Hindawi
Discrete Dynamics in Nature and Society
Volume 2022, Article ID 1241761, 44 pages
https://doi.org/10.1155/2022/1241761

Research Article

Mathematical Analysis of the Prey-Predator System with Immigrant Prey Using the Soft Computing Technique

Naveed Ahmad Khan,[1] **Muhammad Sulaiman**,[1] **Jamel Seidu**,[2] **and Fahad Sameer Alshammari**[3]

[1]*Department of Mathematics, Abdul Wali Khan University Mardan, Mardan, KP, Pakistan*
[2]*School of Railways and Infrastructure Development, University of Mines and Technology (UMaT) Essikado, Sekondi-Takoradi, Ghana*
[3]*Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia*

Correspondence should be addressed to Jamel Seidu; jseidu@umat.edu.gh

Received 28 May 2022; Revised 20 August 2022; Accepted 19 September 2022; Published 13 October 2022

Academic Editor: Chenguang Yang

In this paper, a mathematical model for the system of prey-predator with immigrant prey has been analyzed to find an approximate solution for immigrant prey population density, local prey population density, and predator population density. Furthermore, we present a novel soft computing technique named LeNN-WOA-NM algorithm for solving the mathematical model of the prey-predator system with immigrant prey. The proposed algorithm uses a function approximating ability of Legendre polynomials based on Legendre neural networks (LeNNs), global search ability of the whale optimization algorithm (WOA), and a local search mechanism of the Nelder–Mead algorithm. The LeNN-WOA-NM algorithm is applied to study the effect of variations on the growth rate, the force of interaction, and the catching rate of local prey and immigrant prey. The statistical data obtained by the proposed technique establish the effectiveness of the proposed algorithm when compared with techniques in the latest literature. The efficiency of solutions obtained by LeNN-WOA-NM is validated through performance measures including absolute errors, MAD, TIC, and ENSE.

1. Introduction

A prey-predator system is one of the prevalent phenomena in nature. The interaction between predators and prey in any environment was first introduced by Lotka and Volterra in 1926 [1]. Holling in 1966 [2] elaborated prey-predictor models with different kinds of functional responses for predation. All these models are inspired by biological phenomena and presented by nonlinear ordinary and partial differential equations. Danca et al. [3] study the detailed analysis of a nonlinear prey-predator model. A connection between predators and prey has a long history and will continue as a governing theme in biomathematics because of its universal significance [4]. Researchers have made many changes by introducing different facets to the predator-prey system such as delay in predator growth, harvesting prey and predators, and providing additional food to a predator for sustaining prey population and prey diseases [5–7]. Cai et al. [8] studied the dynamical system of prey-predator with Allee effects in prey growth. A stage-structured predator-prey model with gestation delay is investigated by [9]. In [10, 11], bifurcation, chaos and dynamic behavior of the nonlinear discrete-time predator-prey system is studied. Huang et al. [12] study the stability analysis of the model with consideration of the prey refuge. Studies by Din [13], Weide et al. [14], and Gong et al. [15, 16] are presented for the discrete-time nonlinear prey-predator types of model. Hadeler and Freedman [17] provide extensive references to real-world examples of three-species ecoepidemiological systems of sound prey, infected prey, and predators.

In recent years, there has been a substantial increase in the amount of attention paid to population biology by scientists due to the significant applications it has in ecology. It bridges the gap between mathematics and biology. The dynamical systems in biology have been investigated to interpret various problems. The Lotka–Volterra population model is a well-known mathematical model that describes biological systems [18]. In real life, there is a strong correlation between size, age, and developmental stages of different populations of species. It is an important strategy to incorporate all these variables in the mathematical modeling of populations of different spices. To develop more realistic and accurate mathematical models, many scientists have suggested noise-induced models [19, 20] and spatial models [21]. In biomathematical problems, researchers utilize mathematical modeling and simulations along with biological structures to describe the phenomena by the system of nonlinear differential equations. These nonlinear models are considered stiff and unrealistic, and therefore, finding exact and semianalytical solutions for such problems is challenging because of nonlinearity. Analytical approaches for solving nonlinear problems are mostly based on Laplace or Fourier transformation, Laguerre's integral formula, and the Grunwald–Letnikov concept [22, 23]. When tackling complex problems, these techniques may be challenging to use; in addition, the solution is provided in a closed form that necessitates the evaluation of special functions using complex expressions, such as the Mittag-Leffler function [24].

In recent years, researchers have been working to develop new techniques for finding approximate solutions to nonlinear models; e.g., the Laplace Adomian decomposition method [25], the new coupled fractional reduced differential transform method [26], the Runge–Kutta–Fehlberg method [27], the finite element method [28], the Sumudu decomposition method [29], the implicit Adams methods [30], the confidence domain technique [31], and the homotopy analysis method [32] have become much more significant to get accurate solutions. All these deterministic approaches have their own advantages, applicability, and drawbacks. With great interest, it is noted that such techniques are gradient-based and call for information about the problem beforehand. The availability of several local optima, which leads to solutions where global optimality cannot be easily ensured, is one of the fundamental limitations of gradient-based approaches. Global optimality is sought in gradient-based approaches by randomly scanning the design space from various starting points. However, this causes the technique to become sluggish and computationally inefficient for complex nonlinear optimization problems [33, 34]. In addition, the Runge–Kutta methods are self-starting and stable techniques that can be easily implemented to calculate the solution to different problems. The main drawbacks of the Runge–Kutta methods are that they take longer time to calculate solutions than other multistep methods with equivalent precision, and it is difficult for them to get accurate global estimates of the truncation error [35]. To overcome these drawbacks, a stochastic metaheuristic approach with artificial neural networks is developed, which is free of a gradient and does not require any prior information about the problem. The majority of metaheuristic techniques are inspired by natural, physical, or biological processes and make use of a variety of operators to mimic the fundamental behavior. The harmony between exploration and exploitation is a recurring subject in all metaheuristics.

In recent times, artificial neural network (ANN)-based stochastic algorithms with global and local search optimizers have been designed to solve differential equations representing physical phenomena including flow in a circular cylindrical conduit via electrohydrodynamics [36], a model of an immobilized enzyme system that follows the Michaelis–Menten (MM) kinetics for a microdisk biosensor [37], flow of Johnson–Segalman fluid on the surface of an infinitely long vertical cylinder [38], and beam-column designs [39]. The abovementioned techniques motivate authors to design a new soft computing algorithm, the LeNN-WOA-NM algorithm, to find approximate series solutions using Legendre polynomials for the model presenting the prey-predator system with immigrant prey. The salient features of the paper are summarized as follows:

(i) A mathematical model for a prey-predator system with immigrant prey is formulated and analyzed to study the influence of variations on the growth rate, force of interaction, and the catching rate of local and immigrant prey.

(ii) Artificial neural network-based weighted Legendre polynomials are used to construct the model of approximate solutions for the prey-predator model. A fitness function based on mean square errors is designed to assess unknown parameters with the help of global search ability of whale optimization and a local search mechanism of the Nelder–Mead algorithm.

(iii) The suggested technique can result in adequate solutions for nonlinear hard problems for which no exact algorithm exists that can solve them in a reasonable amount of time.

(iv) Performance of the proposed algorithm is validated in terms of absolute errors, mean absolute deviation, Theil's inequality coefficient, Nash–Sutcliffe efficiency, and error in Nash–Sutcliffe efficiency.

(v) Approximate solutions, convergence of fitness, and performance measures obtained by the LeNN-WOA-NM algorithm for prey-predator systems with immigrant prey are shown through different graphs and tables, which shows the dominance and robustness of the proposed algorithm in solving real-world problems.

(vi) Unlike most classical methods, the LeNN-WOA-NM algorithm requires no gradient information and therefore can be used with nonanalytic, black-box, or simulation-based objective functions to approximate complex problems.

2. Problem Formulation

Goteti developed the mathematical model for a prey-predator system with immigrant prey to understand the interaction and communication between predators and immigrant prey. The model is assumed to follow mass action theory and consists of prey population density, which is defined as follows:

$$N(t) = X(t) + S(t), \tag{1}$$

where $S(t)$ and $X(t)$ denote the population density of local and immigrant prey, respectively. Population density of the predator is denoted by $Y(t)$.

The following assumptions are used in formulating the mathematical model for the prey-predator system with immigrant prey:

(i) In the presence of a predator, the population of prey is classified into two subcategories named local prey $S(t)$ and immigrant prey $X(t)$.

(ii) In absence of the predator, the population growth of local prey logistically increases with an intrinsic growth rate α_1 with environmental carrying capacity denoted by k_1.

(iii) With availability of the local and immigrant prey population, the population of the predator grows logistically with growth rates c_1 and c_2, while suffering loss of populations is denoted by μ_1 and μ_2.

(iv) Immigrant and local prey can reproduce, and therefore, it is assumed that birth rates should be positive. The growth rate of immigrant prey increases at the rate α_2 with environmental carrying capacity denoted by k_2.

(v) It is assumed that immigrant prey is a natural choice of predators. β_1 and β_2 denote the positive and negative force of interaction between local and immigrant prey.

(vi) It is assumed that the local prey and immigrant prey are caught by the predator at the rate of γ_1 and γ_2, respectively.

A mathematical model for a prey-predator system with immigrant prey is given by the following system of differential equations:

$$\begin{aligned} &\frac{dS}{dt} - S\left(\alpha_1 - \frac{\alpha_1 S}{k_1}\right) - \beta_1 SX + \gamma_1 SY = 0,\\ &\frac{dX}{dt} - X\left(\alpha_2 - \frac{\alpha_2 X}{k_2}\right) + \beta_2 SX + \gamma_2 XY = 0, \end{aligned} \tag{2}$$

$$\frac{dY}{dt} - c_1 SY - c_2 XY + \mu_1 Y + \mu_2 Y^2 = 0, \tag{3}$$

with initial populations

$$S = S_0, X = X_0, Y = Y_0 \quad at \quad t = 0. \tag{4}$$

3. Approximate Solutions and Weighted Legendre Polynomials

The Legendre polynomials are denoted by $L_n(t)$, where n denotes the order of Legendre polynomials. These polynomials constitute the set of orthogonal polynomials on $[-1, 1]$. The first eleven Legendre polynomials are given in Table 1. High-order Legendre polynomials are generated by the following recursive formula:

$$L_{n+1}(r) = \frac{1}{n+1}\left[(2n+1)rL_n(r) - nL_{n-1}(r)\right]. \tag{5}$$

We consider an approximate series solution for equations (2)–(4) representing the prey-predator model with immigrant prey as follows:

$$\begin{cases} S_{\text{approx}}(t) = \sum_{n=1}^{34} \zeta_n L_n(\psi_n(t) + \theta_n),\\ X_{\text{approx}}(t) = \sum_{n=35}^{68} \zeta_n L_n(\psi_n(t) + \theta_n),\\ Y_{\text{approx}}(t) = \sum_{n=69}^{102} \zeta_n L_n(\psi_n(t) + \theta_n), \end{cases} \tag{6}$$

where ζ_n, ψ_n, and θ_n are unknown parameters.

Since nth order continuous derivatives of system (6) exist, we consider the first derivative of system (6) as follows:

$$\begin{cases} \frac{d}{dt}\left(S_{\text{approx}}\right) = \sum_{n=1}^{34} \zeta_n L'_n(\psi_n(t) + \theta_n),\\ \frac{d}{dt}\left(X_{\text{approx}}\right) = \sum_{n=35}^{68} \zeta_n L'_n(\psi_n(t) + \theta_n),\\ \frac{d}{dt}\left(Y_{\text{approx}}\right) = \sum_{n=69}^{102} \zeta_n L'_n(\psi_n(t) + \theta_n), \end{cases} \tag{7}$$

and we plug equations (6)–(7) in governing ordinary differential equations. Equations (2)–(4) will be transformed into an equivalent algebraic system of equations that can be solved for unknown parameters ζ_n, ψ_n, and θ_n using the LeNN-WOA-NM algorithm.

4. Fitness Function Formulation

The mean square error (MSE)-based fitness function for solving the prey-predator model for equations (2)-(4) can be written as follows:

$$\text{Minimize}\, \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6, \tag{8}$$

where ϵ_1 to ϵ_6 are the fitness-based errors for equations (2)–(4) along with the initial conditions and are given as follows:

Table 1: Eleven Legendre polynomials with an independent variable t.

n	$L_n(t)$
0	1
1	t
2	$1/2\,(3t^2 - 1)$
3	$1/2\,(5t^3 - 3t)$
4	$1/8\,(35t^4 - 30t^2 + 3)$
5	$1/8\,(63t^5 - 70t^3 + 15t)$
6	$1/16\,(231t^6 - 315t^4 + 105t^2 - 5)$
7	$1/16\,(429t^7 - 693t^5 + 315t^3 - 35t)$
8	$1/128\,(6435t^8 - 12012t^6 + 6930t^4 - 1260t^2 + 35)$
9	$1/128\,(12155t^9 - 25740t^7 + 18018t^5 - 4620t^3 + 315t)$
10	$1/256\,(46189t^{10} - 109395t^8 + 90090t^6 - 30030t^4 + 3465t^2 - 63)$

$$
\begin{cases}
\varepsilon_1 = \dfrac{1}{N}\displaystyle\sum_{n=1}^{34}\left(\dfrac{dS_n}{dt} - S_n\left(\alpha_1 - \dfrac{\alpha_1 S_n}{k_1}\right) - \beta_1 S_n X_n + \gamma_1 S_n Y_n\right)^2, \\
\varepsilon_2 = \dfrac{1}{N}\displaystyle\sum_{n=35}^{68}\left(\dfrac{dX_n}{dt} - X_n\left(\alpha_2 - \dfrac{\alpha_2 X_n}{k_2}\right) + \beta_2 S_n X_n + \gamma_2 X_n Y_n\right)^2, \\
\varepsilon_3 = \dfrac{1}{N}\displaystyle\sum_{n=69}^{102}\left(\dfrac{dY_n}{dt} - c_1 S_n Y_n - c_2 X_n Y_n + \mu_1 Y_n + \mu_2 Y_n^2\right)^2, \\
\varepsilon_4 = (S_0 - 0.5)^2, \\
\varepsilon_5 = (X_0 - 0.5)^2, \\
\varepsilon_6 = (Y_0 - 0.5)^2.
\end{cases}
\tag{9}
$$

5. Optimization Network

5.1. Whale Optimization Algorithm. The whale optimization algorithm is a nature-inspired metaheuristic algorithm designed by Mirjalili and Lewis [40]. The working strategy of WOA is inspired by foraging behavior of humpback whales. The humpback whales chase prey or krill by swimming around them in a molded way as shown in Figure 1. The mathematical model of each phase is explained below.

