3 \geq 0$, then $f_1 > 0$. Similarly, we also prove $f_2 > 0$ under the condition of $\gamma_3 \geq 0$. □

A.2. Proof Process of Theorem 3

Proof. Without loss of generality, we prove this theorem in two steps. The scheme (48) can be rewritten as

$$\left[1 - (\gamma - r)\Delta t - \frac{a\Delta t}{\Delta z} - \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}(2g_1 - f_1 - f_2)\right]U_j^i = U_j^{i+1} - \frac{a\Delta t}{\Delta z}U_{j-1}^i + \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(\sum_{k=0,k\neq1}^{N_1-j+2} g_k U_{j+k-1}^i + \sum_{k=0,k\neq1}^{\infty} g_k U_{j-k+1}^i\right) + \frac{\Delta t\varepsilon H}{U_j^i - \varepsilon + q_j}. \quad (A.3)$$

In order to prove $U_j^i \geq q_j^i$, for all i, j, we introduce

$$u_j^i = U_j^i - q_j, \quad (A.4)$$

and it is straightforward to obtain $u_j^{N_1+1} = U_j^{N_1+1} - q_j \geq 0$. Hence, by substituting u_j^i into (A.3), it yields

$$\left[1 - (\gamma - r)\Delta t - \frac{a\Delta t}{\Delta z} - \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}(2g_1 - f_1 + f_2)\right]\left(u_j^i + q_j\right) = u_j^{i+1} + q_j + \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(\sum_{k=0,k\neq1}^{N_1-j+2} g_k\left(u_{j+k-1}^i + q_{j+k-1}\right)\right) - \frac{a\Delta t}{\Delta z}\left(u_{j-1}^i + q_{j-1}\right) + \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y} \cdot \left(\sum_{k=0,k\neq1}^{\infty} g_k\left(u_{j-k+1}^i + q_{j-k+1}\right)\right) + \frac{\Delta t\varepsilon H}{u_j^i - \varepsilon}. \quad (A.5)$$

After simplification one can obtain

$$\left[1 - (\gamma - r)\Delta t - \frac{a\Delta t}{\Delta z}u_j^i = u_j^{i+1} - \frac{a\Delta t}{\Delta z}u_{j-1}^i + \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(\sum_{k=0,k\neq1}^{N_1-j+2} g_k u_{j+k-1}^i + \sum_{k=0,k\neq1}^{\infty} g_k u_{j-k+1}^i\right) + \frac{\Delta t\varepsilon H}{u_j^i - \varepsilon} - \Delta t E_j - \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}(2g_1 - f_1 - f_2)\right], \quad (A.6)$$

where

$$E_j = -\frac{C\Gamma(-Y)}{(\Delta z)^Y}\left[\sum_{k=0}^{N_1-j+2} g_k q_{j+k-1} - f_1 q_j + \sum_{k=0}^{\infty} g_k q_{j-k+1} - f_2 q_j\right] + (r-\gamma)q_j + \frac{a}{\Delta z}\left(q_{j-1} - q_j\right) = -\frac{C\Gamma(-Y)}{(\Delta z)^Y}\left[\sum_{k=0}^{N_1-j+2} g_k\left(e^{z_{j+k-1}} - K\right) + \sum_{k=0}^{\infty} g_k\left(e^{z_{j-k+1}} - K\right)\right] + (r-\gamma)\left(e^{z_j} - K\right) + \frac{a}{\Delta z}\left(e^{z_{j-1}} - e^{z_j}\right) + \frac{C\Gamma(-Y)}{(\Delta z)^Y}\left(f_1 + f_2\right)\left(e^{z_j} - K\right). \quad (A.7)$$

