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Research Article

Rough Hyperfilters in Po-LA-Semihypergroups

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This paper concerns the study of hyperfilters of ordered LA-semihypergroups, and presents some examples in this respect. Furthermore, we study the combination of rough set theory and hyperfilters of an ordered LA-semihypergroup. We define the concept of rough hyperfilters and provide useful examples on it. A rough hyperfilter is a novel extension of hyperfilter of an ordered LA-semihypergroup. We prove that the lower approximation of a left (resp., right, bi) hyperfilter of an ordered LA-semihypergroup becomes left (resp., right, bi) hyperfilter of an ordered LA-semihypergroup. Similarly we prove it for upper approximation.

1. Introduction

Theory of hyperstructures (also known as multialgebras) was introduced in 1934 at the "8th congress of Scandinavian Mathematicians", where a French mathematician Marty [1] presented some definitions and results on the hypergroup theory, which is a generalization of groups. He gave applications of hyperstructures to rational functions, algebraic functions and noncommutative groups. Hyperstructure theory is an extension of the classical algebraic structure. In algebraic hyperstructures, the composition of two elements is a set, while in a classical algebraic structure, the composition of two elements is an element. Algebraic hyperstructure theory has many applications in other disciplines. Because of new viewpoints and importance of this theory, many authors applied this theory in nuclear physics and chemistry. Around 1940s, several mathematicians added many results to the theory of hyperstructures, especially in United States, France, Russia, Italy, Japan, and Iran. Several papers and books have been written on hypergroups. One of these is "Applications of hyperstructure theory" written by Corsini and Leoreanu-Fotea [2]. Another book "Hyperstructures and Their Representations" written by Vougiouklis [3], was published in 1994. Recently, Hila et al. [4] and Yaqoob et al. [5] introduced the concept of nonassociative semihypergroups which is a generalization of the work of Kazim and Naseeruddin [6]. Later, Yaqoob and Gulistan [7] studied partially ordered left almost semihypergroups.

Filter theory in algebraic structures has been studied by many mathematicians. Hila [8] studied filters in ordered Γ-semigroups. Jakubík [9] studied filters of ordered semigroups. Jirojkul and Chinram [10] studied fuzzy quasi-ideal subsets and fuzzy quasi-filters of ordered semigroup. Jun et al. [11] studied bifilters of ordered semigroups. Kehayopulu [12] introduced filters generated in poe-semigroups. Ren et al. [13] introduced principal filters of po-semigroups. Tang et al. [14] studied hyperfilters of ordered semihypergroups.

Rough set theory, introduced in 1982 by Pawlak [15], is a mathematical approach to imperfect knowledge. The methodology of rough set is concerned with the classification and analysis of imprecise, uncertain, or incomplete information and knowledge. There are several authors who considered the theory of rough sets in algebraic structures, for instance, Biswas and Nanda [16] developed some results on rough subgroups. Jun [17] applied the rough set theory to gamma-subsemigroups/ideals in gamma-semigroups. Shabir and Irshad [18] defined roughness in ordered semigroups. Rough set theory is also considered by some authors in different hyperstructures, for instance, in semihypergroups [19], Γ-semihypergroups [20], hyperrings [21], Hv-groups [22], Hv-modules [23], Γ-semihyperrings [24], hyperlattices [25], hypergroup [26, 27], quantales [28], quantale modules [29], and nonassociative po-semihypergroups [30]. The rough set theory has been applied to the filters of some well known algebraic structures, like BE-algebras [31], ordered semigroups [32, 33], residuated lattices [34], and BL-algebras [35].

2. Preliminaries and Basic Definitions

Definition 1 [4, 5]. A hypergroupoid $(H, \circ)$ is said to be an LA-semihypergroup if for all $a, b, c \in H$,

$$(a \circ b) \circ c = (c \circ b) \circ a. \tag{1}$$

Definition 2 [7]. Let H be a nonempty set and "$\leq$" be an ordered relation on H. Then $(H, \circ, \leq)$ is called an ordered LA-semihypergroup if

(1) $(H, \circ)$ is an LA-semihypergroup;
(2) $(H, \leq)$ is a partially ordered set;
(3) for every $a, b, c \in H$, $a \leq b$ implies $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$, where $a \circ c \leq b \circ c$ means that for every $x \in a \circ c$ there exists $y \in b \circ c$ such that $x \leq y$.

Definition 3 [7]. If A is a nonempty subset of $(H, \circ, \leq)$, then $(A]$ is the subset of H defined as follows:

$$(A] = \{t \in H : t \leq a, \text{ for some } a \in A\}. \tag{2}$$

Definition 4 [7]. A nonempty subset A of an ordered LA-semihypergroup $(H, \circ, \leq)$ is called an LA-subsemihypergroup of H if $(A \circ A] \subseteq (A]$.

Definition 5 [7]. A nonempty subset A of an ordered LA-semihypergroup $(H, \circ, \leq)$ is called a right (resp., left) hyperideal of H if

(1) $A \circ H \subseteq A$ (resp., $H \circ A \subseteq A$),
(2) for every $a \in H, b \in A$ and $a \leq b$ implies $a \in A$.