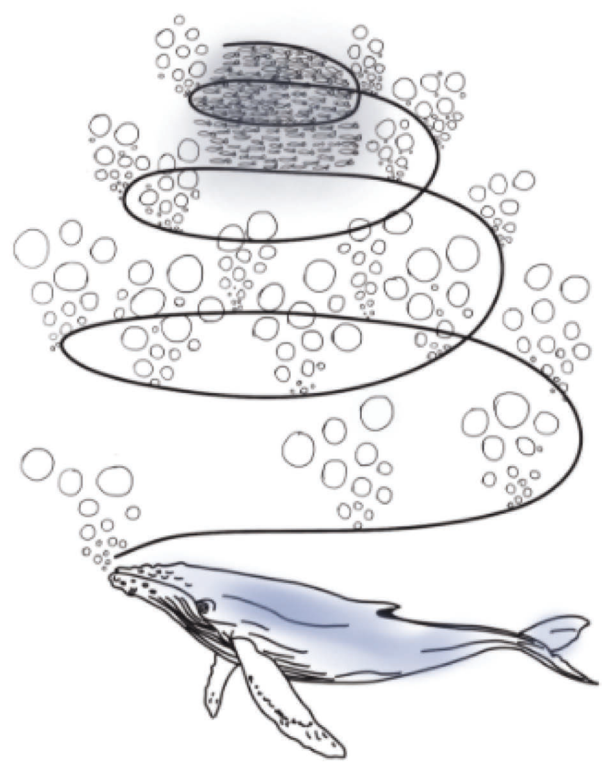


Figure 1: Unique bubble-net feeding method and spiral moment for updating the position of humpback whales.

5.1.1. Exploration Phase. Humpback whales encircle prey for hunting. Equations (10) and (11) mathematically model this behavior as follows:

$$\overrightarrow{E} = \left|\overrightarrow{C} \cdot \overrightarrow{Z}^*(t) - \overrightarrow{Z}(t)\right|, \tag{10}$$

$$\overrightarrow{Z}^*(t+1) = \overrightarrow{Z}^*(t) - \overrightarrow{A} \cdot D, \tag{11}$$

where t denotes the current iteration, Z^* has provided the best solution so far, and D gives the location of humpback whales to prey at each step. A and C are coefficient vectors which are defined as follows:

$$\overrightarrow{A} = 2\overrightarrow{a} \cdot \overrightarrow{r} - \overrightarrow{a}, \tag{12}$$

$$\overrightarrow{C} = 2 \cdot \overrightarrow{r}. \tag{13}$$

where $r \in [0, 1]$ is an arbitrary nominated vector. The value of a reduces from two to zero during exploration as well as in the exploitation phase. During helix-shaped movement, the distance between prey and the humpback whale is given as follows:

$$\overrightarrow{Z}(t+1) = D' \cdot e^{bl} \cdot \cos(2\pi l) + \overrightarrow{Z}^*(t), \tag{14}$$

where $D\prime$ represents the location of the ith whale to the prey and is defined as follows:

$$D\prime = \left|\vec{Z}^{*}(t) - \vec{Z}(t)\right|. \tag{15}$$

Furthermore, b denotes the state of the logarithmic helix and $l \in [-1, 1]$ is any arbitrary number. The shrinking surrounding technique of humpback whales during contracting loop is summarized as follows:

$$\vec{Z}(t+1) = \begin{cases} \vec{Z}(t) - \vec{A} \cdot D & \text{if } p \leq 0.5, \\ \vec{D}\prime \cdot e^{bl} \cdot \cos(2\pi l) + \vec{Z}^{*}(t) & \text{if } p \geq 0.5. \end{cases} \tag{16}$$

5.1.2. Exploitation Phase. In the exploration phase (searching for prey), the heterogeneity of the vector A is utilized. If $|A| > 1$, the position of the search agent is updated, and the entire mechanism is modeled by the following equations:

$$\vec{D} = \vec{C} \cdot \vec{Z}^{*}_{\text{rand}} - \vec{Z}, \tag{17}$$

$$\vec{Z}(t+1) = \vec{Z}_{\text{rand}} - \vec{A} \cdot \vec{D}, \tag{18}$$

where $\overrightarrow{Z_{\text{rand}}}$ is an arbitrary position vector taken from the current population.

Figure 2 shows the flowchart of the WOA. It can be seen that the WOA creates a random initial population from candidate space and evaluates it using an error-based fitness function when the optimization process starts. After finding the best solution, the algorithm repeatedly executes the following steps until ending criterion is achieved.

5.2. Nelder–Mead Algorithm. A Nelder–Mead (NM) algorithm is a direct search method also known as a downhill simplex method developed by Nelder and Mead in 1965 to solve different problems without any information about the gradient [41]. NM is a single path following a local search optimizer that can find good results if initialized with a better initial solution. A simplex consisting of $n + 1$ vertices is set up to minimize a function f with dimensions n [42]. The NM algorithm generates a sequence of simplices by following four basic procedures, namely, reflection, expansion, contraction, and shrink. Further details about the NM algorithm can be found in [43]. Figure 2 shows the working procedure of the NM algorithm.

In recent times, the most successful and effective trend in optimization is the action of integrating components from different methods. The foremost motivation behind the hybridization of diverse algorithmic ideas is to acquire better performing systems, which exploit and coalesce benefits of different techniques. Therefore, in this study, we have combined the global and local search optimization algorithms to achieve more efficient and robust solutions. The hybridized working procedure of the designed LeNN-WOA-NM algorithm is illustrated in Figure 2.

6. Performance Measures

To examine the performance of the proposed algorithm, performance indices are defined in terms of mean absolute deviation (MAD), Theil's inequality coefficient (TIC), and error in Nash–Sutcliffe efficiency (ENSE). Mathematical formulation for $S_{\text{approx}}(t)$, $X_{\text{approx}}(t)$, and $Y_{\text{approx}}(t)$ of the predator and prey model in the case of MAD, TIC, and ENSE is presented as follows:

$$[\text{MAD}_S, \text{MAD}_X, \text{MAD}_Y] = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n}\left|S(t_i) - S_{\text{approx}}(t_i)\right| \\ \frac{1}{n}\sum_{i=1}^{n}\left|X(t_i) - X_{\text{approx}}(t_i)\right| \\ \frac{1}{n}\sum_{i=1}^{n}\left|Y(t_i) - Y_{\text{approx}}(t_i)\right| \end{bmatrix}^{T}, \tag{19}$$

$$[TIC_S, TIC_X, TIC_Y] = \begin{bmatrix} \dfrac{\sqrt{(1/n)\sum_{i=1}^{n}\left(S(t_i) - S_{\text{approx}}(t_i)\right)^2}}{\sqrt{(1/n)\sum_{i=1}^{n}\left(S(t_i)\right)^2} + \sqrt{(1/n)\sum_{i=1}^{n}\left(S_{\text{approx}}(t_i)\right)^2}} \\ \dfrac{\sqrt{(1/n)\sum_{i=1}^{n}\left(X(t_i) - X_{\text{approx}}(t_i)\right)^2}}{\sqrt{(1/n)\sum_{i=1}^{n}\left(X(t_i)\right)^2} + \sqrt{(1/n)\sum_{i=1}^{n}\left(X_{\text{approx}}(t_i)\right)^2}} \\ \dfrac{\sqrt{(1/n)\sum_{i=1}^{n}\left(Y(t_i) - Y_{\text{approx}}(t_i)\right)^2}}{\sqrt{(1/n)\sum_{i=1}^{n}\left(Y(t_i)\right)^2} + \sqrt{(1/n)\sum_{i=1}^{n}\left(Y_{\text{approx}}(t_i)\right)^2}} \end{bmatrix}^{T}, \tag{20}$$

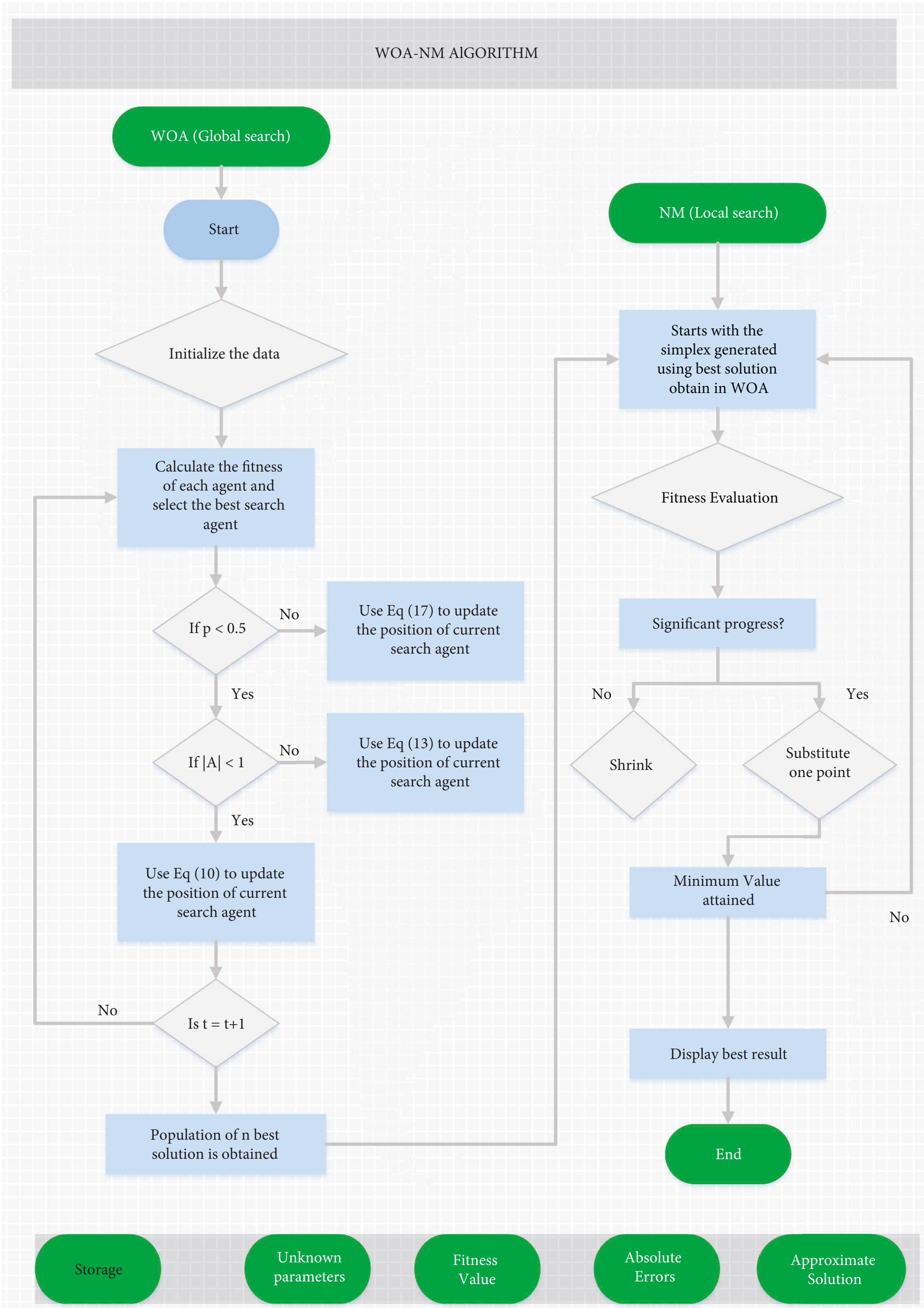


FIGURE 2: Graphical illustration of the working steps of the whale optimization algorithm and Nelder–Mead algorithm for the training of neurons in the LeNN architecture and the minimization of fitness functions.

$$[NSE_S, NSE_X, NSE_Y] = \begin{bmatrix} 1 - \dfrac{\sum_{i=1}^{n} \left(S(t_i) - S_{\text{approx}}(t_i)\right)^2}{\sum_{i=1}^{n} \left(S(t_i) - S\prime(t_i)\right)^2} & S\prime(t_i) = \dfrac{1}{n}\sum_{i=1}^{n} S(t_i) \\ 1 - \dfrac{\sum_{i=1}^{n} \left(X(t_i) - X_{\text{approx}}(t_i)\right)^2}{\sum_{i=1}^{n} \left(X(t_i) - X\prime(t_i)\right)^2} & X\prime(t_i) = \dfrac{1}{n}\sum_{i=1}^{n} X(t_i) \\ 1 - \dfrac{\sum_{i=1}^{n} \left(Y(t_i) - Y_{\text{approx}}(t_i)\right)^2}{\sum_{i=1}^{n} \left(Y(t_i) - Y\prime(t_i)\right)^2} & Y\prime(t_i) = \dfrac{1}{n}\sum_{i=1}^{n} Y(t_i) \end{bmatrix}^T, \tag{21}$$

$$[ENSE_S, ENSE_X, ENSE_Y] = [\,1 - NSE_S, 1 - NSE_X, 1 - NSE_Y\,], \tag{22}$$

where n shows total input grid points. The values of performance measures of MAD, RMSE, and ENSE should be equal to zero for perfect modeling, while NSE should be equal to 1. The global versions of MAD, RMSE, and ENSE for the given mathematical model of the prey-predator system are formulated by given equations:

$$[GMAD_S, GMAD_X, GMAD_Y] = \begin{bmatrix} \dfrac{1}{R}\sum_{r=1}^{R}\left(\dfrac{1}{n}\sum_{i=1}^{n}\left|S(t_i) - S_{\text{approx}}(t_i)\right|\right) \\ \dfrac{1}{R}\sum_{r=1}^{R}\left(\dfrac{1}{n}\sum_{i=1}^{n}\left|X(t_i) - X_{\text{approx}}(t_i)\right|\right) \\ \dfrac{1}{R}\sum_{r=1}^{R}\left(\dfrac{1}{n}\sum_{i=1}^{n}\left|Y(t_i) - Y_{\text{approx}}(t_i)\right|\right) \end{bmatrix}^T, \tag{23}$$

$$[GTIC_S, GTIC_X, GTIC_Y] = \begin{bmatrix} \dfrac{1}{R}\sum_{r=1}^{R}\left(\dfrac{\sqrt{(1/n)\sum_{i=1}^{n}\left(S(t_i) - S_{\text{approx}}(t_i)\right)^2}}{\sqrt{(1/n)\sum_{i=1}^{n}\left(S(t_i)\right)^2} + \sqrt{(1/n)\sum_{i=1}^{n}\left(S_{\text{approx}}(t_i)\right)^2}}\right) \\ \dfrac{1}{R}\sum_{r=1}^{R}\left(\dfrac{\sqrt{(1/n)\sum_{i=1}^{n}\left(X(t_i) - X_{\text{approx}}(t_i)\right)^2}}{\sqrt{(1/n)\sum_{i=1}^{n}\left(X(t_i)\right)^2} + \sqrt{(1/n)\sum_{i=1}^{n}\left(X_{\text{approx}}(t_i)\right)^2}}\right) \\ \dfrac{1}{R}\sum_{r=1}^{R}\left(\dfrac{\sqrt{(1/n)\sum_{i=1}^{n}\left(Y(t_i) - Y_{\text{approx}}(t_i)\right)^2}}{\sqrt{(1/n)\sum_{i=1}^{n}\left(Y(t_i)\right)^2} + \sqrt{(1/n)\sum_{i=1}^{n}\left(Y_{\text{approx}}(t_i)\right)^2}}\right) \end{bmatrix}^T, \tag{24}$$

$$[GNSE_S, GNSE_X, GNSE_Y] = \begin{bmatrix} \frac{1}{R}\sum_{r=1}^{R}\left(1-\frac{\sum_{i=1}^{n}\left(S(t_i)-S_{\text{approx}}(t_i)\right)^2}{\sum_{i=1}^{n}\left(S(t_i)-S\prime(t_i)\right)^2}\right) & S\prime(t_i)=\frac{1}{n}\sum_{i=1}^{n}S(t_i) \\ \frac{1}{R}\sum_{r=1}^{R}\left(1-\frac{\sum_{i=1}^{n}\left(X(t_i)-X_{\text{approx}}(t_i)\right)^2}{\sum_{i=1}^{n}\left(X(t_i)-X\prime(t_i)\right)^2}\right) & X\prime(t_i)=\frac{1}{n}\sum_{i=1}^{n}X(t_i) \\ \frac{1}{R}\sum_{r=1}^{R}\left(1-\frac{\sum_{i=1}^{n}\left(Y(t_i)-Y_{\text{approx}}(t_i)\right)^2}{\sum_{i=1}^{n}\left(Y(t_i)-Y\prime(t_i)\right)^2}\right) & Y\prime(t_i)=\frac{1}{n}\sum_{i=1}^{n}Y(t_i) \end{bmatrix}^T, \tag{25}$$

$$[GENSE_S, GENSE_X, GENSE_Y] = [\, 1-GNSE_S, 1-GNSE_X, 1-GNSE_Y \,], \tag{26}$$

where R denotes the number of independent runs. Global fitness (GFIT) is defined as the mean of fitness values attained in independent runs. Mathematically, GFIT is given as follows:

$$GFIT = \frac{1}{R}\sum_{r=1}^{R}\epsilon_r. \tag{27}$$

7. Result and Discussion

The numerical solutions of the prey-predator system with immigrant prey under the influence of variations in various parameters are investigated with the proposed methodology. The exact solution for these nonlinear models is not known, so the comparative study is conducted with MATLAB solver ode45 and the homotopy perturbation method [44]. Six problems of the prey-predator model are considered for different cases depending on values of coefficients, i.e, $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$, and γ_2. A graphical abstract of the paper is shown in Figure 3.