Since,

$$\left|\frac{e^{\Delta z} - 1}{\Delta z}\right| \leq \frac{e^{z_{\max}} - 1}{z_{\max}}, \quad \sum_{k=0}^{\infty} g_k = 0, \quad (A.8)$$

we have

$$\left|E_j\right| \leq \left|\frac{C\Gamma(-Y)}{(\Delta z)^Y}\left[\sum_{k=0}^{N_1-j+2} g_k q_{j+k-1} + \sum_{k=0}^{\infty} g_k q_{j-k+1}\right]\right| + \left|\frac{a}{\Delta z}\left(e^{z_{j-1}} - e^{z_j}\right)\right| + \left|\frac{C\Gamma(-Y)}{(\Delta z)^Y}\left(f_1 + f_2\right)\left(e^{z_j} - K\right)\right| + \left|(r-\gamma)\left(e^{z_j} - K\right)\right| \leq \left|\frac{C\Gamma(-Y)}{(\Delta z)^Y}\left[\sum_{k=0}^{N_1-j+2} g_k q_{j+k-1} + \sum_{k=0}^{\infty} g_k q_{j-k+1}\right]\right| + |a|e^{z_{\max}}\frac{e^{z_{\max}} - 1}{z_{\max}} + R(G,M) + (\gamma - r)e^{z_{\max}}, \quad (A.9)$$

where

$$R(G,M) = C\Gamma(-Y)\left[\left(\gamma_1 e^{z_{\max}G} + \gamma_2 + \gamma_3\right)G^Y + \left(\gamma_1 e^{z_{\max}M} + \gamma_2 + \gamma_3\right)M^Y\right]. \quad (A.10)$$

In addition, from the fact $\sum_{k=0}^{\infty} g_k e^{x_{j-k+1}} = e^{x_{j+1}}\sum_{k=0}^{\infty} g_k e^{-k\Delta x}$, and according to [15], when $|z| \leq 1$, $(1-z)^\alpha = \sum_{k=0}^{\infty} g_k z^{-k}$, we get

$$\left|\frac{C\Gamma(-Y)}{(\Delta z)^Y}\sum_{k=0}^{\infty} g_k e^{z_{j-k+1}}\right| = \left|e^{z_{j+1}}C\Gamma(-Y)\frac{\left(1 - e^{-\Delta z}\right)^Y}{(\Delta z)^Y}\right| \leq e^{z_{\max}}C\Gamma(-Y). \quad (A.11)$$

And according to Eq. (50) one can obtain

$$\left|\sum_{k=0}^{N_1-j+2} g_k\left(e^{z_{j+k-1}}-K\right)\right| \le \left|\sum_{k=0}^{N_1-j+2} g_k e^{z_{j+k-1}}\right| + K\left|\sum_{k=0}^{N_1-j+2} g_k\right| = \left|\sum_{k=0,k\neq 1}^{N_1-j+2} g_k e^{z_{j+k-1}} + g_1 e^{z_1}\right| + K\left|\sum_{k=0,k\neq 1}^{N_1-j+2} g_k + g_1\right| \le \left|\sum_{k=0,k\neq 1}^{N_1-j+2} g_k e^{z_{\max}}\right| + \left|g_1 e^{z_{\max}}\right| + K\left|\sum_{k=0,k\neq 1}^{N_1-j+2} g_k\right| + K\left|g_1\right| \le Y e^{z_{\max}} + Y e^{z_{\max}} + Y K + Y K = 2Ye^{z_{\max}} + 2Y K. \tag{A.12}$$

Finally, the following inequality can be obtained

$$\left|E_j\right| \le 2Y\lambda\left(e^{z_{\max}}+K\right) + |a|e^{z_{\max}}\frac{e^{z_{\max}}-1}{e^{z_{\max}}} + R(G,M) + (\gamma-r)e^{z_{\max}}, \tag{A.13}$$

where $\lambda = C\Gamma(-Y)/(\Delta z)^Y$.
Define

$$u^i = \min_j \; u_j^i, \tag{A.14}$$

take J be an index such that $u_J^i = u^i$. Noticing the fact that $0 < \Delta t \le 1/(\gamma - r), \Gamma(-Y) < 0, f_i(i=1,2) < 0$ and $a = r - w - \gamma - D < 0$, therefore