If A is both right hyperideal and left hyperideal of H, then A is called a hyperideal (or two sided hyperideal) of H.

Definition 6 [7]. A nonempty subset P of $(H, \circ, \leq)$ is called a prime hyperideal of H if the following conditions hold:

(i) $A \circ B \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$ for any two hyperideals A and B of H;
(ii) if $a \in P$ and $b \leq a$, then $b \in P$ for every $b \in H$.

Definition 7 [30]. A relation δ on an ordered LA-semihypergroup H is called a pseudohyperorder if

(1) $\leq \subseteq \delta$.
(2) δ is transitive, that is $(a, b), (b, c) \in \delta$ implies $(a, c) \in \delta$ for all $a, b, c \in H$.
(3) δ is compatible, that is if $(a, b) \in \delta$ then $(a \circ x, b \circ x) \in \delta$ and $(x \circ a, x \circ b) \in \delta$ for all $a, b, x \in H$.

3. Hyperfilters in Ordered LA-Semihypergroups

Definition 8. Let $(H, \circ, \leq)$ be an ordered LA-semihypergroup. An LA-subsemihypergroup F of H is called a left (resp., right) hyperfilter of H if

(i) for all $a, b \in H, a \circ b \subseteq F \Rightarrow a \in F$ (resp., $b \in F$),
(ii) for all $b \in H$ and $a \in F, a \leq b \Rightarrow b \in F$.

Definition 9. Let $(H, \circ, \leq)$ be an ordered LA-semihypergroup. An LA-subsemihypergroup F of H is called a hyperfilter of H if

(i) for all $a, b \in H, a \circ b \subseteq F \Rightarrow a \in F$ and $b \in F$,
(ii) for all $b \in H$ and $a \in F, a \leq b \Rightarrow b \in F$.

Example 10. Consider a set $H = \{a, b, c\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":

$\circ$	a	b	c
a	a	$\{a, b, c\}$	$\{a, c\}$
b	$\{a, b, c\}$	b	$\{a, b, c\}$
c	$\{a, c\}$	$\{a, b\}$	$\{a, c\}$

$\leq := \{(a, a), (b, b), (c, a), (c, c)\}$.

We give the covering relation "$\prec$" as:

$$\prec := \{(c, a)\}. \tag{3}$$

The figure of H is shown in Figure 1. Then $(H, \circ, \leq)$ is an ordered LA-semihypergroup. Here $\{a\}, \{b\}, \{a, c\}$ and H are hyperfilters of H.

Definition 11. Let $(H, \circ, \leq)$ be an ordered LA-semihypergroup. An LA-subsemihypergroup F of H is called a bi-hyperfilter of H if

(i) for all $a, b \in H, (a \circ b) \circ a \subseteq F \Rightarrow a \in F$,
(ii) for all $b \in H$ and $a \in F, a \leq b \Rightarrow b \in F$.

Example 12. Consider a set $H = \{a, b, c, d, e\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":

$\circ$	a	b	c	d	e
a	a	a	a	a	a
b	a	$\{b, c\}$	c	$\{a, d\}$	c
c	a	b	$\{b, c\}$	$\{a, d\}$	b
d	a	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	d
e	a	b	c	d	e

$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (d, d), (e, e),\}$

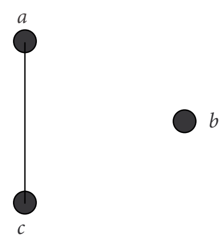

Figure 1: Figure of H for Example 10.

We give the covering relation "$\prec$" as:

$$\prec = \{(a,b),(a,c),(a,d)\}. \tag{4}$$

The figure of H is shown in Figure 2. Then $(H,\circ,\leq)$ is an ordered LA-semihypergroup. The bi-hyperfilters of H are $\{e\}$, $\{b,c,e\}$ and H.

Definition 13. Let $(H,\circ,\leq)$ be an ordered LA-semihypergroup. An LA-subsemihypergroup F of H is called a strong left (resp., strong right) hyperfilter of H if

(i) for all $a,b \in H$, $(a\circ b)\cap F \neq \emptyset \Rightarrow a \in F$ (resp., $b \in F$),

(ii) for all $b \in H$ and $a \in F$, $a \leq b \Rightarrow b \in F$.

Definition 14. Let $(H,\circ,\leq)$ be an ordered LA-semihypergroup. An LA-subsemihypergroup F of H is called a strong hyperfilter of H if

(i) for all $a,b \in H$, $(a\circ b)\cap F \neq \emptyset \Rightarrow a \in F$ and $b \in F$,

(ii) for all $b \in H$ and $a \in F$, $a \leq b \Rightarrow b \in F$.