7.1. Problem I: Influence of Variations in α_1 on the Prey-Predator Model. In this problem, the effect of variations in the intrinsic growth rate of local prey α_1 on population density is discussed. An error-based fitness function along with initial populations is given as follows:

$$\text{Minimize}\, \epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6, \tag{28}$$

where ϵ_1 to ϵ_6 are defined by

$$\begin{cases} \epsilon_1 = \frac{1}{N}\sum_{n=1}^{34}\left(\frac{dS_n}{dt} - S_n\left(\alpha_1 - \frac{\alpha_1 S_n}{50}\right) - (0.2)S_nX_n + (0.01)S_nY_n\right)^2, \\ \epsilon_2 = \frac{1}{N}\sum_{n=35}^{68}\left(\frac{dX_n}{dt} - X_n\left((0.2) - \frac{(0.2)X_n}{k_2}\right) + (0.1)S_nX_n + (0.9)X_nY_n\right)^2, \\ \epsilon_3 = \frac{1}{N}\sum_{n=69}^{102}\left(\frac{dY_n}{dt} - (0.9)S_nY_n - (0.8)X_nY_n + (0.01)Y_n + (0.01)Y_n^2\right)^2, \\ \epsilon_4 = (S_0 - 0.5)^2, \\ \epsilon_5 = (X_0 - 0.5)^2, \\ \epsilon_6 = (Y_0 - 0.5)^2. \end{cases} \tag{29}$$

Five cases are considered, depending on the value of α_1.

Case I: $\alpha_1 = 0.06$

Case II: $\alpha_1 = 0.08$

Case III: $\alpha_1 = 0.10$

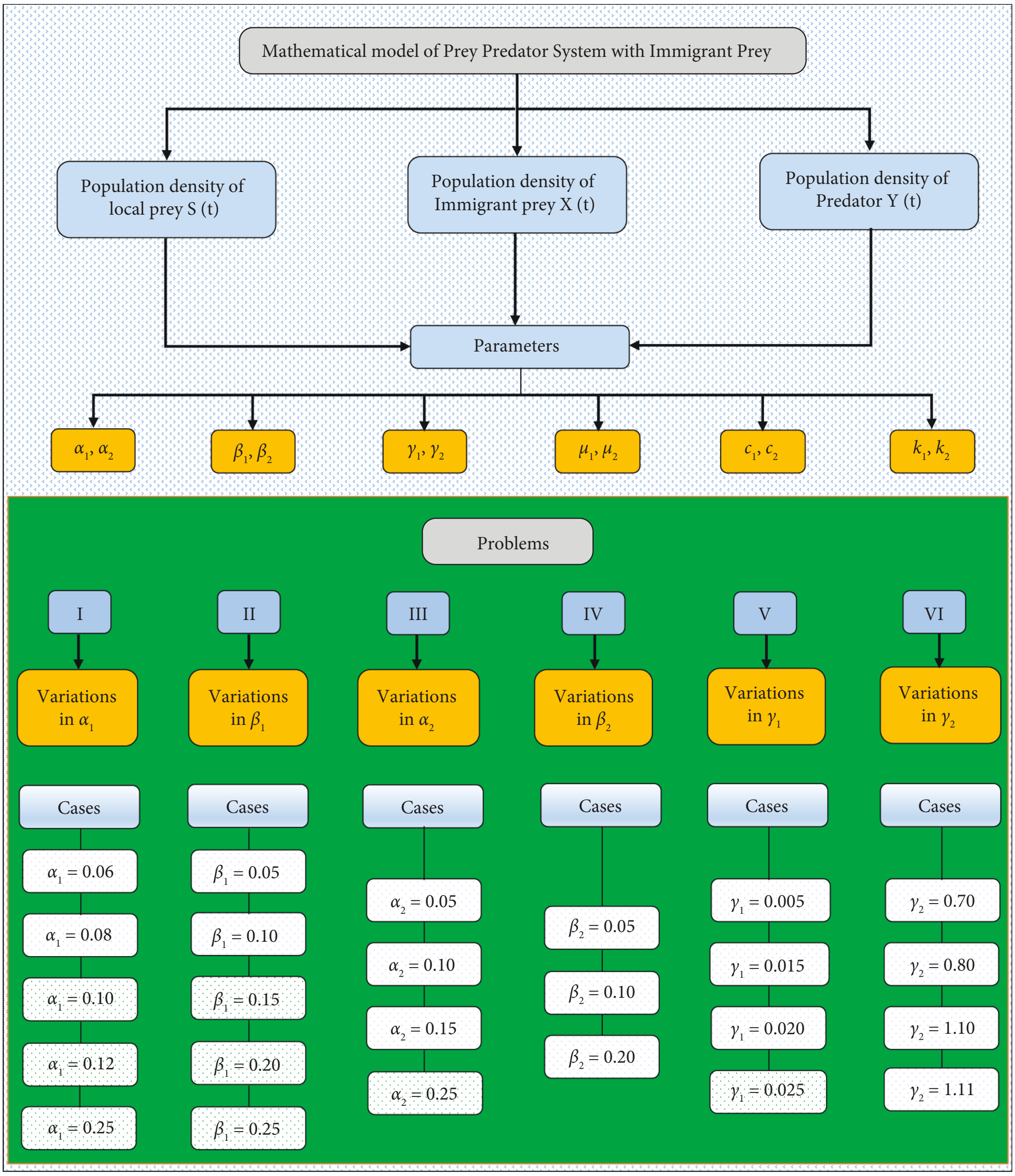


Figure 3: Graphical overview of the paper.

Table 2: Comparison between approximate solutions obtained by the LeNN-WOA-NM algorithm for the prey-predator system with the homotopy perturbation method [44] for $\alpha_1 = 0.12$.

t	$S(t)$		$X(t)$		$Y(t)$	
	HPO	LeNN-WOA-NM	HPO	LeNN-WOA-NM	HPO	LeNN-WOA-NM
0.0	0.500000	0.50000000	0.500000	0.50000000	0.500000	0.50000000
0.2	0.521458	0.52145956	0.466762	0.46676321	0.590440	0.59114765
0.4	0.542942	0.54294366	0.427969	0.42797117	0.695777	0.69922187
0.6	0.564244	0.56424422	0.384329	0.38433104	0.817468	0.82687284
0.8	0.585155	0.58515597	0.337008	0.33701139	0.956844	0.97704917
1.0	0.605478	0.60547960	0.287606	0.28761008	1.115127	1.15310441

Table 3: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population density of local prey $S(t)$ under the influence of the intrinsic growth rate of local prey α_1.

t	$\alpha_1 = 0.06$	$\alpha_1 = 0.08$	$\alpha_1 = 0.10$	$\alpha_1 = 0.12$	$\alpha_1 = 0.25$
0.0	0.50000029	0.50000024	0.50000001	0.50000000	0.50000000
0.2	0.51530461	0.51734394	0.51940166	0.52145956	0.53504743
0.4	0.53020804	0.53441582	0.53866566	0.54294366	0.57158810
0.6	0.54451936	0.55101532	0.55759565	0.56424422	0.60943884
0.8	0.55806566	0.56695084	0.57598624	0.58515597	0.64838425
1.0	0.57069056	0.58205878	0.59365221	0.60547960	0.68819372

Table 4: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population density of immigrant prey $X(t)$ under the influence of the intrinsic growth rate of local prey α_1.

t	$\alpha_1 = 0.06$	$\alpha_1 = 0.08$	$\alpha_1 = 0.10$	$\alpha_1 = 0.12$	$\alpha_1 = 0.25$
0.0	0.49999979	0.49999997	0.49999999	0.50000000	0.50000000
0.2	0.46679789	0.46677660	0.46677363	0.46676321	0.46668320
0.4	0.42814941	0.42808998	0.42803011	0.42797117	0.42757426
0.6	0.38480296	0.38463971	0.38448897	0.38433104	0.38327075
0.8	0.33796050	0.33764489	0.33732977	0.33701139	0.33485636
1.0	0.28923060	0.28868911	0.28815986	0.28761008	0.28389875

Table 5: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population density of the predator $Y(t)$ under the influence of the intrinsic growth rate of local prey α_1.

t	$\alpha_1 = 0.06$	$\alpha_1 = 0.08$	$\alpha_1 = 0.10$	$\alpha_1 = 0.12$	$\alpha_1 = 0.25$
0.0	0.49999991	0.49999919	0.49999999	0.50000000	0.50000000
0.2	0.59011694	0.59022292	0.59032756	0.59047007	0.59114765
0.4	0.69423291	0.69474410	0.69525628	0.69577735	0.69922187
0.6	0.81332517	0.81468984	0.81607383	0.81746843	0.82687284
0.8	0.94811217	0.95097827	0.95389198	0.95684491	0.97704917
1.0	1.09907344	1.10432082	1.10967602	1.11512837	1.15310441

Table 6: Absolute errors obtained by the LeNN-WOA-NM algorithm for population density of local prey $S(t)$ under the influence of the intrinsic growth rate of local prey α_1.

t	$\alpha_1 = 0.06$	$\alpha_1 = 0.08$	$\alpha_1 = 0.10$	$\alpha_1 = 0.12$	$\alpha_1 = 0.25$
0.0	$4.25E-10$	$9.71E-11$	$4.27E-11$	$9.01E-11$	$3.05E-13$
0.2	$3.87E-10$	$3.87E-11$	$1.80E-09$	$8.37E-11$	$9.38E-14$
0.4	$5.99E-11$	$1.16E-10$	$1.85E-09$	$3.57E-11$	$3.30E-13$
0.6	$1.46E-10$	$1.33E-10$	$2.21E-09$	$3.05E-11$	$1.05E-13$
0.8	$1.37E-10$	$2.15E-11$	$3.80E-09$	$7.04E-11$	$1.82E-12$
1.0	$9.93E-11$	$2.11E-10$	$6.82E-11$	$6.49E-11$	$4.90E-14$

Case IV: $\alpha_1 = 0.12$

Case V: $\alpha_1 = 0.25$

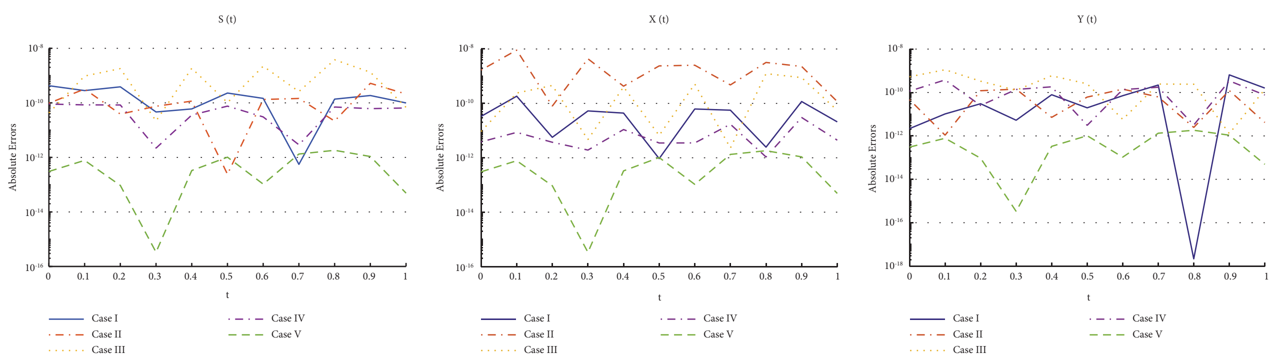
The LeNN-WOA-NM algorithm is applied to prey-predator model equation (29) to study the influence of variations in the intrinsic growth rate of local prey. A comparison between approximate solutions obtained by the LeNN-WOA-NM algorithm for the prey-predator system with the homotopy perturbation method [44] for $\alpha_1 = 0.12$ is illustrated in Table 2. Approximate solutions for population densities of local prey, immigrant prey, and predator are given in Tables 3–5, respectively. Absolute errors are

Table 7: Absolute errors obtained by the LeNN-WOA-NM algorithm for population density of immigrant prey $X(t)$ under the influence of the intrinsic growth rate of local prey α_1.

t	$\alpha_1 = 0.06$	$\alpha_1 = 0.08$	$\alpha_1 = 0.10$	$\alpha_1 = 0.12$	$\alpha_1 = 0.25$
0.0	$3.33E-11$	$1.68E-09$	$9.24E-12$	$3.85E-12$	$3.05E-13$
0.2	$5.66E-12$	$7.98E-11$	$4.55E-10$	$3.72E-12$	$9.38E-14$
0.4	$4.40E-11$	$4.29E-10$	$4.03E-10$	$1.09E-11$	$3.30E-13$
0.6	$6.22E-11$	$2.53E-09$	$5.15E-10$	$3.52E-12$	$1.05E-13$
0.8	$2.46E-12$	$3.20E-09$	$1.24E-09$	$1.07E-12$	$1.82E-12$
1.0	$2.09E-11$	$1.17E-10$	$5.55E-11$	$4.44E-12$	$4.90E-14$

Table 8: Absolute errors obtained by the LeNN-WOA-NM algorithm for population density of the predator $Y(t)$ under the influence of the intrinsic growth rate of local prey α_1.

