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Research Article

The Generalized Pomeron Functional Equation

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This paper investigates the linear functional equation with constant coefficients $\varphi(t) = \kappa\varphi(\lambda t) + f(t)$, where both $\kappa > 0$ and $1 > \lambda > 0$ are constants, f is a given continuous function on $\mathbb{R}$, and $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ is unknown. We present all continuous solutions of this functional equation. We show that (i) if $\kappa > 1$, then the equation has infinite many continuous solutions, which depends on arbitrary functions; (ii) if $0 < \kappa < 1$, then the equation has a unique continuous solution; and (iii) if $\kappa = 1$, then the equation has a continuous solution depending on a single parameter $\varphi(0)$ under a suitable condition on f.

1. Introduction

Recently, Mickens [1] considered a linear functional equation

$$\varphi(t) = N\varphi\left(\frac{t}{N^m}\right) - (N-1), \tag{1}$$

where $N \geq 1$ is a positive integer, m is a positive real number, and φ is an unknown function with the domain $\mathbb{R}$. When $m = 2$, this equation is called *pomeron functional equation*. The functional equation comes from some phenomena in physics (cf. [2, 3, 4]). Mickens gives an exact solution of (1) in [1].

$$\varphi(t) = \begin{cases} 1 - ct^{1/m}, & t \in [0, \infty), \\ 1 - c(-t)^{1/m}, & t \in (-\infty, 0), \end{cases} \tag{2}$$

where c is any positive constant. Mickens mistakenly thought solution (2) as the general solution. We recall that the general solution of functional equations, which in general depends on arbitrary functions, are quite different from the one of differential equations, which in general depends on arbitrary constants.

This paper considers the generalized pomeron functional equation:

$$\varphi(t) = \kappa\varphi(\lambda t) + f(t), \tag{3}$$

where $\kappa > 0$ and $1 > \lambda > 0$, f is a given continuous function on $\mathbb{R}$, and $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ is unknown. Note that equation (3) is a linear, inhomogeneous, and functional equation. Bessenyei in [5] and Shi and Gong in [6] considered cyclic cases for some linear functional equations. Brydak in [7] showed the stability of the linear equation $\varphi[f(x)] = g(x)\varphi(x) + F(x)$. Czerwik in [8] studied the continuous dependence on given functions for solutions. We use the recurrence method to present the general solution of equation (3). Compared with Kuczma's monograph ([9], Chapter II), our method follows more directly and easily.

2. The General Solution to Equation (3)

Note that, if $\kappa \neq 1$ and φ satisfies (3), then $\varphi(0) = f(0)/(1-\kappa)$. Furthermore, if $\kappa = 1$ and (3) has a solution, then $f(0) = 0$.

According to the range of κ, we consider the following three cases to investigate continuous solutions of equation (3).

2.1. The Case $\kappa > 1$. For this case, equation (3) has infinitely many continuous solutions, which depend on arbitrary functions.

Theorem 1. *Choose arbitrarily* $t_0 \in (0, +\infty)$ *and* $s_0 \in (-\infty, 0)$. *Let* φ_0 *and* ψ_0 *be any continuous function defined in* $[\lambda t_0, t_0]$ *and* $[s_0, \lambda s_0]$, *respectively, such that*

$$\varphi_0(t_0) = \kappa\varphi_0(\lambda t_0) + f(t_0),$$
$$\psi_0(s_0) = \kappa\psi_0(\lambda s_0) + f(s_0). \tag{4}$$

Define φ^+ *on* $[0, +\infty)$ *by*

$$\varphi^+(t) = \begin{cases} \dfrac{f(0)}{1-\kappa}, & t = 0, \\ \varphi_0(t), & t \in [\lambda t_0, t_0], \\ \varphi_k(t), & t \in [\lambda^k t_0, \lambda^{k-1} t_0],\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{5}$$

where

$$\varphi_k(t) = \kappa^{-1}\varphi_{k-1}\left(\frac{t}{\lambda}\right) + \kappa^{-1} f\left(\frac{t}{\lambda}\right),$$
$$\text{for } t \in [\lambda^k t_0, \lambda^{k-1} t_0],\ k = \pm 1, \pm 2, \ldots. \tag{6}$$