Example 15. Consider a set $H = \{a,b,c,d,e\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":

$\circ$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	$\{b,c\}$	$\{b,c\}$
c	c	c	c	c	c
d	e	$\{b,c\}$	c	$\{b,c\}$	$\{b,c\}$
e	d	$\{b,c\}$	c	$\{b,c\}$	$\{b,c\}$

$\leq := \{(a,a),(b,b),(c,b),(c,c),(d,b),(d,d),(e,b),(e,e)\}$

We give the covering relation "$\prec$" as:

$$\prec = \{(c,b),(d,b),(e,b)\}. \tag{5}$$

The figure of H is shown in Figure 3. Then $(H,\circ,\leq)$ is an ordered LA-semihypergroup. The hyperfilters of H are $\{a\}$, $\{a,b\}$ and H. Here $\{a\}$ and H are also strong hyperfilters of H. While $\{a,b\}$ is not a strong hyperfilter of H, because $(d\circ e)\cap\{a,b\} = \{b\} \neq \emptyset$, but $d \notin \{a,b\}$ and $e \notin \{a,b\}$.

Definition 16. Let $(H,\circ,\leq)$ be an ordered LA-semihypergroup. An LA-subsemihypergroup F of H is called a strong bi-hyperfilter of H if

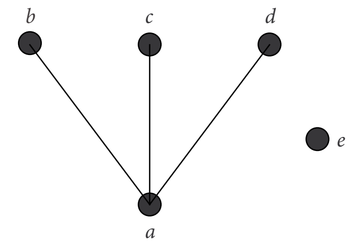

Figure 2: Figure of H for Example 12.

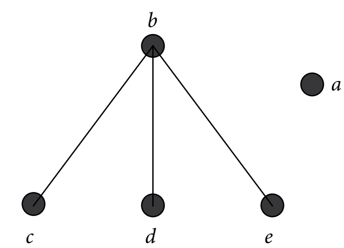

Figure 3: Figure of H for Example 15.

(i) for all $a,b \in H$, $((a\circ b)\circ a)\cap F \neq \emptyset \Rightarrow a \in F$,

(ii) for all $b \in H$ and $a \in F$, $a \leq b \Rightarrow b \in F$.

Example 17. Consider a set $H = \{a,b,c\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":

$\circ$	a	b	c
a	a	$\{a,b\}$	$\{a,c\}$
b	$\{a,c\}$	$\{b,c\}$	c
c	$\{a,b\}$	b	$\{b,c\}$

$\leq := \{(a,a),(a,b),(a,c),(b,b),(c,c)\}.$

We give the covering relation "$\prec$" as:

$$\prec = \{(a,b),(a,c)\}. \tag{6}$$

The figure of H is shown in Figure 4. Then $(H,\circ,\leq)$ is an ordered LA-semihypergroup. The bi-hyperfilters of H are $\{a\}$, $\{b,c\}$ and H. But H is the only strong bi-hyperfilter of H.

Lemma 18. *Let $(H,\circ,\leq)$ be an ordered LA-semihypergroup and S be an LA-subsemihypergroup of H. Then for a left (resp., right, bi) hyperfilter F of H, either $F\cap S = \emptyset$ or $F\cap S$ is a left (resp., right, bi) hyperfilter of S.*

Proof. Consider S is an LA-subsemihypergroup of H and F is a bi-hyperfilter of H. Suppose that $F\cap S = \emptyset$. Now $(F\cap S)^2 \subseteq F^2 \subseteq F$ and $(F\cap S)^2 \subseteq S^2 \subseteq S$. Therefore, $(F\cap S)^2 \subseteq F\cap S$. So $F\cap S$ is an LA-subsemihypergroup of S. Suppose that, for any $a,b \in S$, $(a\circ b)\circ a \subseteq F\cap S$. Therefore, $(a\circ b)\circ a \subseteq F$. As $a \in S$ and F is a bi-hyperfilter of H, then $a \in F$. Thus, $a \in F\cap S$. Finally take any $a \in F\cap S$ and $b \in S$ such that $a \leq b$. As F is a bi-hyperfilter of H with $a \in F$ and $a \leq b \in S$, $b \in F$. Therefore, $b \in F\cap S$. Hence $F\cap S$ is a bi-hyperfilter of S. The other cases can be seen in similar way. ☐

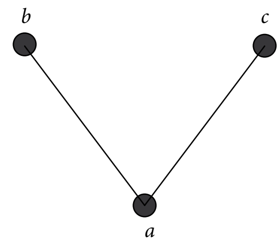

Figure 4: Figure of H for Example 17.

Lemma 19. *Let* $(H, \circ, \leq)$ *be an ordered LA-semihypergroup and* S *be an LA-subsemihypergroup of* H. *Then for a strong left (resp., strong right, strong bi) hyperfilter* F *of* H, *either* $F \cap S = \emptyset$ *or* $F \cap S$ *is a strong left (resp., strong right, strong bi) hyperfilter of* S.

Proof. Proof is straightforward. □

Theorem 20. *Let* $(H, \circ, \leq)$ *be an ordered LA-semihypergroup and* $\{F_i | i \in I\}$ *a family of left (resp., right, bi) hyperfilters of* H. *Then* $\bigcap_{i \in I} F_i$ *is a left (resp., right, bi) hyperfilter of* H *if* $\bigcap_{i \in I} F_i \neq \emptyset$, *where* $|I| \geq 2$.