t	$\alpha_1 = 0.06$	$\alpha_1 = 0.08$	$\alpha_1 = 0.10$	$\alpha_1 = 0.12$	$\alpha_1 = 0.25$
0.0	$2.17E-12$	$4.68E-11$	$5.32E-10$	$1.16E-10$	$3.05E-13$
0.2	$3.04E-11$	$1.21E-10$	$3.32E-10$	$2.52E-11$	$9.38E-14$
0.4	$7.91E-11$	$7.16E-12$	$5.91E-10$	$1.83E-10$	$3.30E-13$
0.6	$7.12E-11$	$1.43E-10$	$5.80E-12$	$1.36E-10$	$1.05E-13$
0.8	$2.17E-18$	$2.46E-12$	$2.41E-10$	$3.16E-12$	$1.82E-12$
1.0	$1.60E-10$	$4.16E-12$	$1.40E-10$	$7.66E-11$	$4.90E-14$



(a) (b) (c)

Figure 4: Absolute errors obtained by the LeNN-WOA-NM algorithm under the influence of variations in the intrinsic growth rate of local prey on population densities of the prey-predator model. (a) Population density of local prey. (b) Population density of immigrant prey. (c) Population density of the predator.

presented in Tables 6–8 and are graphically illustrated in Figure4. Table 9 represents the statistics of global values of performance indicators during 100 independent trails. Figure 5 shows the convergence of the fitness value. It can be seen that the fitness value for each case lies around 10^{-6} to 10^{-10}. The percentage convergence of the fitness value and performance indicators during multiple runs is shown in Table 10. Trained neurons in the LeNN structure for obtaining best solutions are shown in Table 11. From Figure 6, the following conclusions are drawn:

(i) Population density of local prey has a direct relation with the intrinsic growth rate of local prey

(ii) Population density of immigrant prey has an inverse relation with the intrinsic growth rate of local prey

(iii) Population density of the predator varies directly with the intrinsic growth rate of local prey

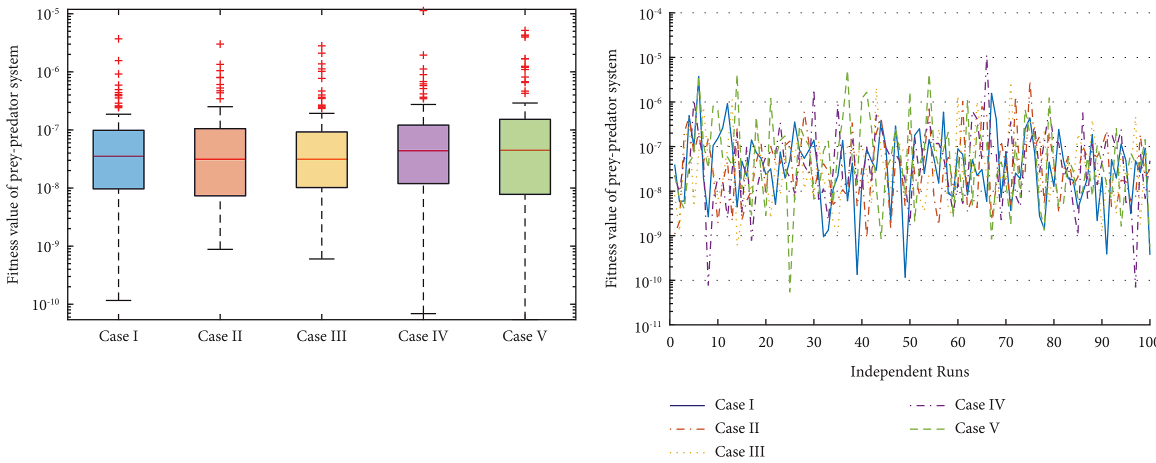
7.2. Problem II: Effect of Variations in β_1 on the Prey-Predator Model. In this problem, the effect of variations in the positive impact of force of interaction β_1 between local and immigrant prey on population densities of the prey-predator model is discussed. An error-based fitness function along with initial populations is given as follows:

$$\text{Minimize}\, \epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6, \qquad (30)$$

where ϵ_1 to ϵ_6 are defined as follows:

Table 9: Statistics of global performance indices for variations in the intrinsic growth rate of local prey on the prey-predator model.

Cases	$S(t)$				$X(t)$				$Y(t)$			
	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE
I	1.21E – 07	2.47E – 05	1.30E – 05	3.82E – 05	1.21E – 07	2.26E – 05	1.61E – 05	3.21E – 06	1.55E – 07	3.70E – 05	1.30E – 05	2.00E – 06
II	5.15E – 08	1.64E – 05	8.52E – 06	8.13E – 06	5.25E – 08	1.69E – 05	1.19E – 05	1.15E – 06	5.66E – 08	2.67E – 05	9.42E – 06	3.87E – 07
III	8.12E – 07	2.74E – 05	1.39E – 05	2.61E – 05	1.45E – 07	2.14E – 05	1.51E – 05	2.90E – 06	1.61E – 07	3.89E – 05	1.40E – 05	1.15E – 06
IV	5.57E – 07	1.88E – 05	9.59E – 06	7.77E – 06	5.86E – 08	1.62E – 05	1.17E – 05	1.19E – 06	9.01E – 08	3.19E – 05	1.14E – 05	5.85E – 07
V	6.95E – 07	2.20E – 05	1.02E – 05	3.59E – 06	1.05E – 07	1.99E – 05	1.43E – 05	2.04E – 06	1.37E – 07	3.72E – 05	1.30E – 05	7.78E – 07



(a) (b)

Figure 5: Comparison between the box plots and the convergence graph of fitness evaluation over 100 independent runs for the prey-predator model with variations in the intrinsic growth rate of local prey. (a) Box plots for fitness evaluation of the prey-predator model with variations in α_1 (b) Convergence of the fitness value of the prey-predator model with variations in α_1.

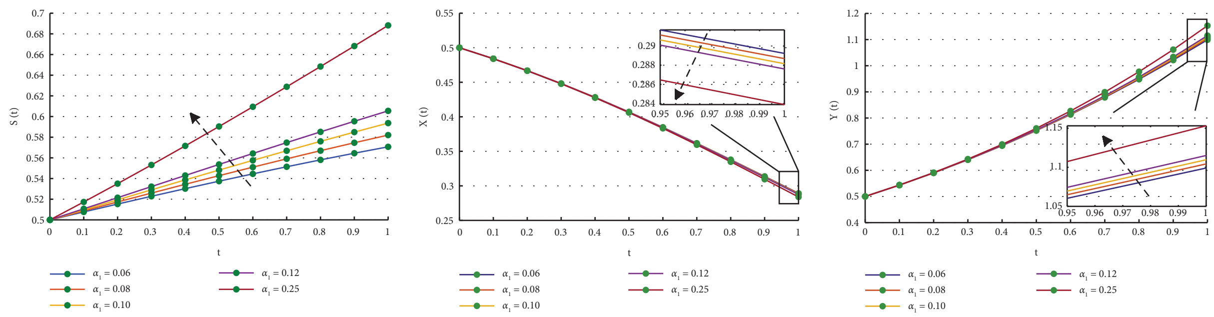
Table 10: Convergence analysis for variation in α_1 on the prey-predator model.

Cases		FIT				MAD			TIC			ENSE		
		$\le 10^{-7}$	$\le 10^{-8}$	$\le 10^{-9}$	$\le 10^{-10}$	$\le 10^{-4}$	$\le 10^{-5}$	$\le 10^{-6}$	$\le 10^{-4}$	$\le 10^{-5}$	$\le 10^{-6}$	$\le 10^{-6}$	$\le 10^{-7}$	$\le 10^{-8}$
$S(t)$	I	98	75	25	5	100	99	35	100	99	61	39	22	3
	II	80	30	3	0	100	100	49	100	100	56	68	36	8
	III	78	34	2	0	100	94	41	100	99	63	74	33	3
	IV	81	37	4	0	100	99	43	100	100	68	83	44	16
	V	77	27	5	0	100	97	42	100	100	66	87	60	27
$X(t)$	I	81	37	13	3	96	44	2	98	54	3	65	23	6
	II	85	41	11	0	100	43	1	100	100	58	71	20	3
	III	74	41	17	1	97	39	0	100	98	52	64	24	8
	IV	86	47	16	4	100	46	4	100	100	55	73	37	9
	V	80	47	12	1	98	43	3	100	99	57	68	26	6
$Y(t)$	I	80	41	16	3	100	95	21	99	57	9	47	14	7
	II	86	47	9	1	100	98	28	100	68	5	46	13	0
	III	78	36	8	2	100	91	20	99	57	4	45	10	0
	IV	76	42	9	3	100	94	26	100	60	8	44	10	6
	V	70	29	8	1	100	95	21	100	57	7	41	16	1

$$\begin{cases} \epsilon_1 = \dfrac{1}{N}\displaystyle\sum_{n=1}^{34}\left(\dfrac{dS_n}{dt} - S_n\left((0.12) - \dfrac{(0.12)S_n}{50}\right) - (\beta_1)S_nX_n + (0.01)S_nY_n\right)^2, \\ \epsilon_2 = \dfrac{1}{N}\displaystyle\sum_{n=35}^{68}\left(\dfrac{dX_n}{dt} - X_n\left((0.2) - \dfrac{(0.2)X_n}{k_2}\right) + (0.1)S_nX_n + (0.9)X_nY_n\right)^2, \\ \epsilon_3 = \dfrac{1}{N}\displaystyle\sum_{n=69}^{102}\left(\dfrac{dY_n}{dt} - (0.9)S_nY_n - (0.8)X_nY_n + (0.01)Y_n + (0.01)Y_n^2\right)^2, \\ \epsilon_4 = (S_0 - 0.5)^2, \\ \epsilon_5 = (X_0 - 0.5)^2, \\ \epsilon_6 = (Y_0 - 0.5)^2. \end{cases} \quad (31)$$

Table 11: Unknown parameters obtained by the LeNN-WOA-NM algorithm for the prey-predator model under the influence of variations in the intrinsic growth rate of local prey α_1.

	Index	$S(t)$			$X(t)$			$Y(t)$		
		ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
	1	0.24298626	−0.2668655	0.09328582	0.17524653	0.0657111	0.45518402	0.62050563	0.107119458	−0.184535
	2	0.12478022	0.39026984	0.13404058	−0.1205176	−0.504835	−0.152368	0.29193615	−0.06286814	0.043174485
	3	0.15946384	0.47205052	0.04884412	−0.0142591	−0.0346883	0.98886623	0.09715236	0.181551811	0.868732734
	4	0.01120683	0.24610287	−0.2837988	0.14946792	0.16097393	−0.074499	0.46391737	−0.16073035	0.113397979
	5	−0.022669	0.0943905	0.0992974	0.34191594	0.58292848	0.00959628	0.06797262	−0.11244358	0.013444379
Case I	6	−0.3216158	0.1410155	0.18325041	0.03228638	0.70287879	0.12845754	−0.1361841	0.158063474	−0.05836861
	7	0.48768209	0.0293772	0.17094612	−0.0018354	0.14860632	0.65130564	0.67663964	0.314984108	0.261128949
	8	0.08329406	0.04117897	−0.0205859	−0.1064367	−0.1326281	−0.0481534	−0.0486244	−0.33359642	0.564727586
	9	0.87211207	−0.0743821	0.18882808	0.69221376	−0.1213543	0.01584571	0.00730463	−0.07959673	0.248981123
	10	0.74644658	0.01575884	0.17044653	0.13151592	0.13622307	0.26101806	−0.1087744	−0.09419355	−32.2098458
	11	0.26313561	0.09923455	−0.0506162	0.01009165	0.13795634	0.05968636	0.42506235	0.302798399	0.094716386
	1	0.33259121	−0.3657449	0.21062276	0.25900442	0.15918437	0.41026629	0.29040796	−1.70842597	−0.22039024
	2	0.17889024	−0.0431343	−0.6411116	0.23110108	0.42625725	0.08496042	−0.0683335	−0.01537247	0.252533232
	3	0.07297637	0.1667513	−0.0811545	−0.2048892	−0.1120065	−0.0826385	−0.6647359	−0.16394699	−0.00997236
	4	−0.1365861	0.47090897	−0.0639777	$3.43E-05$	0.1546024	−0.178235	0.4646494	0.190521738	0.195440966
	5	−0.007358	0.13102291	0.24249896	−0.1088503	−0.1671249	0.16954851	−0.7315874	−0.03059029	−0.07106493
Case II	6	0.43362447	−0.155685	0.44563542	0.22079963	−0.6459135	−0.0878247	−0.5471167	−0.36068591	−0.90966495
	7	−0.4072594	0.43108895	−0.0973015	0.00859421	0.19519171	0.05831302	0.00294865	−0.03440871	0.372338435
	8	−0.015323	−0.0127997	−0.079715	0.41948941	−0.2878455	0.21735868	−0.839503	−0.04815065	0.025011228
	9	0.26217328	0.20028492	0.2222801	−0.1630391	0.17520391	0.0974704	0.34891726	−0.13212935	−0.1419514
	10	0.09242525	0.19371262	−0.1574267	0.34189851	0.28434772	0.08494461	0.46694404	−0.12329874	39.39674102
	11	0.13225195	−0.0846055	−0.0281568	0.14217372	0.31887231	0.42910377	0.26716207	0.088173159	0.085561799
	1	0.08792719	−0.02998	1.2174462	0.69165686	0.06374932	−0.0488322	0.11491028	0.090075909	−0.32604325
	2	0.11610049	1.56406569	0.0795106	−0.0902964	0.09097774	−0.1017928	0.06371619	0.492255996	0.406557357
	3	0.03262421	−0.2983161	0.4361223	0.71023061	0.27130756	0.53574029	0.74546663	0.731579954	−0.11925334
	4	−0.8534256	0.27102507	0.09970798	−0.4630868	0.19588223	0.25655668	−0.1733402	0.4924538	−0.02738845
	5	−0.4373573	0.06892556	0.11113751	0.22424973	0.18678342	−0.0095592	0.13860986	0.342147207	−0.12816846
Case III	6	0.85585046	−0.0086572	0.70804298	0.16532453	0.46387591	0.19167055	−0.4155845	0.358174513	0.519586436
	7	0.4731327	−0.0131084	0.53818072	0.00107541	−0.2371714	0.21538868	0.27067076	0.007722887	−0.11651201
	8	−0.1730569	−0.0542656	1.00463015	0.33304716	0.42111507	0.54031591	−0.0982948	−0.30964889	0.225264957
	9	0.01950551	0.3195719	−0.1781725	0.00222254	−0.0527906	0.33879162	0.14536108	0.185060402	0.022547703
	10	−0.2758316	0.27171296	−0.0428651	0.07565986	0.29894632	−0.1182305	0.19684946	0.626442716	0.062105445
	11	0.28574702	−0.2628512	0.06102257	0.04403502	0.22414615	0.40364307	0.55206157	0.429077344	−0.00256072
	1	0.25369644	−0.4777306	−0.8361847	−0.0621264	0.27351167	−0.6065099	0.68147388	−1.43275825	−0.10932386
	2	0.06991497	−0.1139906	−0.4735192	−0.5854725	−0.1661107	0.07770566	−0.2460336	0.052462785	0.191077917
	3	−0.2351744	−0.4870632	−0.0716903	−0.2556044	$9.83E-05$	0.02156199	−0.3140772	−0.4208503	−0.13182798
	4	−0.0489805	0.03121692	−0.2705836	−0.2176759	−0.3388043	−0.1570093	−0.2947292	0.000861238	0.091135849
	5	0.12066069	0.00528409	−0.1876653	0.18129324	−0.2476741	−0.0843931	0.33223353	0.037877215	0.552486838
Case IV	6	−0.2917694	0.09885985	−0.655587	−0.5821936	0.02484774	−0.2678441	−0.1209825	0.019527337	−0.2041232
	7	0.2180106	0.10156035	−0.0839076	−0.0408137	0.20227179	0.08783601	−1.1472874	0.088467804	−0.1006823
	8	−0.3594382	−0.0329465	−0.0085748	−0.076245	0.071221	−0.2677479	−0.6912671	0.072673015	−0.60129414
	9	0.12647778	−0.0321146	−0.1925977	−0.1982251	−0.1306954	−0.4290151	−0.4035842	−0.01597822	−0.09388657
	10	−0.7218529	−0.127281	−0.253891	−0.520109	−0.148345	−0.1301849	0.45904036	−0.14555693	0.210878261
	11	−0.0415557	−0.0626802	−0.0135924	−0.2376142	−0.2602334	0.31718881	−0.0300309	−0.07780308	0.119273654
	1	0.11535169	0.09331411	0.06459598	0.13708955	0.02523131	0.14440187	0.53678594	0.289671331	0.265496244
	2	1.2018728	0.66566882	−0.0195754	0.55314202	0.20920125	0.53535701	0.3092534	0.73571248	0.320987524
	3	0.20319881	−0.8256885	1.09540267	0.09636349	0.12623634	−0.1361461	0.29987708	−0.14730456	−0.1852485
	4	0.0272977	0.50122086	−0.2264851	−0.0410539	0.34541103	−0.1572514	0.88612589	0.219231183	0.296041826
	5	0.13436663	0.09124816	0.31502947	0.03670816	−0.2842128	0.28047538	0.30493881	0.228728298	−0.83846218
Case V	6	0.59090535	0.2219209	0.35836933	0.13287532	0.07279467	0.5746159	0.05242013	0.22070811	0.059736502
	7	0.57147768	0.17459027	0.74963519	0.96594705	0.18063006	−0.2505621	0.11584087	−0.02191061	0.160702925
	8	0.14083549	0.5151422	0.06170226	0.18189487	0.19412865	0.22132998	0.9645296	−0.15578224	0.422131366
	9	−0.0987012	−0.3447324	0.30547697	0.22752865	0.56631904	0.08438966	0.0392108	−0.15270349	0.520225963
	10	0.05337954	0.38978704	0.32410767	0.04067255	0.40603952	0.21526779	−0.0360576	−0.00016971	−1.80718751
	11	0.16882868	0.42247401	0.06696436	−0.0534127	0.38690363	0.40861018	−0.1032548	0.039874898	0.21583529