Define φ^- *on* $(-\infty, 0]$ *by*

$$\varphi^-(t) = \begin{cases} \dfrac{f(0)}{1-\kappa}, & t = 0, \\ \psi_0(t), & t \in [s_0, \lambda s_0], \\ \psi_k(t), & t \in [\lambda^{k-1} s_0, \lambda^k s_0],\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{7}$$

where

$$\psi_k(t) = \kappa^{-1}\psi_{k-1}\left(\frac{t}{\lambda}\right) + \kappa^{-1} f\left(\frac{t}{\lambda}\right),$$
$$\text{for } t \in [\lambda^{k-1} s_0, \lambda^k s_0],\ k = \pm 1, \pm 2, \ldots. \tag{8}$$

Then,

$$\varphi(t) = \begin{cases} \varphi^+(t), & t \in [0, +\infty), \\ \varphi^-(t), & t \in (-\infty, 0], \end{cases} \tag{9}$$

is the general continuous solution on $\mathbb{R}$ *of equation (3).*

Proof. One can check that φ defined above satisfies equation (3).

Firstly, we prove that φ^+ is continuous in $(0, +\infty)$. By the continuity of φ_0 in $[\lambda t_0, t_0]$, we have $\lim_{t \longrightarrow \lambda t_0 + 0} \varphi^+(t) = \varphi^+(\lambda t_0)$. And

$$\begin{aligned} \lim_{t \longrightarrow \lambda t_0 - 0} \varphi^+(t) &= \lim_{t \longrightarrow t_0 - 0} \varphi_1(\lambda t) \\ &= \lim_{t \longrightarrow t_0 - 0} \kappa^{-1}\varphi_0(t) + \kappa^{-1} f(t) \\ &= \varphi_0(\lambda t_0) \\ &= \varphi^+(\lambda t_0). \end{aligned} \tag{10}$$

Thus, φ^+ is continuous for $t \in [\lambda t_0, t_0)$. Now $(0, +\infty) = \cup[\lambda^{k+1} t_0, \lambda^k t_0)$. Assume that φ^+ is continuous for $t \in [\lambda^k t_0, \lambda^{(k-1)} t_0)$, $k > 0$. We have for $t \in [\lambda^{k+1} t_0, \lambda^k t_0)$,

$$\varphi_{k+1}(t) = \kappa^{-1}\varphi_k\left(\frac{t}{\lambda}\right) + \kappa^{-1} f\left(\frac{t}{\lambda}\right). \tag{11}$$

Thus, φ^+ is continuous for $t \in [\lambda^{k+1} t_0, \lambda^k t_0)$. Similarly, we can show that φ^+ is continuous for $t \in [\lambda^{k+1} t_0, \lambda^k t_0)$ for $k < 0$.

Secondly, we shall prove that

$$\lim_{t \longrightarrow 0^+} \varphi^+(t) = \frac{f(0)}{1-\kappa}. \tag{12}$$

Let $t \in (0, \lambda t_0)$. Then, $t = \lambda^n x$ for some $n \in \mathbb{N}$ and $x \in [\lambda t_0, t_0]$. Hence, we obtain

$$\begin{aligned} \varphi^+(t) &= \varphi^+(\lambda^n x) = \varphi_n(\lambda^n x) \\ &= \kappa\varphi_{n-1}(\lambda^{n-1} x) + f(\lambda^{n-1} x) \\ &= \kappa^2\varphi_{n-2}(\lambda^{n-2} x) + \kappa f(\lambda^{n-2} x) + f(\lambda^{n-1} x) \\ &= \vdots \\ &= \kappa^n\varphi_0(x) + \kappa^{n-1} f(x) + \cdots + \kappa f(\lambda^{n-2} x) + f(\lambda^{n-1} x). \end{aligned} \tag{13}$$

Let $\varepsilon > 0$. Since f is continuous, there exists $N \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n > N$, we have $|f(\lambda^n x) - f(0)| < \varepsilon$. Therefore, we obtain

$$\begin{aligned} \left|\sum_{i=1}^{n} \kappa^{i-1} f(\lambda^{n-i} x) - \frac{f(0)}{1-\kappa}\right| &= \left|\sum_{i=1}^{n} \kappa^{i-1} f(\lambda^{n-i} x) - f(0)\sum_{i=1}^{\infty} \kappa^{i-1}\right| \\ &\le \sum_{i=1}^{n-N} \kappa^{i-1}\left|f(\lambda^{n-i} x) - f(0)\right| \\ &\quad + \left|\sum_{i=n-N+1}^{n} \kappa^{i-1} f(\lambda^{n-i} x)\right| \\ &\quad + |f(0)|\left|\sum_{i=n-N+1}^{\infty} \kappa^{i-1}\right| \\ &\le \varepsilon\sum_{i=1}^{n-N} \kappa^{i-1} + \sum_{i=n-N+1}^{n} \kappa^{i-1}\left|f(\lambda^{n-i} x)\right| \\ &\quad + |f(0)|\left|\sum_{i=n-N+1}^{\infty} \kappa^{i-1}\right| \\ &\le \frac{\varepsilon}{1-\kappa} + \left(|f(0)| + \sup_{|t|\le|x|} |f(t)|\right) \\ &\quad \cdot \sum_{i=n-N+1}^{\infty} \kappa^{i-1} \\ &= \frac{\varepsilon}{1-\kappa} + \frac{\kappa^{n-N}}{1-\kappa}\left(|f(0)| + \sup_{|t|\le|x|} |f(t)|\right). \end{aligned} \tag{14}$$