Proof. Consider F_i is a bi-hyperfilter of H for each $i \in I$. Assume that $\bigcap_{i \in I} F_i \neq \emptyset$. Let $a, b \in \bigcap_{i \in I} F_i$. Then $a, b \in F_i$ for all $i \in I$. As, for all $i \in I$, F_i is a bi-hyperfilter of H, so $a \circ b \subseteq F_i$. Hence $a \circ b \subseteq \bigcap_{i \in I} F_i$, i.e., $\bigcap_{i \in I} F_i$ is an LA-subsemihypergroup of H. Suppose that, for any $a, b \in H$, $(a \circ b) \circ a \subseteq \bigcap_{i \in I} F_i$. Therefore, $(a \circ b) \circ a \subseteq F_i$ for all $i \in I$. As F_i is a bi-hyperfilter of H, then $a \in F_i$ for all $i \in I$. Thus, $a \in \bigcap_{i \in I} F_i$. Finally take any $a \in \bigcap_{i \in I} F_i$ and $b \in H$ such that $a \leq b$. Therefore, $a \in F_i$ for all $i \in I$. As F_i is a bi-hyperfilter of H with $a \in F_i$ and $a \leq b \in S$, this implies $b \in F_i$ for all $i \in I$. Therefore, $b \in \bigcap_{i \in I} F_i$. Hence $\bigcap_{i \in I} F_i$ is a bi-hyperfilter of H. The other cases can be seen in similar way. □

Theorem 21. *Let* $(H, \circ, \leq)$ *be an ordered LA-semihypergroup and* $\{F_i | i \in I\}$ *a family of strong left (resp., strong right, strong bi) hyperfilters of* H. *Then* $\bigcap_{i \in I} F_i$ *is a strong left (resp., strong right, strong bi) hyperfilter of* H *if* $\bigcap_{i \in I} F_i \neq \emptyset$, *where* $|I| \geq 2$.

Proof. Proof is straightforward. □

Remark 22. Generally, the union of two hyperfilters of an ordered LA-semihypergroup H need not be a hyperfilter of H.

The following example shows that, the union of two hyperfilters of an ordered LA-semihypergroup H is not a hyperfilter of H.

Example 23. Consider Example 1. One can easily see that $F_1 = \{a\}$ and $F_2 = \{b\}$ are both hyperfilters of H. But $F_1 \cup F_2 = \{a, b\}$ is not a hyperfilter of H, because $(F_1 \cup F_2)^2 \nsubseteq (F_1 \cup F_2)$, i.e., $F_1 \cup F_2$ is not an LA-subsemihypergroup of H.

Lemma 24. *Let* $(H, \circ, \leq)$ *be an ordered LA-semihypergroup and* F_1, F_2 *hyperfilters of* H. *Then* $F_1 \cup F_2$ *is a hyperfilter of* H *if and only if* $F_1 \subseteq F_2$ *or* $F_2 \subseteq F_1$.

Proof. Proof is straightforward. □

Theorem 25. *Let* H *be an ordered LA-semihypergroup and* F *be a nonempty subset of* H. *Then the following statements are equivalent;*

(i) F is a left hyperfilter of H.

(ii) $H \setminus F = \emptyset$ or $H \setminus F$ is a prime right hyperideal of H.

Proof. (i) ⇒ (ii) Assume that $H \setminus F \neq \emptyset$. Let $x \in H \setminus F$ and $y \in H$. If $x \circ y \nsubseteq H \setminus F$, then $x \circ y \subseteq F$. Since F is a left hyperfilter, so $x \in F$ which is not possible. Thus, $x \circ y \subseteq H \setminus F$, and so $(H \setminus F) \circ H \subset H \setminus F$. In order to show that $H \setminus F$ is right hyperideal of H, we need to show that $(H \setminus F] \circ H \subseteq (H \setminus F]$. Let $x \in (H \setminus F] \circ H$, which implies that $x \in y \circ z$ for some $y \in (H \setminus F]$ and $z \in H$. Now consider $x \leq y \circ z \subseteq H \setminus F$. Thus, $(H \setminus F] \circ H \subseteq (H \setminus F]$, which shows that $H \setminus F$ is right hyperideal of H.

Next we shall show that $H \setminus F$ is prime. Let $x \circ y \subseteq H \setminus F$ for $x, y \in H$. Suppose that $x \notin H \setminus F$ and $y \notin H \setminus F$. Then $x \in F$ and $y \in F$. Since F is an LA-subsemihypergroup of H, so $x \circ y \subseteq F$. It is not possible. Thus, $x \in H \setminus F$ and $y \in H \setminus F$. Hence, $H \setminus F$ is prime, and so $H \setminus F$ is a prime right hyperideal.