(a) (b) (c)

Figure 6: Comparison between solutions obtained by the LeNN-WOA-NM algorithm under the influence of variations in the intrinsic growth rate of local prey on population densities of the prey-predator model. (a) Approximate solutions for population density of local prey. (b) Approximate solutions for population density of immigrant prey. (c) Approximate solutions for population density of the predator.

Table 12: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population density of local prey $S(t)$ under the influence of β_1.

t	$\beta_1 = 0.05$	$\beta_1 = 0.10$	$\beta_1 = 0.15$	$\beta_1 = 0.20$	$\beta_1 = 0.25$
0.0	0.49999811	0.50000565	0.49999994	0.50000001	0.49999994
0.2	0.51394192	0.51644576	0.51893794	0.52145440	0.52399164
0.4	0.52796160	0.53293741	0.53790411	0.54293644	0.54801843
0.6	0.54203480	0.54936883	0.55674985	0.56424287	0.57183842
0.8	0.55610994	0.56564641	0.57531194	0.58515293	0.59516204
1.0	0.57009452	0.58167704	0.59345033	0.60547330	0.61773135

Table 13: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population density of immigrant prey $X(t)$ under the influence of β_1.

t	$\beta_1 = 0.05$	$\beta_1 = 0.10$	$\beta_1 = 0.15$	$\beta_1 = 0.20$	$\beta_1 = 0.25$
0.0	0.49999992	0.50000514	0.49999988	0.50000001	0.49999969
0.2	0.46680042	0.46679364	0.46677626	0.46676260	0.46673440
0.4	0.42817965	0.42812547	0.42804193	0.42797008	0.42789521
0.6	0.38488898	0.38470693	0.38451903	0.38432922	0.38413261
0.8	0.33810661	0.33775597	0.33737496	0.33701285	0.33663106
1.0	0.28943501	0.28883454	0.28823065	0.28760701	0.28697657

Table 14: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population density of the predator $Y(t)$ under the influence of β_1.

t	$\beta_1 = 0.05$	$\beta_1 = 0.10$	$\beta_1 = 0.15$	$\beta_1 = 0.20$	$\beta_1 = 0.25$
0.0	0.50000032	0.49999215	0.50000005	0.50000003	0.49999973
0.2	0.59004112	0.59016271	0.59030647	0.59044066	0.59057741
0.4	0.69390817	0.69451560	0.69516439	0.69578033	0.69640936
0.6	0.81260140	0.81419598	0.81584261	0.81746743	0.81912365
0.8	0.94690117	0.95016524	0.95348538	0.95684239	0.96025853
1.0	1.09742754	1.10322425	1.10912457	1.11513013	1.12125476

Table 15: Absolute errors obtained by the LeNN-WOA-NM algorithm for population density of local prey $S(t)$ under the influence of β_1.

t	$\beta_1 = 0.05$	$\beta_1 = 0.10$	$\beta_1 = 0.15$	$\beta_1 = 0.20$	$\beta_1 = 0.25$
0.0	1.38$E-08$	1.77$E-09$	1.10$E-09$	1.24$E-09$	1.04$E-10$
0.2	7.06$E-09$	1.51$E-09$	8.29$E-10$	9.85$E-10$	3.47$E-10$
0.4	4.21$E-09$	3.33$E-11$	2.05$E-10$	2.81$E-10$	1.92$E-10$
0.6	4.01$E-09$	2.43$E-10$	2.77$E-10$	2.79$E-10$	2.07$E-09$
0.8	1.44$E-08$	2.19$E-10$	6.01$E-10$	4.68$E-10$	1.57$E-09$
1.0	2.13$E-08$	4.83$E-10$	1.02$E-09$	3.60$E-10$	7.30$E-10$

Table 16: Absolute errors obtained by the LeNN-WOA-NM algorithm for population density of immigrant prey $X(t)$ under the influence of β_1.

t	$\beta_1 = 0.05$	$\beta_1 = 0.10$	$\beta_1 = 0.15$	$\beta_1 = 0.20$	$\beta_1 = 0.25$
0.0	1.39$E-09$	3.44$E-11$	3.07$E-10$	1.07$E-11$	2.30$E-09$
0.2	2.29$E-09$	4.35$E-09$	5.04$E-10$	5.02$E-12$	4.24$E-10$
0.4	7.01$E-10$	5.98$E-10$	6.77$E-10$	9.67$E-11$	1.96$E-10$
0.6	1.41$E-11$	2.14$E-10$	4.69$E-10$	6.77$E-11$	1.29$E-09$
0.8	1.61$E-09$	2.21$E-10$	8.72$E-12$	7.39$E-11$	1.42$E-09$
1.0	1.01$E-09$	2.03$E-09$	1.13$E-09$	3.02$E-10$	8.42$E-11$

Table 17: Absolute errors obtained by the LeNN-WOA-NM algorithm for population density of the predator $Y(t)$ under the influence of β_1.

t	$\beta_1 = 0.05$	$\beta_1 = 0.10$	$\beta_1 = 0.15$	$\beta_1 = 0.20$	$\beta_1 = 0.25$
0.0	3.95$E-11$	3.48$E-09$	4.48$E-09$	1.83$E-10$	4.83$E-10$
0.2	8.22$E-10$	1.51$E-09$	3.01$E-09$	3.19$E-10$	3.25$E-10$
0.4	3.03$E-11$	1.96$E-10$	1.20$E-09$	2.60$E-11$	5.34$E-10$
0.6	9.03$E-10$	3.60$E-10$	2.64$E-09$	5.37$E-10$	8.30$E-12$
0.8	1.36$E-09$	1.83$E-09$	7.57$E-10$	5.75$E-10$	2.25$E-10$
1.0	3.84$E-09$	7.37$E-09$	1.40$E-09$	9.52$E-10$	2.47$E-10$

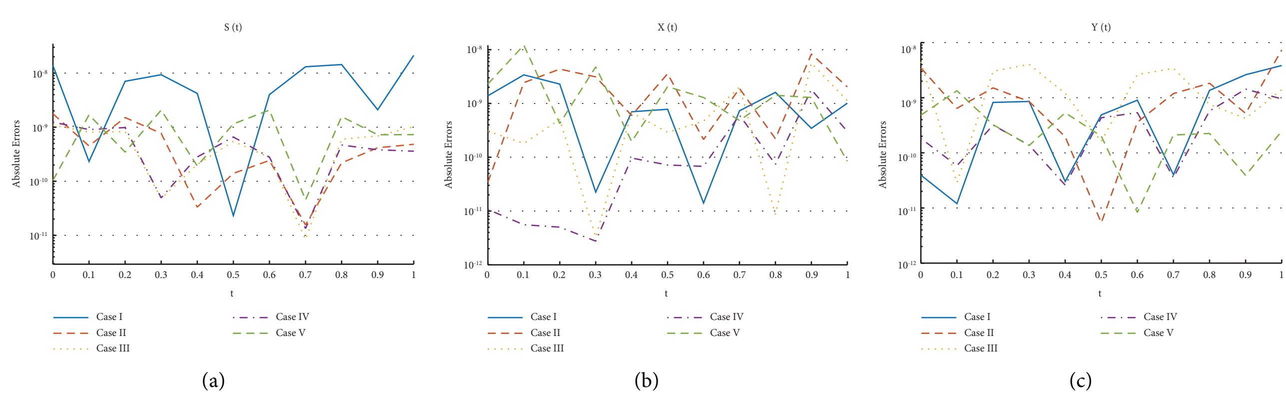


Figure 7: Absolute errors obtained by the LeNN-WOA-NM algorithm under the influence of variations in β_1 on population densities of the prey-predator model. (a) Population density of local prey. (b) Population density of immigrant prey. (c) Population density of the predator.

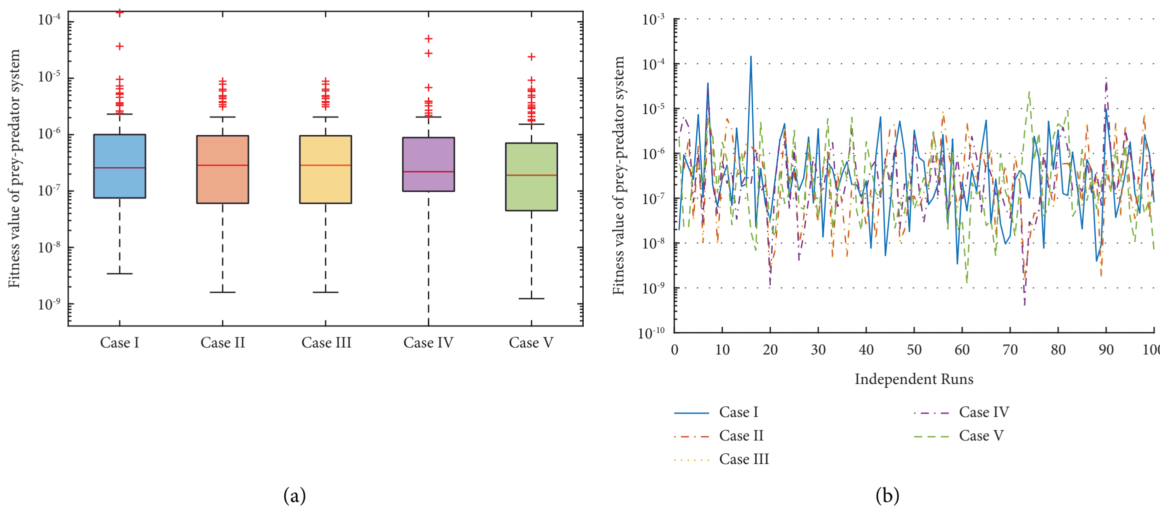


Figure 8: Comparison between the box plots and convergence graphs of fitness evaluation over 100 independent runs for the prey-predator model with variations in force of interaction between local and immigrant prey. (a) Box plots for fitness evaluation of the prey-predator model with variations in β_1. (b) Convergence of the fitness value of the prey-predator model with variations in β_1.

Five cases are considered, depending on the value of β_1.

Case I: $\beta_1 = 0.05$

Case II: $\beta_1 = 0.10$

Case III: $\beta_1 = 0.15$

Case IV: $\beta_1 = 0.20$

Case V: $\beta_1 = 0.25$

Approximate solutions for the effect of variations in β_1 on population densities of the prey-predator model are given in Tables 12–14. Absolute errors in our solution for population densities are presented in Tables 15–17, respectively. The solution of the design scheme overlaps the exact solution with absolute errors that lie between 10^{-9} and 10^{-11} as illustrated in Figure 7. Figure 8 shows the behavior of fitness evaluation for the prey-predator model under the influence of β_1. Convergence analysis of performance measures is given in Tables 18 and 19. Trained neurons obtained by the LeNN-WOA-NM algorithm for different cases of Eq (31) are shown in Table 20. From the solutions (see Figure 9), the following conclusions can be drawn:

(i) The negative force of interaction between local and immigrant prey has a direct relation with population density of local prey and the predator, while population density is inversely related to β_1

7.3. Problem III: Effect of Variations in α_2 on the Prey-Predator Model. In this problem, the effect of variations in the increasing rate of immigrant prey α_2 on population densities of the prey-predator model is discussed. An error-based fitness function along with initial populations is given as follows:

$$\text{Minimize } \epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6, \tag{32}$$

where ϵ_1 to ϵ_6 are defined as follows:

$$\begin{cases} \epsilon_1 = \dfrac{1}{N}\displaystyle\sum_{n=1}^{34}\left(\dfrac{dS_n}{dt} - S_n\left((0.12) - \dfrac{(0.12)S_n}{50}\right) - (0.2)S_nX_n + (0.01)S_nY_n\right)^2, \\ \epsilon_2 = \dfrac{1}{N}\displaystyle\sum_{n=35}^{68}\left(\dfrac{dX_n}{dt} - X_n\left(\alpha_2 - \dfrac{\alpha_2 X_n}{k_2}\right) + (0.1)S_nX_n + (0.9)X_nY_n\right)^2, \\ \epsilon_3 = \dfrac{1}{N}\displaystyle\sum_{n=69}^{102}\left(\dfrac{dY_n}{dt} - (0.9)S_nY_n - (0.8)X_nY_n + (0.01)Y_n + (0.01)Y_n^2\right)^2, \\ \epsilon_4 = (S_0 - 0.5)^2, \\ \epsilon_5 = (X_0 - 0.5)^2, \\ \epsilon_6 = (Y_0 - 0.5)^2. \end{cases} \tag{33}$$

Table 18: Comparison through global performance indices for variations in β_1 on the prey-predator model.