Together with (13) and (14), we have

$$\lim_{t\to 0^+}\left|\varphi^+(t)-\frac{f(0)}{1-\kappa}\right| = \lim_{n\to +\infty}\left|\varphi^+(\lambda^n x)-\frac{f(0)}{1-\kappa}\right| = 0. \quad (15)$$

Therefore, φ^+ is continuous in $[0,+\infty)$. Similarly, one can prove that φ^- is continuous in $(-\infty,0]$.

On the other hand, every continuous solution can be obtained in this manner. □

2.2. The Case $0<\kappa<1$. The corresponding homogeneous equation of (3) is given by

$$\varphi(t)=\kappa\varphi(\lambda t). \quad (16)$$

According to the superposition principle for the linear equation, the general solution of (3) is given by the additions of the general solution to the corresponding homogeneous equation (16) and a particular solution of the original inhomogeneous equation (3), cf. [9].

Lemma 1. *Suppose $\kappa,\lambda\in(0,1)$. Then, $\varphi=0$ is the only continuous solution of equation (16).*

Proof. We use the proof by contradiction.

Assume $\varphi(0)\neq 0$. Then, $\varphi(t)\neq 0$ in $\mathbb{R}$. If fact, if there exists a t_0 such that $\varphi(t_0)=0$, then by $\varphi(t)=\kappa\varphi(\lambda t)$,

$$\varphi(0)=\lim_{n\to\infty}\varphi(\lambda^n t_0)=\lim_{n\to\infty}\kappa^{-n}\varphi(t_0)=0, \quad (17)$$

which contradicts the assumption. Thus, it follows from $\varphi(t)=\kappa\varphi(\lambda t)$ that

$$\lim_{n\to\infty}\kappa^{-n}=\lim_{n\to\infty}\frac{\varphi(\lambda^n t)}{\varphi(t)}=\frac{\varphi(0)}{\varphi(t)}. \quad (18)$$

This is a contradiction. Therefore, $\varphi(0)=0$.

Assume that there exists an interval $J=[a,b]\subset\mathbb{R}\backslash\{0\}$ such that

$$|\varphi(t)|>c>0, \quad \text{for } t\in J. \quad (19)$$

Given an $\varepsilon>0$, we can find a $\delta>0$ such that

$$|\varphi(t)|<\varepsilon c, \quad \text{for } t\in(0,\delta). \quad (20)$$

Thus, there exists an $N>0$ such that $\lambda^n b<\delta$ for $n\geq N$, which implies the inequality $\lambda^n t<\delta$ for $t\in J$ and $n\geq N$. Hence, we have

$$\left|\kappa^{-n}\right|=\frac{|\varphi(\lambda^n t)|}{|\varphi(t)|}<\varepsilon. \quad (21)$$

This is a contradiction. Therefore, $\varphi(t)\equiv 0$ on $\mathbb{R}$. □

Theorem 2. *Suppose $\kappa,\lambda\in(0,1)$. Then, equation (3) has a unique continuous solution:*

$$\varphi(t)=\sum_{n=0}^{\infty}\kappa^n f(\lambda^n t). \quad (22)$$

Proof. We can obtain the uniqueness from Lemma 1. It suffices to prove that (22) is a continuous solution of equation (3).