$$1-(\gamma-r)\Delta t - \frac{a\Delta t}{\Delta z} - \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(2g_1 - f_1 - f_2\right) > 0, \tag{A.15}$$

$$-\frac{a\Delta t}{\Delta z} > 0, \tag{A.16}$$

and $g_k \ge 0$ for $k \ge 2$, for $j = J$, it follows from (A.3) that

$$\left[1-(\gamma-r)\Delta t - \frac{a\Delta t}{\Delta z} - \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(2g_1 - f_1 - f_2\right)\right] \ge u_J^{i+1} - \frac{a\Delta t}{\Delta z}u^i + \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(\sum_{k=0,\neq 1}^{N_1-j+2} g_k u^i + \sum_{k=0,\neq 1}^{\infty} g_k u^i\right) + \frac{\Delta t\varepsilon H}{u^i+\varepsilon} - \Delta t E_j. \tag{A.17}$$

After simple algebra operation, one obtains

$$\left[1-(\gamma-r)\Delta t - \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(g_1 - f_1 - f_2\right)\right]u^i - \frac{\Delta t\varepsilon H}{u^i+\varepsilon} + \Delta t E_j \ge u_J^{i+1} \ge u^{i+1}. \tag{A.18}$$

Define a function

$$F(x) = Ax - \frac{\varepsilon\Delta t H}{x+\varepsilon} + \Delta t E_J, \tag{A.19}$$

where $A = 1-(\gamma-r)\Delta t - \Delta t C\Gamma(-Y)/(\Delta z)^Y\left(g_1 - f_1 - f_2\right) > 0$. So if we assume that $u^{i+1} \ge 0$, directly, we get

$$F\left(u^i\right) \ge 0. \tag{A.20}$$

According the Eq. (53), we have

$$F(0) = \Delta t\left(E_J - H\right) \le 0. \tag{A.21}$$

In addition

$$F'(0) = A + \frac{\varepsilon\Delta t H}{(x+\varepsilon)^2} \ge 0 \tag{A.22}$$

based on the condition $u_j^{N_1+1} \ge 0$, we can obtain $u^i \ge 0$, naturally $u_j^i \ge 0$. Consequently

$$U_i^j \ge q_j, \tag{A.23}$$

for all i, j.
Next, we should prove that $U_j^i \ge 0$ for all i, j. Following the above idea, we define

$$U^i = \min_j U_j^i, \tag{A.24}$$

and let J be an index such that $U_J^i = U^i$. It follows from (48) that

$$\left[1-(\gamma-r)\Delta t - \frac{a\Delta t}{\Delta z} - \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(2g_1 - f_1 - f_2\right)\right]U^i \ge U_J^{i+1} - \frac{a\Delta t}{\Delta z}U^i + \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(\sum_{k=0,\neq 1}^{N_1-j+2} g_k U^i + \sum_{k=0,\neq 1}^{\infty} g_k U^i\right) + \frac{\Delta t\varepsilon H}{U_J^i - \varepsilon + q_J}. \tag{A.25}$$

After arrangement, it gives

$$\left[1-(\gamma-r)\Delta t - \frac{a\Delta t}{\Delta z} - \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(2g_1 - f_1 - f_2\right)\right]U^i \ge U_J^{i+1} + \frac{\Delta t\varepsilon H}{U_J^i - \varepsilon + q_J}. \tag{A.26}$$

In first step, $U_j^i \ge q_j$ for all i, j have been proved, then $\Delta t\varepsilon H/\left(U_J^i - \varepsilon + q_J\right) < 0$, so

$$\left[1-(\gamma-r)\Delta t - \frac{a\Delta t}{\Delta z} - \frac{\Delta t C\Gamma(-Y)}{(\Delta z)^Y}\left(2g_1 - f_1 - f_2\right)\right]U^i \ge U_J^{i+1}. \tag{A.27}$$

Noticing the formula in the square brackets in the above equation is more than 0, and $U_j^{N_2+1} = \max\left(\exp\left(z_j\right) - K, 0\right) \ge 0$ for all j, namely $U^{N_2+1} \ge 0$. Hence, by mathematical induction, one obtains

$$U_j^i \ge 0, \tag{A.28}$$

for all i, j, which completes the proof.