Cases	$S(t)$				$X(t)$				$Y(t)$			
	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE
I	$8.39E-06$	$7.18E-05$	$3.75E-05$	$4.26E-04$	$2.61E-06$	$2.74E-04$	$2.14E-04$	$2.14E-04$	$1.63E-06$	$1.21E-04$	$4.34E-05$	$1.12E-05$
II	$1.00E-06$	$6.17E-05$	$3.16E-05$	$1.30E-04$	$8.45E-07$	$6.17E-05$	$4.37E-05$	$1.99E-05$	$1.54E-06$	$1.23E-04$	$4.33E-05$	$1.09E-05$
III	$1.64E-06$	$7.26E-05$	$3.67E-05$	$1.93E-04$	$2.27E-06$	$7.43E-05$	$5.27E-05$	$4.08E-05$	$1.49E-06$	$1.41E-04$	$4.91E-05$	$2.10E-05$
IV	$1.44E-06$	$6.46E-05$	$3.23E-05$	$1.17E-04$	$1.73E-06$	$6.86E-05$	$4.88E-05$	$3.59E-05$	$1.91E-06$	$1.23E-04$	$4.33E-05$	$1.55E-05$
V	$5.98E-07$	$5.97E-05$	$2.96E-05$	$7.46E-05$	$1.07E-06$	$6.29E-05$	$4.45E-05$	$2.23E-05$	$1.63E-06$	$1.04E-04$	$3.72E-05$	$9.00E-06$

Table 19: Convergence analysis of population density of local, immigrant prey, and predator under the influence of variations in β_1.

	Cases	FIT				MAD			TIC			ENSE		
		$\le 10^{-6}$	$\le 10^{-7}$	$\le 10^{-8}$	$\le 10^{-9}$	$\le 10^{-4}$	$\le 10^{-5}$	$\le 10^{-6}$	$\le 10^{-4}$	$\le 10^{-5}$	$\le 10^{-6}$	$\le 10^{-5}$	$\le 10^{-6}$	$\le 10^{-7}$
$S(t)$	I	89	55	16	0	99	83	6	100	91	35	71	38	4
	II	87	61	25	6	100	84	10	100	94	29	73	39	7
	III	98	80	27	2	100	79	12	100	94	25	78	37	12
	IV	98	77	25	3	100	85	8	100	93	22	86	43	9
	V	90	84	40	4	100	81	19	100	94	41	83	60	21
$X(t)$	I	97	75	36	11	99	77	15	100	85	21	94	66	27
	II	99	81	48	14	100	83	11	100	90	17	96	69	27
	III	98	84	33	12	100	82	7	100	90	9	96	65	19
	IV	99	84	39	11	100	80	10	100	92	14	71	17	3
	V	77	27	5	0	100	82	8	100	89	16	93	61	32
$Y(t)$	I	97	75	36	11	99	62	3	100	92	23	99	80	41
	II	94	78	36	8	100	68	4	100	90	24	97	79	38
	III	81	37	4	0	99	65	5	100	87	20	80	49	9
	IV	97	74	34	7	99	67	4	100	90	21	81	43	11
	V	77	27	5	0	100	97	42	100	100	66	87	60	27

Four cases are considered, depending on the value of α_2.

Case I: $\alpha_2 = 0.05$

Case II: $\alpha_2 = 0.10$

Case III: $\alpha_2 = 0.15$

Case IV: $\alpha_2 = 0.25$

The LeNN-WOA-NM algorithm is used to optimize the population densities of equation (33). Table 21 represents the comparison between the Ranga–Kutta method and the proposed technique LeNN-WOA-NM. A comparison between population density of local prey $S(t)$, immigrant prey $X(t)$, and predator $Y(t)$ with variations in α_2 is shown in Table 22. Statistics of absolute errors are given in Table 23 and graphically presented in Figure 10. The behavior of the fitness function for different cases is shown on boxplots as demonstrated in Figure 11. The mean values of the fitness function lie around 10^{-7} as shown in Tables 24 and 25. Unknown parameters achieved by the LeNN-WOA-NM algorithm for the given problem are given in Table 26. From solutions (see Figure 12), the following conclusion can be drawn:

(i) Population density of local prey, immigrant prey, and predator varies directly with an increasing rate of immigrate prey α_2

7.4. Problem IV. In this problem, the effect of variations in the negative force of interaction between local prey and immigrant prey on population densities of the prey-predator model is discussed. An error-based fitness function along with initial populations is given as follows:

$$\text{Minimize}\, \epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6, \tag{34}$$

where ϵ_1 to ϵ_6 are defined as follows:

$$\begin{cases} \epsilon_1 = \dfrac{1}{N}\displaystyle\sum_{n=1}^{34}\left(\dfrac{dS_n}{dt} - S_n\left((0.12) - \dfrac{(0.12)S_n}{50}\right) - (0.2)S_nX_n + (0.01)S_nY_n\right)^2, \\ \epsilon_2 = \dfrac{1}{N}\displaystyle\sum_{n=35}^{68}\left(\dfrac{dX_n}{dt} - X_n\left((0.2) - \dfrac{(0.2)X_n}{k_2}\right) + \beta_2 S_nX_n + (0.9)X_nY_n\right)^2, \\ \epsilon_3 = \dfrac{1}{N}\displaystyle\sum_{n=69}^{102}\left(\dfrac{dY_n}{dt} - (0.9)S_nY_n - (0.8)X_nY_n + (0.01)Y_n + (0.01)Y_n^2\right)^2, \\ \epsilon_4 = (S_0 - 0.5)^2, \\ \epsilon_5 = (X_0 - 0.5)^2, \\ \epsilon_6 = (Y_0 - 0.5)^2. \end{cases} \tag{35}$$

Table 20: Unknown parameters obtained by the LeNN-WOA-NM algorithm for the prey-predator model under the influence of variations in β_1.

	Index	$S(t)$			$X(t)$			$Y(t)$		
		ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
Case I	1	0.43006915	0.1953118	0.58526278	0.2039733	1.09461748	−0.3042383	0.45137375	0.613472621	0.21300391
	2	−0.1817023	0.12967851	−0.2238481	−0.1299324	−0.2205129	−0.2434037	0.89383923	0.347571185	0.192691905
	3	−0.1639676	−0.1906971	0.01654308	−0.2683275	0.21586942	−0.0650826	0.15826136	−0.38637064	0.544846185
	4	0.6432707	−0.2243964	−0.0644709	−0.1415483	0.00139286	−0.1221445	0.21055624	−0.28460075	0.325283188
	5	−0.063381	0.20479542	−0.0837701	0.37097529	−0.2551801	−0.1457444	0.85822173	−0.35898385	0.095809867
	6	−0.0587215	0.09571579	−0.2151334	0.23857131	0.30661251	0.34850165	−0.005745	0.620231649	0.59934855
	7	0.31197992	−0.0344083	−0.0542888	0.01997741	−0.1641653	−0.0515449	0.00074005	0.000287462	−0.37381232
	8	0.85365924	−0.0169789	0.25572233	0.80492801	−0.0023113	−0.2852425	−0.0522238	−0.2275238	0.146324864
	9	−0.2837938	−0.069528	−0.2221892	0.31692949	0.34731804	−0.0409553	−0.168709	−0.12570174	0.029229593
	10	−0.2720053	−0.0083273	0.71303279	0.0130666	0.04347241	0.10490579	1.16491711	−0.14620103	0.240782136
	11	0.0232457	0.04413909	0.57180799	−0.3961779	0.13805184	−0.0250827	0.1189892	−0.09976743	0.676863876
Case II	1	0.20084257	0.01848105	0.01151112	0.18834185	0.07714385	1.00199221	0.33090711	0.591954522	0.172081801
	2	0.77731512	0.20280783	−0.1534665	−0.0898678	−0.0222164	0.05097659	0.39013928	−0.16168208	−0.87414288
	3	−0.8932687	0.01562423	−0.8790331	−0.8785872	−0.2400978	0.01126939	0.62600687	−0.07469029	0.293200012
	4	−0.1254086	0.08769516	−0.4580886	−0.2608019	−0.0870728	0.23316388	0.49916445	−0.27253148	−0.00841655
	5	0.05410689	−0.0916165	−0.4295043	−0.5942252	−0.0340513	0.03471072	−0.0634013	−0.11877784	−0.77212021
	6	0.22564844	0.11645954	−0.0194417	0.06716894	−0.0067884	−0.2770185	−0.3074844	−0.17361157	−0.02940934
	7	0.25778166	0.02802331	0.32826015	0.0874406	−0.9239147	0.5776072	0.12893563	−0.07494518	0.101314911
	8	−0.6152252	0.08467719	−0.005507	−0.0016885	0.26959676	0.00027392	0.01072972	−0.05967189	0.794011281
	9	−0.4916591	0.03423008	0.35221385	0.1673546	0.20097946	0.03970559	−0.5329031	0.05685715	0.093143062
	10	0.09372071	0.45237283	−0.4270023	0.12270954	0.01633451	−0.0244912	0.32114085	−0.08529571	0.896172613
	11	$1.84E-05$	−0.1783898	−0.0967566	−0.032303	−0.3339006	−0.0531862	0.05299049	0.00250836	0.0227459
Case III	1	0.32865927	0.73232241	−0.0426226	0.3680448	0.12592558	−0.1711813	0.6895093	0.365675977	0.1815117
	2	0.06352326	0.01615348	0.71943001	0.02765614	0.5770961	0.12508308	0.72579383	0.234207148	0.278978323
	3	0.05902058	0.74414566	−0.0387953	−0.1285802	0.0259792	0.02360138	−0.0390772	0.334378871	0.373885739
	4	0.01672986	0.30780827	−0.0305187	0.76478835	0.00922025	−0.0003176	0.39416458	−0.14105866	0.231267475
	5	0.16886906	0.16663601	−0.1557756	0.14372478	−0.0053584	0.39867936	0.53867137	0.083576797	−0.0930085
	6	−0.1209806	−0.3749588	0.21620292	0.04682217	0.16582226	−0.1813835	0.31094709	−0.11844016	0.003817292
	7	−0.0756049	0.26868517	0.11232029	0.21554735	0.08172268	0.33302842	0.25825317	0.129740332	−0.27092193
	8	0.17875086	−0.0130043	0.7761683	−0.2683637	−0.0568285	0.08575422	−0.3405182	0.078523738	−0.1038394
	9	0.52179302	0.03458665	0.06475427	0.06956231	0.25072313	0.02715948	0.25907261	0.038549325	−0.02300298
	10	0.44795101	0.14647434	0.16811506	−0.0008598	0.31689128	0.21717058	0.55924027	0.030536512	−0.63332453
	11	0.07886187	−0.1241851	0.13943947	0.01341213	−0.6267738	0.34912288	−0.1931707	−0.01298631	0.579458203
Case IV	1	0.36926007	0.34021884	−0.0935731	0.16169213	0.09246816	1.38553805	0.44058544	0.499926759	0.127460796
	2	0.32481477	0.21004215	0.17360292	0.06611299	−0.152444	0.17230255	0.4871752	0.240390062	−0.13247269
	3	−0.1570069	0.04398714	−0.1372116	0.10393927	0.27712388	0.21929305	0.35934559	0.072436543	0.599680184
	4	0.10042725	−0.0405233	−0.0276124	0.00768115	−0.1152797	0.46671001	0.32697635	0.218813803	0.11578542
	5	0.2981599	0.07895408	0.05929431	−0.2575791	0.18210718	0.27949753	0.11208642	−0.13550479	−0.19303355
	6	0.01774278	0.0794438	0.34301322	0.3079287	−0.0288429	0.35285173	0.08051491	−0.0383289	0.301524551
	7	0.15448802	−0.0290092	0.37921177	0.40577398	0.02317734	0.19760018	0.5453665	0.190128144	0.134972259
	8	0.07303209	0.08335033	−0.0272924	−0.0268407	0.11495061	0.17565862	0.14548013	0.050459546	0.999559566
	9	0.06638016	0.09365192	0.46046701	0.31157149	0.24788627	0.39745699	0.14745623	−0.0203899	0.11010541
	10	0.14574411	0.27019012	0.08964107	−0.013653	0.6999887	0.14706426	0.14311693	0.085587473	0.406653134
	11	0.00422723	−0.0202326	0.09754264	$4.84E-05$	0.01130254	0.4544351	0.20276643	−0.02176553	0.041899718
Case V	1	0.41668155	0.55141417	0.31705647	0.53356928	−2.0883662	−0.2610704	0.24747335	−0.49568348	0.15116601
	2	0.28111104	0.29873199	0.12415861	0.01035183	0.3912644	−0.2250856	0.11463835	0.572594198	0.639529712
	3	−0.2880051	0.47632466	0.02148248	0.42831754	0.25753792	−0.1067969	0.49570912	−0.56574524	0.140211616
	4	0.28382154	0.07492232	0.25762031	−0.0051616	0.449995	−0.2583676	0.00862342	0.042826998	−0.24532638
	5	−0.1828993	0.05610035	0.41157899	0.10823296	−0.0022137	0.43840826	0.14531787	0.504082565	0.073357981
	6	−0.3536545	0.4045564	0.05410188	−0.1039612	−0.4487177	0.18545137	0.00207541	0.071838234	−0.18198509
	7	0.23523025	−0.1197378	0.22737042	0.27683837	−0.4502542	0.53741643	−0.1929315	0.094579762	0.334936445
	8	−0.0101984	0.60796793	0.06730913	−0.0740784	0.1513789	0.63740825	−0.3265244	−0.27219855	0.235174666
	9	−0.0038892	−0.3239558	0.10405389	0.41194902	0.01303423	−0.058557	0.25579472	0.192432911	−0.11924924
	10	0.20983244	0.24683053	0.09135039	0.0475623	0.44987648	−0.0005977	−0.2977078	0.013114946	0.599953402
	11	0.17699185	0.19078854	−0.4635923	−0.0023057	−0.2080841	0.13883293	0.21279433	−0.19995201	0.280170884

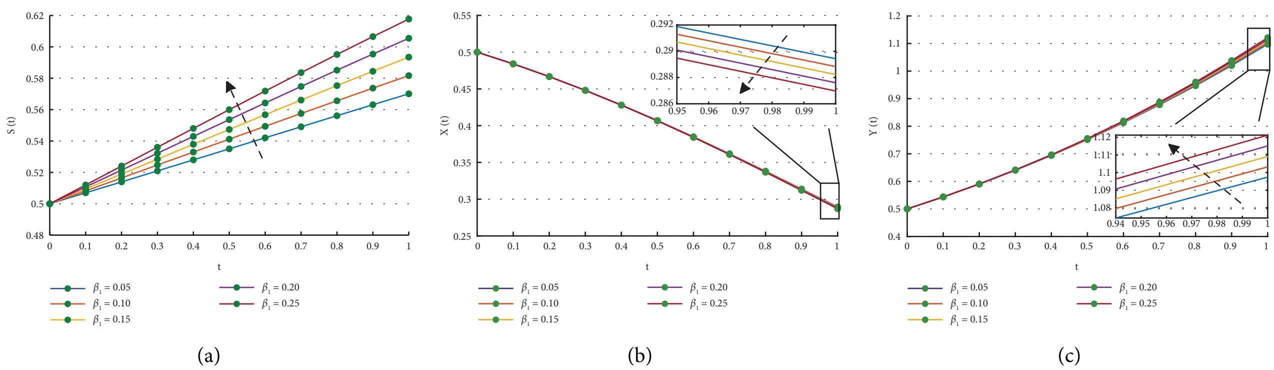


Figure 9: Comparison between solutions obtained by the LeNN-WOA-NM algorithm under the influence of variations in β_1 on population densities of the prey-predator model. (a) Approximate solutions for population density of local prey. (b) Approximate solutions for population density of immigrant prey. (c) Approximate solutions for population density of the predator.