Choose an arbitrary $c\in(0,\infty)$. Put $t\in[-c,c]$. One can see that $\lambda^n t\in[-c,c]$ for $n\geq 0$. Let

$$M=\sup_{[0,c]}|f(t)|. \quad (23)$$

We have

$$\left|\kappa^n f(\lambda^n t)\right|\leq\kappa^n M, \quad \text{for } n\geq 0,\ t\in[-c,c]. \quad (24)$$

Consequently, the series $\sum_{n=1}^{\infty}\kappa^n f(\lambda^n t)$ in (22) converges uniformly in $[-c,c]$ for every $c\in(0,\infty)$. Therefore, $\varphi(t)$ in (22) is a continuous function in $\mathbb{R}$. Further,

$$\kappa\varphi(\lambda t)+f(t)=\sum_{n=1}^{\infty}\kappa^n f(\lambda^n t)+f(t)=\sum_{n=0}^{\infty}\kappa^n f(\lambda^n t)=\varphi(t). \quad (25)$$

Thus, $\varphi(t)$ is the only continuous solution of equation (3). □

2.3. The Case $\kappa=1$. If $\kappa=1$, then equation (3) becomes the following form:

$$\varphi(t)=\varphi(\lambda t)+f(t). \quad (26)$$

Lemma 2. *Suppose $\varphi(t)$ is a continuous solution of equation (26). Then,*

$$\varphi(t)=\eta+\sum_{j=0}^{\infty}f\left(\lambda^j t\right), \quad \text{where } \eta=\varphi(0). \quad (27)$$

Proof. Let φ be a continuous solution of equation (26). Then,

$$\begin{aligned}\varphi(t)&=\varphi(\lambda t)+f(t)\\&=\varphi\left(\lambda^2 t\right)+f(\lambda t)+f(t)\\&=\cdots\\&=\varphi(\lambda^n t)+\sum_{j=0}^{n-1}f\left(\lambda^j t\right)\\&=\varphi(0)+\sum_{j=0}^{\infty}f\left(\lambda^j t\right).\end{aligned} \quad (28)$$

Substituting $t=0$ into equation (26), we have $\varphi(0)=\varphi(0)+f(0)$. Consequently, $f(0)=0$. □

Note that the series $\sum_{j=0}^{\infty}f(\lambda^j t)$ in (27) may not converge. Some additional assumption on f should be made to guarantee the existence of continuous solutions of equation (26).

We present one sufficient condition on f in the following theorem. □

Theorem 3. *Suppose that there exists positive constants δ, σ, and L such that*

$$|f(t)|\leq L|t|^{\sigma}, \quad \text{for } t\in(-\delta,\delta). \quad (29)$$

Then, equation (26) has a continuous solution.

Proof. Since

$$\sum_{j=0}^{\infty}\left|f\left(\lambda^j t\right)\right| \leq \sum_{j=0}^{\infty} L\left|\lambda^j t\right|^{\sigma} = \frac{Lt^{\sigma}}{1-\lambda^{\sigma}}, \quad (30)$$

the series (27) converges uniformly in the interval $[-c, c]$ for any fixed $c \in (0, \infty)$. It follows from Lemma 2 that $\varphi(t)$ in (27) is a continuous solution. □

3. Conclusion

This paper investigates one kind of linear functional equation with constant coefficients $\varphi(t) = \kappa\varphi(\lambda t) + f(t)$, where $\kappa > 0$ and $\lambda \in (0, 1)$, f is a given continuous function on $\mathbb{R}$, and $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ is unknown. If $\kappa > 1$, then the equation has infinite many continuous solutions, which depend on arbitrary functions. If $0 < \kappa < 1$, then the equation has the unique continuous solution:

$$\varphi(t) = \sum_{n=0}^{\infty} \kappa^n f(\lambda^n t). \quad (31)$$

If $\kappa = 1$ and the equation has a continuous solution, then $f(0) = 0$ and

$$\varphi(t) = \eta + \sum_{j=0}^{\infty} f\left(\lambda^j t\right), \quad \text{where } \eta = \varphi(0). \quad (32)$$

For the last case, we only give one sufficient condition for the existence of continuous solutions. The problem for the necessary and sufficient condition still remains open.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] R. E. Mickens, "A note on solutions to pomeron functional equations," *Journal of Difference Equations and Applications*, vol. 23, no. 7, pp. 1219–1221, 2017.

[2] H. D. I. Abarbanel and J. B. Bronzan, "Pomeranchuk singularity in a Reggon theory with quartic couplings," *Physical Review D*, vol. 9, no. 12, pp. 3304–3312, 1974.

[3] H. Fujisaki, "Self-reproducing conditions on the Pomeranchuk singularity," *Progress of Theoretical Physics*, vol. 44, no. 1, pp. 219–229, 1970.

[4] R. E. Mickens, "Self-consistent vacuum trajectory," *Il Nuovo Cimento A*, vol. 56, no. 3, pp. 799–803, 1968.