A.3. Newton-Type Iteration Approach to the System (48)

$$\begin{aligned}&\left\{[\beta-\eta(f_1+f_2)]\mathbf{I}+\xi\mathbf{B}+\eta(\mathbf{A}+\mathbf{A}^\tau)-\mathbf{J}_\mathbf{F}\left(\omega^{l-1}\right)\right\}\delta\omega^l\\&\quad=-\{[\beta-\eta(f_1+f_2)]\mathbf{I}+\xi\mathbf{B}+\eta(\mathbf{A}+\mathbf{A}^\tau)\}\omega^{l-1}\\&\qquad+f\left(U^{i+1}\right)-\mathbf{E}^i+F\left(\omega^{l-1}\right),\end{aligned}\tag{A.29}$$

$$\omega^l=\omega^{l-1}+\kappa\left(\omega^l-\omega^{l-1}\right),\tag{A.30}$$

where $l=1,2,\ldots$, with the initial value $\omega^0=\mathbf{U}^{i+1}$ for each time level as the given initial guess and $\delta\omega^l=\omega^l-\omega^{l-1}$. $\mathbf{J}_\mathbf{F}$ is Jacobian matrix of column vector $F\left(\omega^l\right)$ and $\kappa\in(0,1)$ is a damping parameter. We choose $\mathbf{U}^i=\omega^l$, if $\left\|\omega^l-\omega^{l-1}\right\|_\infty\le\theta$ for some l as the stopping criterion, where θ is a sufficiently small positive control tolerance number. In this paper, we take $\kappa=0.2,\theta=10^{-4},\varepsilon=10^{-5}$ and $\gamma_3=0$. Now the challenging point that should be emphasized is that both matrices $\mathbf{A}$ and $\mathbf{A}^\tau$ are dense matrix, and the storage requirement and computational efforts are very high, which presents difficulty in capturing the optimal exercise boundary under the CGMY framework. So, the FBi-CGSTAB method should be employed to overcome the challenging point. And finally, the total storage requirement and computational costs have been significantly reduced from $O\left(N_1{}^2\right)$ to $O(N_1)$ and from $O\left(N_1{}^3\right)$ to $O(N_1\log N_1)$ respectively.

Data Availability

There isn't any data in our manuscript, but for the Matlab code, we can provide to anyone who want.

Conflicts of Interest

The author declare that they have no conflicts of interest.

Acknowledgments

We are especially grateful for anonymous referees for comments and suggestions that greatly improved this paper. The work on this paper was partially supported by the Guizhou science and Technology Department Basic Research Projects ([2019]1175); Science and Technology Bureau Project of Baiyun District ([2017]37); PhD Research Fund of Guizhou University of Commerce ([2018]001).

References

[1] J. Xia and X. Y. Zhan, "Stock loans," *Mathematical Finance*, vol. 17, no. 2, pp. 307–317, 2007.

[2] X. P. Lu and E. R. M. Putri, "Finite maturity margin call stock loans," *Operations Research Letters*, vol. 44, no. 1, pp. 12–18, 2016.

[3] N. Cai and L. H. Sun, "Valuation of stock loans with jump risk," *Journal of Economic Dynamics and Control*, vol. 40, pp. 213–241, 2014.

[4] W. T. Chen, L. B. Xu, and S. P. Zhu, "Stock loan valuation under a stochstic interest rate model," *Computers & Mathematics with Applications*, vol. 70, pp. 1757–1771, 2015.