(ii) ⇒ (i) If $H \setminus F = \emptyset$, then $F = H$. Thus, in this case, F is a left hyperfilter of H. Next assume that $H \setminus F$ is a prime right hyperideal, then F is an LA-subsemihypergroup of H. Suppose that $x \circ y \nsubseteq F$ for $x, y \in F$. Then $x \circ y \subseteq H \setminus F$ for $x, y \in F$. Since $H \setminus F$ is prime, $x \in H \setminus F$ and $y \in H \setminus F$. It is not possible. Thus, $x \circ y \subseteq F$ and so F is an LA-subsemihypergroup of H. Let $x \circ y \subseteq F$ for $x, y \in H$. Then, $x \in F$. Indeed, if $x \notin F$, then $x \in H \setminus F$. Since $H \setminus F$ is prime right hyperideal of H, $x \circ y \subseteq H \setminus F \circ H \subset H \setminus F$. It is not possible. Thus, $x \in F$. Therefore, F is a left hyperfilter of H. □

Theorem 26. *Let* H *be an ordered LA-semihypergroup and* F *be a nonempty subset of* H. *Then the following statements are equivalent;*

(1) F is a right hyperfilter of H.

(2) $H \setminus F = \emptyset$ or $H \setminus F$ is a prime left hyperideal.

Proof. The proof is similar to the proof of Theorem 25. □

Theorem 27. *Let* H *be an ordered LA-semihypergroup and* F *be a nonempty subset of* H. *Then the following statements are equivalent;*

(1) F is a hyperfilter of H.

(2) $H \setminus F = \emptyset$ or $H \setminus F$ is a prime hyperideal of H.

Proof. Follows directly from Theorems 25 and 26. □

4. Rough Hyperfilters in Ordered LA-Semihypergroups

In this section, we will study some results on hyperfilters of ordered LA-semihypergroups in terms of rough sets which are based on pseudohyperorder relation.

Definition 28 [30]. Let X be a nonempty set and δ be a binary relation on X. By $\mathscr{P}(X)$ we mean the power set of X. For all $A \subseteq X$, we define δ^- and $\delta^+ : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\begin{aligned} \delta^-(A) &= \{x \in X : \forall y, x\delta y \Rightarrow y \in A\} \\ &= \{x \in X : \delta N(x) \subseteq A\}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \delta^+(A) &= \{x \in X : \exists y \in A, \text{ such that } x\delta y\} \\ &= \{x \in X : \delta N(x) \cap A \neq \emptyset\}, \end{aligned} \tag{8}$$

where $\delta N(x) = \{y \in X : x\delta y\}$. $\delta^-(A)$ and $\delta^+(A)$ are called the lower approximation and the upper approximation operations, respectively.

Definition 29 [30, Definition 4.1]. Let δ be a pseudohyperorder on an ordered LA-semihypergroup H. Then a nonempty subset A of H is called a δ-upper (resp., δ-lower) rough LA-subsemihypergroup of H if $\delta^+(A)$ (resp., $\delta^-(A)$) is an LA-subsemihypergroup of H.

Theorem 30 [30, Theorem 4.3]. *Let δ be a pseudohyperorder on an ordered LA-semihypergroup H and A an LA-subsemihypergroup of H. Then*

(1) $\delta^+(A)$ is an LA-subsemihypergroup of H.
(2) if δ is complete, then $\delta^-(A)$ is, if it is nonempty, an LA-subsemihypergroup of H.

Definition 31. Let δ be a pseudohyperorder on an ordered LA-semihypergroup H. Then a nonempty subset F of H is called a δ-lower rough left hyperfilter (resp., right hyperfilter, hyperfilter) of H if $\delta^-(F)$ is a left hyperfilter (resp., right hyperfilter, hyperfilter) of H.

Theorem 32. *Let δ be a complete pseudohyperorder on an ordered LA-semihypergroup H and F a left hyperfilter (resp., right hyperfilter, hyperfilter) of H. Then $\delta^-(F)$ is, if it is nonempty, a left hyperfilter (resp., right hyperfilter, hyperfilter) of H.*

Proof. Let F be a left hyperfilter of H. Since F is an LA-subsemihypergroup of H, then by Theorem 30(2), $\delta^-(F)$ is an LA-subsemihypergroup of H. Let $x \circ y \subseteq \delta^-(F)$ for some $x, y \in H$. Then $\delta N(x) \circ \delta N(y) = \delta N(x \circ y) \subseteq F$. We suppose that $\delta^-(F)$ is not a left hyperfilter of H. Then there exist $x, y \in H$ such that $x \circ y \subseteq \delta^-(F)$ but $x \notin \delta^-(F)$. Thus, $\delta N(x) \nsubseteq F$. Then there exist $a \in \delta N(x)$ such that $a \notin F$ and $b \in \delta N(y)$. Thus, $a \circ b \subseteq \delta N(x) \circ \delta N(y) \subseteq F$. Since F is a left hyperfilter of H, so we have $a \in F$, which is a contradiction. Now, let $x \in \delta^-(F)$ and $y \in H$ such that $x \leq y$. Then $\delta N(x) \subseteq F$ and $x\delta y$. This implies that $\delta N(x) = \delta N(y)$. Since $\delta N(x) \subseteq F$, so $\delta N(y) \subseteq F$. Thus, $y \in \delta^-(F)$. Thus, $\delta^-(F)$ is a left hyperfilter of H. The case for right hyperfilter and hyperfilter can be proved in a similar way. □

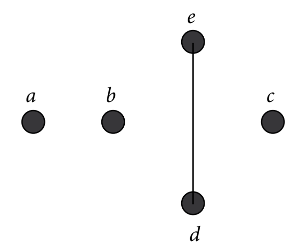

Figure 5: Figure of H for Example 35.