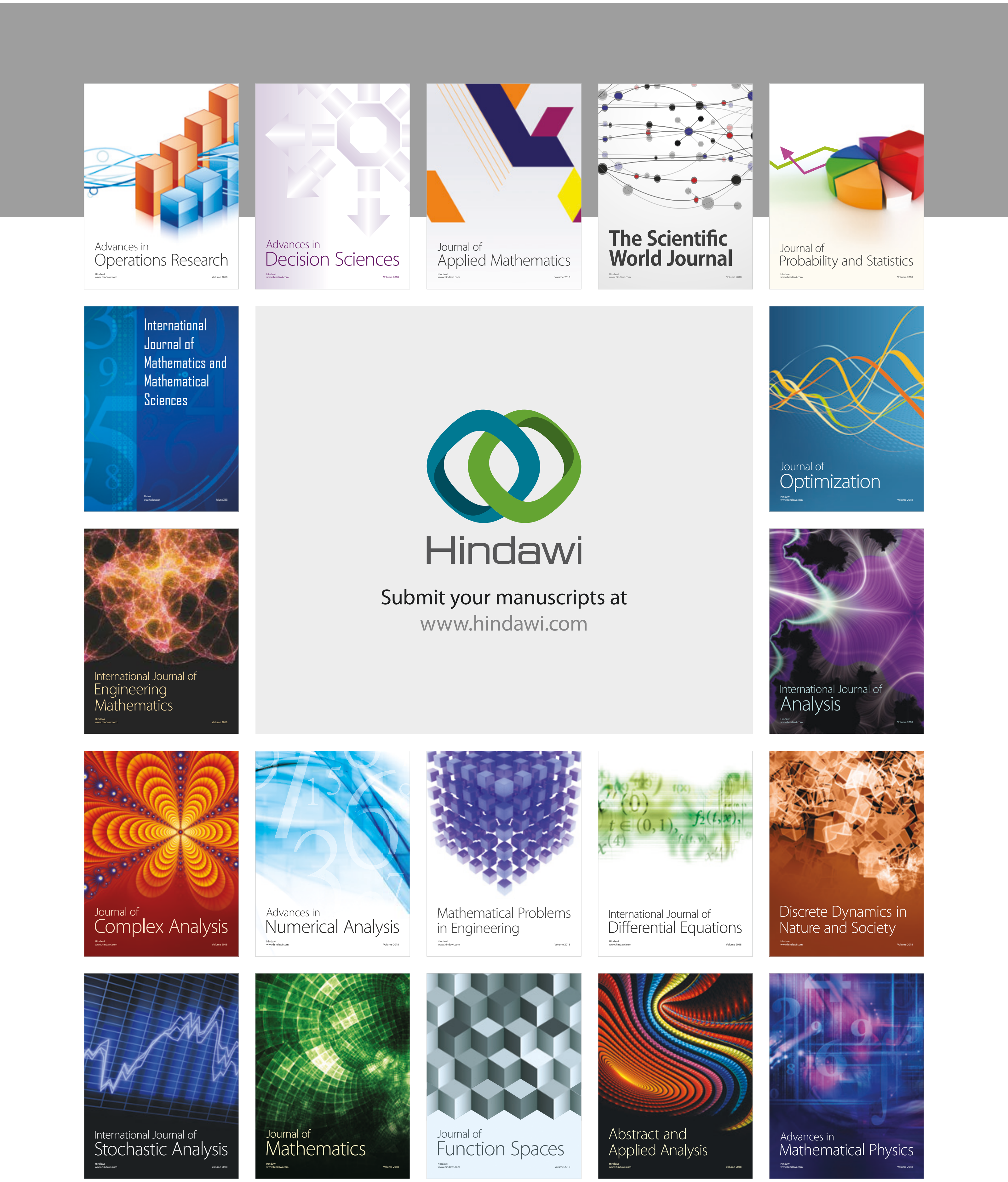
[5] M. Bessenyei, "Functional equations and finite groups of substitutions," *The American Mathematical Monthly*, vol. 117, no. 10, pp. 921–927, 2010.

[6] Y.-G. Shi and X.-B. Gong, "Linear functional equations involving Babbages equation," *Elemente der Mathematik*, vol. 69, no. 4, pp. 195–204, 2014.

[7] D. Brydak, "On the stability of the functional equation $\varphi[f(x)] = g(x)\varphi(x) + F(x)$," *Proceedings of the American Mathematical Society*, vol. 26, no. 3, pp. 455–460, 1970.

[8] S. Czerwik, "On the continuous dependence of solutions of some functional equations on given functions," *Annales Polonici Mathematici*, vol. 24, no. 3, pp. 247–252, 1971.

[9] M. Kuczma, *Functional Equations in a Single Variable*, Polish Scientific Publishers, Warszawa, Poland, 1968.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