Table 21: Comparison between solutions obtained by the LeNN-WOA-NM algorithm and ode45 for the prey-predator model with $\alpha_2 = 0.15$.

t	$S(t)$	ode45	$X(t)$	ode45	$Y(t)$	ode45
0.0	0.50000060	0.5	0.49999713	0.5	0.50000085	0.5
0.2	0.52140043	0.5214836	0.46218751	0.46914875	0.59021206	0.59055511
0.4	0.54273869	0.54304303	0.41967719	0.43242894	0.69476821	0.69630534
0.6	0.56382241	0.56446804	0.37330484	0.39042402	0.81497365	0.81880710
0.8	0.58443923	0.58554139	0.32433283	0.34420213	0.95201479	0.95948647
1.0	0.60442948	0.60605527	0.27440198	0.29530956	1.10702186	1.11963769

Three cases are considered, depending on the value of β_2.

Case I: $\beta_2 = 0.05$

Case II: $\beta_2 = 0.10$

Case III: $\beta_2 = 0.20$

The LeNN-WOA-NM algorithm is used to optimize the population densities of prey-predator model equation (35). Population densities of local prey, immigrant prey, and the predator are given in Tables 27-28, while absolute errors are shown in Table 29. The absolute errors in the solutions of the proposed technique lie between 10^{-9} and 10^{-12} as shown in Figure 13. Table 30 shows global values of performance indices and convergence of the fitness value as shown in Figure 14. The statistics shown in Table 31 illustrate that the global values of different performance functions lie between 10^{-5} and 10^{-7}, which highlights the robustness of the technique. The values of the weights in the LeNN structure for recreation of the approximate solutions are given in Table 32. From Figure 15, the following conclusion can be drawn:

(i) Population density of local prey varies directly with variations in β_2

(ii) Population density of immigrant prey and predator varies inversely with variations in β_2

7.5. Problem V: Effect of Variation in γ_1 on the Prey-Predator Model. In this problem, the effect of variations in the catching rate of local prey γ_1 on population densities of the prey-predator model is discussed. An error-based fitness function along with initial populations is given as follows:

$$\text{Minimize}\, \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6, \tag{36}$$

where ϵ_1 to ϵ_6 are defined as follows:

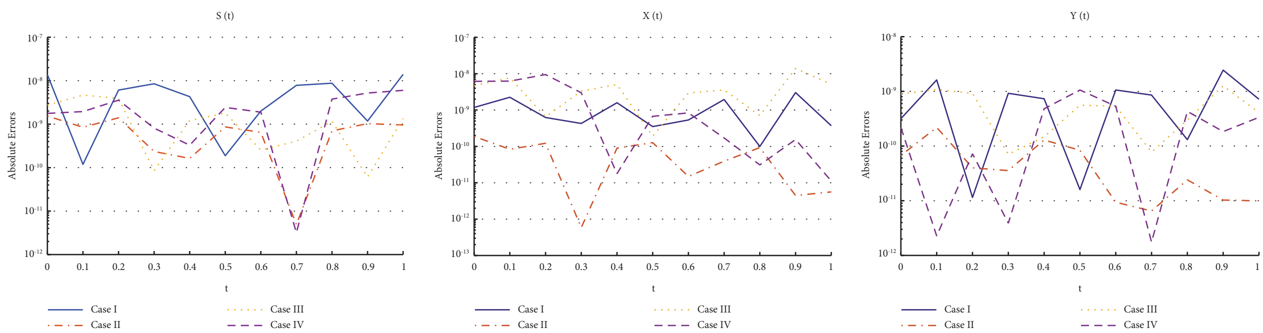
$$\begin{cases} \epsilon_1 = \dfrac{1}{N}\displaystyle\sum_{n=1}^{34}\left(\dfrac{dS_n}{dt} - S_n\left((0.12) - \dfrac{(0.12)S_n}{50}\right) - (0.2)S_nX_n + \gamma_1 S_nY_n\right)^2, \\ \epsilon_2 = \dfrac{1}{N}\displaystyle\sum_{n=35}^{68}\left(\dfrac{dX_n}{dt} - X_n\left((0.2) - \dfrac{(0.2)X_n}{k_2}\right) + (0.1)S_nX_n + (0.9)X_nY_n\right)^2, \\ \epsilon_3 = \dfrac{1}{N}\displaystyle\sum_{n=69}^{102}\left(\dfrac{dY_n}{dt} - (0.9)S_nY_n - (0.8)X_nY_n + (0.01)Y_n + (0.01)Y_n^2\right)^2, \\ \epsilon_4 = (S_0 - 0.5)^2, \\ \epsilon_5 = (X_0 - 0.5)^2, \\ \epsilon_6 = (Y_0 - 0.5)^2. \end{cases} \tag{37}$$

Table 22: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population densities of local, immigrant prey, and predator under the influence of the increasing rate of immigrant prey α_2.

t	$S(t)$				$X(t)$				$Y(t)$			
	$\alpha_2 = 0.05$	$\alpha_2 = 0.10$	$\alpha_2 = 0.15$	$\alpha_2 = 0.25$	$\alpha_2 = 0.05$	$\alpha_2 = 0.10$	$\alpha_2 = 0.15$	$\alpha_2 = 0.25$	$\alpha_2 = 0.05$	$\alpha_2 = 0.10$	$\alpha_2 = 0.15$	$\alpha_2 = 0.25$
0.0	0.49999996	0.49999970	0.50000060	0.50000018	0.49999997	0.50000023	0.49999713	0.49999870	0.499999997	0.500000562	0.500000847	0.500001033
0.2	0.52131291	0.52135689	0.52140043	0.52150112	0.45321205	0.45768261	0.46218751	0.47137924	0.589770367	0.589997848	0.590212055	0.590665856
0.4	0.54235797	0.54255447	0.54273869	0.54312436	0.40356843	0.41154623	0.41967719	0.43644215	0.692819675	0.693794106	0.694768206	0.696794465
0.6	0.56299694	0.56341623	0.56382241	0.56466616	0.35216917	0.36258531	0.37330484	0.39568861	0.810159712	0.812546426	0.814973647	0.820021754
0.8	0.58308161	0.58375622	0.58443923	0.58587662	0.30037587	0.31213132	0.32433283	0.35016888	0.942778570	0.947340120	0.952014792	0.961823425
1.0	0.60245464	0.60342550	0.60442948	0.60653682	0.24968763	0.26176158	0.27440198	0.30144245	1.091704823	1.099236698	1.107021862	1.123532428

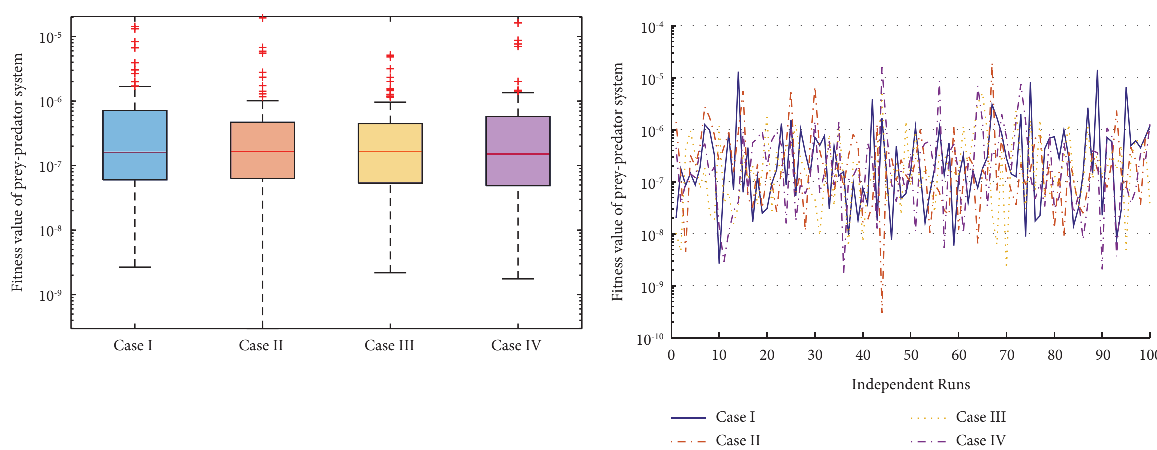
Table 23: Absolute errors obtained by the LeNN-WOA-NM algorithm for population densities of local, immigrant prey, and predator under the influence of α_2.

t	$S(t)$				$X(t)$				$Y(t)$			
	$\alpha_2 = 0.05$	$\alpha_2 = 0.10$	$\alpha_2 = 0.15$	$\alpha_2 = 0.25$	$\alpha_2 = 0.05$	$\alpha_2 = 0.10$	$\alpha_2 = 0.15$	$\alpha_2 = 0.25$	$\alpha_2 = 0.05$	$\alpha_2 = 0.10$	$\alpha_2 = 0.15$	$\alpha_2 = 0.25$
0	1.38$E-08$	1.54$E-09$	2.74$E-09$	1.77$E-09$	1.18$E-09$	1.88$E-10$	4.68$E-09$	6.17$E-09$	3.17$E-10$	6.73$E-11$	8.97$E-10$	2.19$E-10$
0.2	6.09$E-09$	1.40$E-09$	3.92$E-09$	3.57$E-09$	6.24$E-10$	1.22$E-10$	6.51$E-10$	9.47$E-09$	1.15$E-11$	4.00$E-11$	9.21$E-10$	7.22$E-11$
0.4	4.30$E-09$	1.63$E-10$	1.22$E-09$	3.29$E-10$	1.59$E-09$	8.95$E-11$	5.03$E-09$	1.78$E-11$	7.33$E-10$	1.29$E-10$	1.48$E-10$	4.81$E-10$
0.6	2.00$E-09$	6.47$E-10$	2.54$E-10$	1.89$E-09$	5.33$E-10$	1.49$E-11$	2.95$E-09$	8.36$E-10$	1.06$E-09$	9.40$E-12$	5.36$E-10$	5.34$E-10$
0.8	8.76$E-09$	7.02$E-10$	1.16$E-09$	3.77$E-09$	9.98$E-11$	9.28$E-11$	7.06$E-10$	3.10$E-11$	1.31$E-10$	2.41$E-11$	2.99$E-10$	4.31$E-10$
1	1.40$E-08$	9.57$E-10$	1.35$E-09$	6.00$E-09$	3.71$E-10$	5.62$E-12$	5.19$E-09$	1.12$E-11$	7.18$E-10$	9.95$E-12$	4.08$E-10$	3.35$E-10$



(a) (b) (c)

Figure 10: Absolute errors obtained by the LeNN-WOA-NM algorithm under the influence of variations in α_2 on population densities of the prey-predator model. (a) Population density of local prey. (b) Population density of immigrant prey. (c) Population density of the predator.



(a) (b)

Figure 11: Comparison between the box plots and convergence graphs of fitness evaluation over 100 independent runs for the prey-predator model with variations in the increasing rate of immigrant prey. (a) Box plots for fitness evaluation of the prey-predator model with variations in α_2. (b) Convergence of the fitness value of the prey-predator model with variations in α_2.

Four cases are considered, depending on variations in γ_1.

Case I: $\gamma_1 = 0.005$

Case II: $\gamma_1 = 0.015$

Case III: $\gamma_1 = 0.020$

Case IV: $\gamma_1 = 0.025$

The LeNN-WOA-NM algorithm is used to optimize the population densities of equation (37). Table 33 represents the comparison between the Ranga–Kutta method (ode45) and the proposed technique LeNN-WOA-NM algorithm. Approximate solutions for population density of local prey $S(t)$, immigrant prey $X(t)$, and the predator $Y(t)$ with different values of γ_1 are presented in Table 34. Statistics of absolute errors are given in Table 35 and graphically shown in Figure 16. The minimum and mean values of the fitness function for each case lie between 10^{-9} and 10^{-7}, respectively as illustrated in Figure 17. Convergence analysis of the fitness value, MAD, TIC, and ENSE is given in Tables 36-37. Unknown neurons of LeNN are shown in Table 38. From Figure 18, the following conclusion can be drawn:

(i) Population density of local prey and population density of the predator vary inversely with variations in γ_1

(ii) Population density of immigrant prey has a direct relation with variations in γ_1

7.6. Problem VI: Effect of Variations in γ_2 on the Prey-Predator Model. In this problem, the effect of variations in the catching rate of immigrant prey on population densities of

Table 24: Statistics of global performance indices for variations in the intrinsic growth rate of immigrant prey on the prey-predator model.

Cases	$S(t)$				$X(t)$				$Y(t)$			
	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE
I	$2.25E-06$	$5.70E-05$	$2.85E-05$	$7.97E-05$	$9.91E-07$	$6.19E-05$	$4.75E-05$	$7.97E-05$	$6.03E-07$	$8.42E-05$	$3.00E-05$	$5.97E-06$
II	$1.21E-06$	$4.64E-05$	$2.36E-05$	$4.09E-05$	$5.08E-07$	$4.68E-05$	$3.50E-05$	$8.22E-06$	$4.05E-07$	$6.93E-05$	$2.49E-05$	$2.93E-06$
III	$1.24E-06$	$4.33E-05$	$2.17E-05$	$3.13E-05$	$4.31E-07$	$4.76E-05$	$3.46E-05$	$8.99E-06$	$5.75E-07$	$7.72E-05$	$2.82E-05$	$3.20E-06$
IV	$5.22E-07$	$4.55E-05$	$2.29E-05$	$4.13E-05$	$3.65E-07$	$4.91E-05$	$3.39E-05$	$1.15E-05$	$8.39E-07$	$8.64E-05$	$3.06E-05$	$4.58E-06$

Table 25: Convergence analysis of population density of local prey, immigrant prey, and predator under the influence of variations in the increasing rate of immigrant prey α_2.