Definition 33. Let δ be a pseudohyperorder on an ordered LA-semihypergroup H. Then a nonempty subset F of H is called a δ-upper rough left hyperfilter (resp., right hyperfilter, hyperfilter) of H if $\delta^+(F)$ is a left hyperfilter (resp., right hyperfilter, hyperfilter) of H.

Theorem 34. *Let δ be a complete pseudohyperorder on an ordered LA-semihypergroup H and F a left hyperfilter (resp., right hyperfilter, hyperfilter) of H. Then $\delta^+(F)$ is a left hyperfilter (resp., right hyperfilter, hyperfilter) of H.*

Proof. Let F be a left hyperfilter of H. Since F is an LA-subsemihypergroup of H, then by Theorem 30(1), $\delta^+(F)$ is an LA-subsemihypergroup of H. Let $x \circ y \subseteq \delta^+(F)$ for some $x, y \in H$. Then $(\delta N(x) \circ \delta N(y)) \cap F = \delta N(x \circ y) \cap F \neq \emptyset$, so there exist $a \in \delta N(x)$ and $b \in \delta N(y)$ such that $a \circ b \subseteq F$. Since F is left hyperfilter of H, so we have $a \in F$. Thus, $\delta N(x) \cap F \neq \emptyset$ which implies that $x \in \delta^+(F)$. Now, let $x \in \delta^+(F)$ and $y \in H$ such that $x \leq y$. Then $\delta N(x) \cap F \neq \emptyset$ and $x\delta y$. This implies that $\delta N(x) = \delta N(y)$. Since $\delta N(x) \cap F \neq \emptyset$, so $\delta N(y) \cap F \neq \emptyset$. Thus, $y \in \delta^+(F)$. Thus, $\delta^+(F)$ is a left hyperfilter of H, that is, F is a δ-upper rough left hyperfilter of H. The case for right hyperfilter and hyperfilter can be proved in a similar way. □

The converse of Theorems 32 and 34 is not true in general.

Example 35. Consider a set $H = \{a, b, c, d, e\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":

$\circ$	a	b	c	d	e
a	a	b	c	d	e
b	c	d	$\{d, e\}$	$\{d, e\}$	e
c	b	$\{d, e\}$	d	$\{d, e\}$	e
d	d	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$	e
e	e	e	e	e	e

$\leq := \{(a, a), (b, b), (c, c), (d, d), (d, e), (e, e)\}.$

We give the covering relation "$\prec$" as:

$$\prec = \{(d,e)\}. \tag{9}$$

The figure of H is shown in Figure 5. Then $(H,\circ,\leq)$ is an ordered LA-semihypergroup. Now let

$$\delta = \{(a,a),(b,b),(b,c),(c,b),(c,c),(d,d),(d,e),(e,d),(e,e)\} \tag{10}$$

be a complete pseudohyperorder on H, such that

$$\delta N(a) = \{a\}, \delta N(b) = \delta N(c) = \{b,c\} \text{ and } \delta N(d) = \delta N(e) = \{d,e\}. \tag{11}$$

Now for $\{a,b,d\} \subseteq H$,

$$\delta^{-}(\{a,b,d\}) = \{a\} \quad \text{and } \delta^{+}(\{a,b,d\}) = \{a,b,c,d,e\}. \tag{12}$$

It is clear that $\delta^{-}(\{a,b,d\})$ and $\delta^{+}(\{a,b,d\})$ are both hyperfilters of H, but $\{a,b,d\}$ is not a hyperfilter of H.

Definition 35. Let δ be a pseudohyperorder on an ordered LA-semihypergroup H. Then a nonempty subset F of H is called a δ-lower rough bi-hyperfilter of H if $\delta^{-}(F)$ is a bi-hyperfilter of H.

Theorem 36. *Let δ be a complete pseudohyperorder on an ordered LA-semihypergroup H and F a bi-hyperfilter of H. Then $\delta^{-}(F)$ is, if it is nonempty, a bi-hyperfilter of H.*

Proof. Let F be a bi-hyperfilter of H. Since F is an LA-subsemihypergroup of H, then by Theorem 30(2), $\delta^{-}(F)$ is an LA-subsemihypergroup of H. Let $(x\circ y)\circ x \subseteq \delta^{-}(F)$ for some $x,y\in H$. Then $(\delta N(x)\circ\delta N(y))\circ\delta N(x) = \delta N((x\circ y)\circ x) \subseteq F$. We suppose that $\delta^{-}(F)$ is not a bi-hyperfilter of H. Then there exist $x,y\in H$ such that $(x\circ y)\circ x\subseteq\delta^{-}(F)$, but $x\notin\delta^{-}(F)$. Thus, $\delta N(x)\nsubseteq F$. Then there exist $a\in\delta N(x)$ such that $a\notin F$ and $b\in\delta N(y)$. Thus, $(a\circ b)\circ a\subseteq(\delta N(x)\circ\delta N(y))\circ\delta N(x)\subseteq F$. Since F is a bi-hyperfilter of H, so we have $a\in F$, which is a contradiction. Now, let $x\in\delta^{-}(F)$ and $y\in H$ such that $x\leq y$. Then $\delta N(x)\subseteq F$ and $x\delta y$. This implies that $\delta N(x)=\delta N(y)$. Since $\delta N(x)\subseteq F$, so $\delta N(y)\subseteq F$. Thus $y\in\delta^{-}(F)$. Hence $\delta^{-}(F)$ is a bi-hyperfilter of H. □