[5] D. S. Bates, "Dollar jump fears, 1984–1992: distributional abnormalities implicit in currency futures options," *Journal of International Money and Finance*, vol. 15, no. 1, pp. 65–93, 1996.

[6] B. Eraker, "Do stock prices and volatility jump? reconciling evidence from spot and option prices," *The Journal of Finance*, vol. 59, no. 3, pp. 1367–1403, 2004.

[7] P. Jorion, "On jump processes in the foreign exchange and stock markets," *Review of Financial Studies*, vol. 1, no. 4, pp. 427–445, 1988.

[8] P. Carr and L. Wu, "The finite moment log stable process and option pricing," *The Journal of Finance*, vol. 58, no. 2, pp. 753–777, 2003.

[9] E. F. Fama, "The behavior of stock market prices," *The Journal of Business*, vol. 38, no. 1, pp. 34–105, 1965.

[10] B. B. Mandelbrot, "Fractional brownian motions, fractional noises and applications," *SIAM Review*, vol. 10, no. 4, pp. 422–437, 1968.

[11] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman Press, San Francisco, 1982.

[12] B. B. Mandelbrot, *Fractals and Scaling in Finance: Discontinuity*, Risk, Springer Verlag, Concentration, 1997.

[13] J. C. Cox and S. A. Ross, "The valuation of options for alternative stochastic processes," *Journal of Financial Economics*, vol. 3, no. 1-2, pp. 145–166, 1976.

[14] S. Markose and A. Alentorn, "The generalized extreme value distribution, implied tail index, and option pricing," *The Journal of Derivatives*, vol. 18, no. 3, pp. 35–60, 2011.

[15] O. Marom and E. Momoniat, "A comparison of numerical solutions of fractional diffusion models in finance," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 6, pp. 3435–3442, 2009.

[16] P. Carr, H. Geman, D. Madan, and M. Yor, "The fine structure of asset returns: an empirical investigation," *The Journal of Business*, vol. 75, no. 2, pp. 305–333, 2002.

[17] L. Ballotta and I. Kyriakou, "Monte carlo simulation of the CGMY process and option pricing," *Journal of Futures Markets*, vol. 34, no. 12, pp. 1095–1121, 2014.

[18] H. Zhang, F. Liu, I. Turner, and S. Chen, "The numerical simulation of the tempered fractional Black–Scholes equation for european double barrier option," *Applied Mathematical Modelling*, vol. 40, no. 11-12, pp. 5819–5834, 2016.

[19] W. T. Chen, X. Xu, and S. P. Zhu, "Analytically pricing european-style options under the modified Black–Scholes equation with a spatial-fractional derivative," *Quarterly of Applied Mathematics*, vol. 73, no. 3, pp. 597–611, 2014.

[20] Á. Cartea, "Dynamic hedging of financial instruments when the underlying follows a non-gaussian process," *SSRN Electronic Journal*, 2005.

[21] P. Wilmott, J. Dewynne, and S. Howison, *Option Pricing*, Oxford Financial Press, Oxford, 1st edition, 1993.

[22] S. Chen, F. Liu, X. Jiang, I. Turner, and V. Anh, "A fast semi-implicit difference method for a nonlinear two-sided space-fractional diffusion equation with variable diffusivity coefficients," *Applied Mathematics and Computation*, vol. 257, pp. 591–601, 2015.

[23] H. Hejazi, T. Moroney, and F. Liu, "Stability and convergence of a finite volume method for the space fractional advection-dispersion equation," *Journal of Computational and Applied Mathematics*, vol. 255, pp. 684–697, 2014.

[24] W. T. Chen, M. Y. Du, and X. Xu, "An explicit closed-form analytical solution for european options under the CGMY process," *Communications in Nonlinear Science Numerical Simulation*, vol. 42, pp. 285–297, 2017.