Definition 37. Let δ be a pseudohyperorder on an ordered LA-semihypergroup H. Then a nonempty subset F of H is called a δ-upper rough bi-hyperfilter of H if $\delta^{+}(F)$ is a bi-hyperfilter of H.

Theorem 38. *Let δ be a complete pseudohyperorder on an ordered LA-semihypergroup H and F a bi-hyperfilter of H. Then $\delta^{+}(F)$ is a bi-hyperfilter of H.*

Proof. Let F be a bi-hyperfilter of H. Since F is an LA-subsemihypergroup of H, then by Theorem 30(1), $\delta^{+}(F)$ is an LA-subsemihypergroup of H. Let $(x\circ y)\circ x\subseteq\delta^{+}(F)$ for some $x,y\in H$. Then $((\delta N(x)\circ\delta N(y))\circ\delta N(x))\cap F=\delta N((x\circ y)\circ x)\cap F\neq\emptyset$, so there exist $a\in\delta N(x)$ and $b\in\delta N(y)$ such that $(a\circ b)\circ a\subseteq F$. Since F is bi-hyperfilter of H, so we have $a\in F$. Thus, $\delta N(x)\cap F\neq\emptyset$ which implies that $x\in\delta^{+}(F)$. Now, let $x\in\delta^{+}(F)$ and $y\in H$ such that $x\leq y$. Then $\delta N(x)\cap F\neq\emptyset$ and $x\delta y$. This implies that $\delta N(x)=\delta N(y)$. Since $\delta N(x)\cap F\neq\emptyset$, so $\delta N(y)\cap F\neq\emptyset$. Thus, $y\in\delta^{+}(F)$. Thus, $\delta^{+}(F)$ is a bi-hyperfilter of H, that is, F is a δ-upper rough bi-hyperfilter of H. □

Definition 39. Let δ be a pseudohyperorder on an ordered LA-semihypergroup H. Then a nonempty subset F of H is called a δ-lower rough strong left hyperfilter (resp., strong right hyperfilter, strong hyperfilter) of H if $\delta^{-}(F)$ is a strong left hyperfilter (resp., strong right hyperfilter, strong hyperfilter) of H.

Theorem 40. *Let δ be a complete pseudohyperorder on an ordered LA-semihypergroup H and F a strong left hyperfilter (resp., strong right hyperfilter, strong hyperfilter) of H. Then $\delta^{-}(F)$ is, if it is nonempty, a strong left hyperfilter (resp., strong right hyperfilter, strong hyperfilter) of H.*

Proof. Let F be a strong hyperfilter of H. Since F is an LA-subsemihypergroup of H, then by Theorem 30(2), $\delta^{-}(F)$ is an LA-subsemihypergroup of H. Let $(x\circ y)\cap\delta^{-}(F)\neq\emptyset$ for some $x,y\in H$. For $t\in(x\circ y)\cap\delta^{-}(F)$, we have $t\in x\circ y$ and $t\in\delta^{-}(F)$. This implies that $\delta N(t)\subseteq\delta N(x\circ y)=\delta N(x)\circ\delta N(y)$ and $\delta N(t)\subseteq F$. This implies that $\delta N(t)\subseteq(\delta N(x)\circ\delta N(y))\cap F$. Thus $(\delta N(x)\circ\delta N(y))\cap F\neq\emptyset$. We suppose that $\delta^{-}(F)$ is not a strong hyperfilter of H. Then there exist $x,y\in H$ such that $(x\circ y)\cap\delta^{-}(F)\neq\emptyset$ but $x\notin\delta^{-}(F)$ and $y\notin\delta^{-}(F)$. Thus, $\delta N(x)\nsubseteq F$ and $\delta N(y)\nsubseteq F$. Then there exist $a\in\delta N(x)$ and $b\in\delta N(y)$ such that $a\notin F$ and $b\notin F$. Thus, $(a\circ b)\cap F\subseteq(\delta N(x)\circ\delta N(y))\cap F\neq\emptyset$. Since F is a strong hyperfilter of H, so we have $a\in F$ and $b\in F$, which is a contradiction. Now, let $x\in\delta^{-}(F)$ and $y\in H$ such that $x\leq y$. Then $\delta N(x)\subseteq F$ and $x\delta y$. This implies that $\delta N(x)=\delta N(y)$. Since $\delta N(x)\subseteq F$, so $\delta N(y)\subseteq F$. Thus, $y\in\delta^{-}(F)$. Hence $\delta^{-}(F)$ is a strong hyperfilter of H. The case for strong left hyperfilter and strong right hyperfilter can be proved in a similar way. □

Definition 41. Let δ be a pseudohyperorder on an ordered LA-semihypergroup H. Then a nonempty subset F of H is called a δ-upper rough strong left hyperfilter (resp., strong right hyperfilter, strong hyperfilter) of H if $\delta^{+}(F)$ is a strong left hyperfilter (resp., strong right hyperfilter, strong hyperfilter) of H.

Theorem 42. *Let δ be a complete pseudohyperorder on an ordered LA-semihypergroup H and F a strong left hyperfilter (resp., strong right hyperfilter, strong hyperfilter) of H. Then $\delta^{+}(F)$ is a strong left hyperfilter (resp., strong right hyperfilter, strong hyperfilter) of H.*

Proof. Let F be a strong hyperfilter of H. Since F is an LA-subsemihypergroup of H, then by Theorem 30(1), $\delta^{+}(F)$ is an

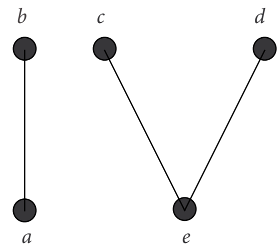

Figure 6: Figure of H for Example 42.

LA-subsemihypergroup of H. Let $(x \circ y) \cap \delta^+(F) \neq \emptyset$ for some $x, y \in H$. Then $(\delta N(x) \circ \delta N(y)) \cap F = \delta N(x \circ y) \cap F \neq \emptyset$, so there exist $a \in \delta N(x)$ and $b \in \delta N(y)$ such that $(a \circ b) \cap F \neq \emptyset$. Since F is strong hyperfilter of H, so we have $a \in F$ and $b \in F$. Thus, $\delta N(x) \cap F \neq \emptyset$ and $\delta N(y) \cap F \neq \emptyset$, which implies that $x \in \delta^+(F)$ and $y \in \delta^+(F)$. Now, let $x \in \delta^+(F)$ and $y \in H$ such that $x \leq y$. Then $\delta N(x) \cap F \neq \emptyset$ and $x \delta y$. This implies that $\delta N(x) = \delta N(y)$. Since $\delta N(x) \cap F \neq \emptyset$, so $\delta N(y) \cap F \neq \emptyset$. Thus, $y \in \delta^+(F)$. Thus, $\delta^+(F)$ is a strong hyperfilter of H, that is, F is a δ-upper rough strong hyperfilter of H. The case for strong left hyperfilter and strong right hyperfilter can be proved in a similar way. □

The converse of Theorems 39 and 41 is not true in general.

Example 43. Consider a set $H = \{a, b, c, d, e\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":

$\circ$	a	b	c	d	e
a	b	b	e	e	e
b	$\{a, b\}$	$\{a, b\}$	e	e	e
c	e	e	$\{c, d\}$	d	e
d	e	e	c	$\{c, d\}$	e
e	e	e	e	e	e

$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, d), (e, c), (e, d), (e, e)\}$.

We give the covering relation "$\prec$" as:

$$\prec = \{(a, b), (e, c), (e, d)\}. \tag{13}$$

The figure of H is shown in Figure 6. Then $(H, \circ, \leq)$ is an ordered LA-semihypergroup. Now let δ be a complete pseudohyperorder on H, such that

$$\delta N(a) = \delta N(b) = \{a, b\} \text{ and } \delta N(c) = \delta N(d) = \delta N(e) = \{c, d, e\}. \tag{14}$$

Now for $\{a, b, c\} \subseteq H$,

$$\delta^-(\{a, b, c\}) = \{a, b\} \text{ and } \delta^+(\{a, b, c\}) = \{a, b, c, d, e\}. \tag{15}$$

It is clear that $\delta^-(\{a, b, c\})$ and $\delta^+(\{a, b, c\})$ are both strong hyperfilters of H but $\{a, b, c\}$ is not a hyperfilter of H.

Definition 44. Let δ be a pseudohyperorder on an ordered LA-semihypergroup H. Then a nonempty subset F of H is called a δ-lower rough strong bi-hyperfilter of H if $\delta^-(F)$ is a strong bi-hyperfilter of H.

Theorem 45. *Let δ be a complete pseudohyperorder on an ordered LA-semihypergroup H and F a strong bi-hyperfilter of H. Then $\delta^-(F)$ is, if it is nonempty, a strong bi-hyperfilter of H.*

Proof. The proof is similar to the proof of Theorem 40. □

Definition 46. Let δ be a pseudohyperorder on an ordered LA-semihypergroup H. Then a nonempty subset F of H is called a δ-upper rough strong bi-hyperfilter of H if $\delta^+(F)$ is a strong bi-hyperfilter of H.

Theorem 47. *Let δ be a complete pseudohyperorder on an ordered LA-semihypergroup H and F a strong bi-hyperfilter of H. Then $\delta^+(F)$ is a strong bi-hyperfilter of H.*

Proof. The proof is similar to the proof of Theorem 42. □

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors of this paper Ferdaous Bouaziz and Naveed Yaqoob declare that they have no conflicts of interest.

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