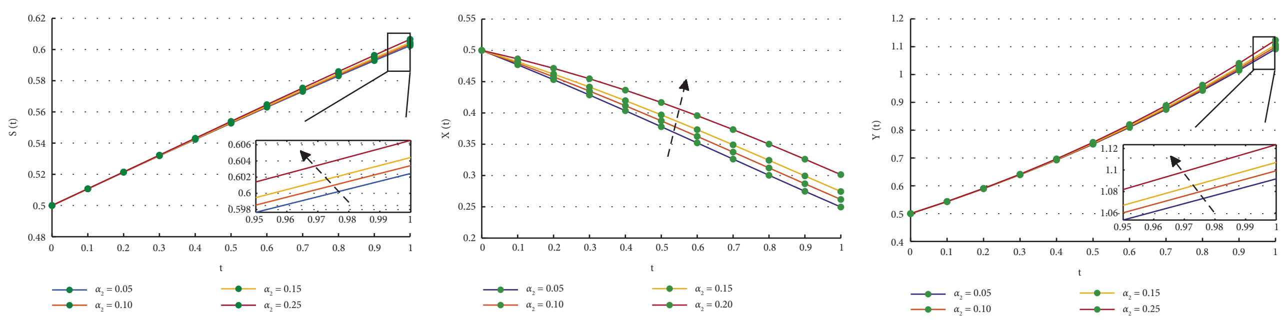
	Cases	FIT				MAD			TIC			ENSE		
		$\leq 10^{-6}$	$\leq 10^{-7}$	$\leq 10^{-8}$	$\leq 10^{-9}$	$\leq 10^{-4}$	$\leq 10^{-5}$	$\leq 10^{-6}$	$\leq 10^{-4}$	$\leq 10^{-5}$	$\leq 10^{-6}$	$\leq 10^{-5}$	$\leq 10^{-6}$	$\leq 10^{-7}$
$S(t)$	I	94	62	20	1	100	89	9	100	96	29	89	46	9
	II	100	70	16	1	100	91	14	100	98	26	91	42	14
	III	99	65	16	1	100	96	11	100	99	27	96	47	11
	IV	100	87	38	6	100	85	20	100	98	42	87	53	21
$X(t)$	I	98	78	45	14	100	85	13	100	90	17	95	65	37
	II	100	89	54	17	100	87	11	100	92	19	100	86	38
	III	99	92	52	11	100	89	8	100	97	17	99	79	32
	IV	100	89	46	16	100	87	12	100	96	21	98	71	27
$Y(t)$	I	99	86	39	12	100	76	5	100	95	27	97	89	72
	II	100	92	46	11	100	79	6	100	97	31	100	93	60
	III	100	83	46	15	100	74	8	100	99	26	100	87	56
	IV	98	78	43	11	100	69	4	100	98	19	99	93	54

Table 26: Unknown parameters obtained by the LeNN-WOA-NM algorithm for the prey-predator model under the influence of variations in the increasing rate of immigrant prey α_2.

	Index	$S(t)$			$X(t)$			$Y(t)$		
		ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
Case I	1	0.29809866	−0.0315366	0.58012735	0.47297503	0.16870771	0.25422827	0.04487536	0.494241827	0.219042842
	2	0.07902509	−0.142623	−0.5326572	0.29933207	0.36334065	0.40051667	0.7993046	0.01500143	0.046498829
	3	0.37716273	−0.0698453	−0.5278093	−0.5919778	0.06003798	−0.6780052	−0.608852	0.609110992	0.116343672
	4	0.06418686	0.24209073	0.11910483	0.29508708	−0.0084275	0.0309784	−0.0262412	−0.34143893	−0.53664746
	5	−0.0499254	−0.0250403	0.05106686	0.06326291	0.38462339	0.07701417	0.12151676	0.119098177	0.09960376
	6	−0.0395958	0.01489951	−0.2883484	0.08480605	0.18102453	−0.0291467	0.00187112	−0.01144597	−0.34122987
	7	0.15684755	0.08609064	0.20735343	0.51895958	0.4033704	0.21052592	0.49856342	0.262653738	0.034559337
	8	−0.0340164	0.04863621	0.22276639	0.04403982	0.21344694	0.31828637	0.24286503	0.043774589	0.529564847
	9	−0.2519217	0.061991	0.13654615	0.1880865	0.14486401	0.37615985	0.46832364	0.174620368	0.266799487
	10	0.54249116	−0.0324119	0.22565323	0.10730714	0.21764342	0.30007676	0.07731738	0.211031089	0.199472911
	11	0.12904085	0.04855899	−0.4666845	0.25677349	−0.0420107	−0.1514589	−0.1778985	0.104252082	−0.3576282
Case II	1	0.22014881	−0.0955389	0.01922733	0.44099269	0.16902098	0.21058167	0.56662276	0.214534099	−0.12547843
	2	−0.510556	0.02522092	0.56596508	0.2853307	−0.0140499	0.1381917	−0.0004646	0.491829768	0.382367631
	3	−0.1657635	0.01472461	−0.1147983	0.39351912	0.21069293	−0.2685875	0.39509887	0.072673362	−0.18283919
	4	0.05282327	0.07711052	−0.2051438	0.12992756	0.19352914	0.2499045	0.43645858	0.382113573	0.058714001
	5	0.64499253	0.25762641	0.23808203	0.8495944	−0.0043216	−0.0017105	0.25457162	0.313158881	−0.13553561
	6	0.09858966	−0.0273145	0.48787987	0.09374791	0.21889407	−0.0062504	−0.1344748	0.044741355	−0.2149526
	7	0.23380371	−0.1260215	0.3289594	0.05799592	−0.333364	0.15201786	−0.4456943	−0.06976927	0.002138263
	8	0.16035874	−0.1136321	−0.0036501	0.01897557	−0.0650362	0.09091915	1.14914925	−0.03585999	0.165910127
	9	0.01683166	0.03027356	0.01366012	0.16775817	0.20057677	0.1459576	0.53828172	0.130328528	0.267951969
	10	0.26771936	0.12204652	0.2996044	−0.1152812	0.18819176	0.17111444	−0.3869172	0.095372589	−1.04663455
	11	−0.1430149	0.0042418	−0.1108715	−0.1402711	0.17676553	−0.2193704	−0.2747091	0.12544106	0.108988227
Case III	1	0.20833477	0.87977164	−0.329864	0.37497171	0.15803512	1.08685499	0.81716797	−0.8285298	0.207619724
	2	−0.009672	−0.0834121	−0.0062989	0.00506333	0.45907436	−0.2830568	−0.055188	−1.29640697	0.383186286
	3	−0.1260443	0.5466368	−0.2547923	−0.1978145	−0.3014792	0.01482453	0.16026646	−0.12624417	−0.14211688
	4	−0.0459971	−0.1850824	0.44010074	0.85262434	−0.2293558	−0.1803312	0.15934035	−0.27875907	0.925897653
	5	0.61609669	−0.0654918	−0.2249982	1.06003476	−0.0605608	0.0377206	−0.2465468	0.095037929	−0.00368046
	6	−0.4221877	0.1915771	0.29040894	−0.1447329	−0.1716272	−0.2814112	0.10252045	0.140572643	0.100666708
	7	0.12304762	−0.0412171	−0.2947437	−0.3344259	0.15047463	−0.0948965	−0.3952389	−0.18667499	0.22033229
	8	0.22471377	−0.1369576	−0.0135929	0.46118375	0.09675102	0.25499179	0.6880506	−0.04146236	−0.02009528
	9	−0.0028525	0.28384483	−0.0137988	0.86855616	0.15119596	0.43141743	−0.310849	0.177047029	0.639644264
	10	0.10012064	−0.0365188	−0.3495845	0.3752446	0.15082062	0.39737423	0.02610775	−0.11632104	0.264941005
	11	0.6577301	−0.1729186	0.28691512	−0.277895	−0.2620078	−0.3183686	0.4666604	0.182729564	−0.40213292

Table 26: Continued.

	Index	$S(t)$			$X(t)$			$Y(t)$		
		ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
	1	0.27584273	0.14440132	0.05159123	−0.0014501	0.02944933	0.43857897	0.41541694	0.364106607	0.137637467
	2	1.0378582	0.22634893	−0.1336449	0.37479484	0.32390138	0.11683191	1.10634191	0.743870281	0.133721081
	3	0.30555234	0.14178372	0.09870863	0.1378461	−0.0134231	0.25235605	0.15847534	0.546747961	0.267557848
	4	0.49396717	−0.0077854	0.12863956	0.3091184	−0.0434135	0.98475694	0.07222518	−0.0015274	0.03284215
	5	0.96461172	0.48761051	0.0267945	0.4431833	0.19500184	0.28889858	0.10389701	−0.06048102	0.012779125
Case IV	6	−0.0311005	−0.2458898	0.01323704	0.39215778	0.07525855	0.0342729	−0.0239009	0.305047682	0.240819746
	7	−0.0009785	0.27803419	0.43494014	0.0813522	0.16527361	0.10128603	−0.0235703	0.131236783	0.342597408
	8	0.03245948	0.05591694	0.9180906	−0.064116	0.19286301	0.46309077	−0.0226751	0.265693128	0.191819022
	9	0.04135314	−0.0595634	−0.018826	−0.0453863	0.10473006	0.09188733	0.12139352	0.281687652	0.039757676
	10	0.06206464	0.10871955	0.21480649	0.12965376	−0.0275439	0.21169344	0.08952384	−0.23209362	−0.11227638
	11	0.41183634	−0.0123283	0.31935826	−0.0629837	0.00702969	0.08486244	0.22578589	0.194356561	0.207691077



(a) (b) (c)

Figure 12: Comparison between solutions obtained by the LeNN-WOA-NM algorithm under the influence of variations in the increasing rate of immigrant prey on population densities of the prey-predator model. (a) Approximate solutions for population density of local prey. (b) Approximate solutions for population density of immigrant prey. (c) Approximate solutions for population density of the predator.

Table 27: Comparison between solutions obtained by the LeNN-WOA-NM algorithm and ode45 for the prey-predator model under the influence of variations in β_2.

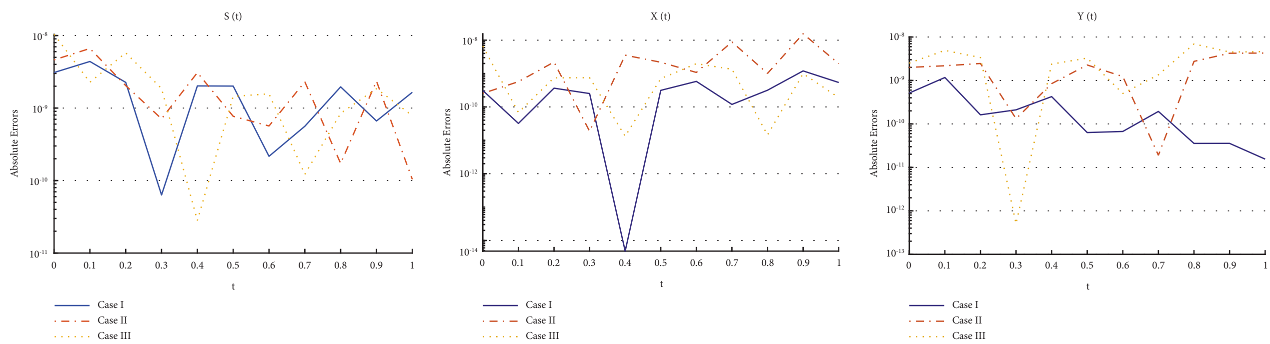
t	$S(t)$	*ode*45	$X(t)$	*ode*45	$Y(t)$	*ode*45
0.0	0.50000260	0.5	0.49999663	0.5	0.50000434	0.5
0.2	0.52140524	0.52140803	0.46202665	0.46202728	0.59020495	0.59020976
0.4	0.54273086	0.54274216	0.41918897	0.41919322	0.69472629	0.69473392
0.6	0.56379821	0.56380333	0.37243267	0.37243138	0.81483980	0.81483944
0.8	0.58439482	0.58439812	0.32308553	0.32307837	0.95168295	0.95168948
1.0	0.60434466	0.60435385	0.27281091	0.27280834	1.10636629	1.10638491

Table 28: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population densities of local prey, immigrant prey, and predator under the influence of the negative impact of the force of interaction on immigrant prey β_2.

t	$S(t)$			$X(t)$			$Y(t)$		
	$\beta_2 = 0.05$	$\beta_2 = 0.10$	$\beta_2 = 0.20$	$\beta_2 = 0.05$	$\beta_2 = 0.10$	$\beta_2 = 0.20$	$\beta_2 = 0.05$	$\beta_2 = 0.10$	$\beta_2 = 0.20$
0.0	0.49999997	0.49999839	0.50000260	0.49999998	0.50000038	0.49999663	0.49999998	0.49999971	0.500004344
0.2	0.52147482	0.52146698	0.52140524	0.46914810	0.46676681	0.46202665	0.59055020	0.59044621	0.590204953
0.4	0.54303474	0.54294731	0.54273086	0.43242621	0.42797254	0.41918897	0.69630031	0.69578968	0.694726289
0.6	0.56446751	0.56424461	0.56379821	0.39042495	0.38432545	0.37243267	0.81880214	0.81747363	0.814839800
0.8	0.58553673	0.58516310	0.58439482	0.34420481	0.33701910	0.32308553	0.95947828	0.95685274	0.951682945
1.0	0.60604682	0.60548065	0.60434466	0.29530830	0.28760242	0.27281091	1.11962732	1.11514838	1.106366286

TABLE 29: Absolute errors obtained by the LeNN-WOA-NM algorithm for population densities of local prey, immigrant prey, and predator under the influence of the negative impact of the force of interaction on immigrant prey β_2.

t	$S(t)$			$X(t)$			$Y(t)$		
	$\beta_2 = 0.05$	$\beta_2 = 0.10$	$\beta_2 = 0.20$	$\beta_2 = 0.05$	$\beta_2 = 0.10$	$\beta_2 = 0.20$	$\beta_2 = 0.05$	$\beta_2 = 0.10$	$\beta_2 = 0.20$
0.0	3.09E − 09	4.62E − 09	1.04E − 08	3.11E − 10	2.55E − 10	6.59E − 09	5.14E − 10	2.00E − 09	2.51E − 09
0.2	2.26E − 09	2.04E − 09	5.78E − 09	3.65E − 10	2.21E − 09	7.44E − 10	1.62E − 10	2.46E − 09	3.36E − 09
0.4	2.02E − 09	3.08E − 09	2.81E − 11	4.80E − 15	3.55E − 09	1.32E − 11	4.25E − 10	8.45E − 10	2.38E − 09
0.6	2.16E − 10	5.66E − 10	1.57E − 09	5.83E − 10	1.08E − 09	1.97E − 09	6.71E − 11	1.23E − 09	5.01E − 10
0.8	1.96E − 09	1.72E − 10	8.46E − 10	3.17E − 10	1.01E − 09	1.45E − 11	3.56E − 11	2.75E − 09	6.92E − 09
1.0	1.65E − 09	1.04E − 10	7.87E − 10	5.40E − 10	1.98E − 09	1.91E − 10	1.55E − 11	4.27E − 09	4.61E − 09



(a) (b) (c)

FIGURE 13: Absolute errors obtained by the LeNN-WOA-NM algorithm under the influence of variations in β_2 on population densities of the prey-predator model. (a) Population density of local prey. (b) Population density of immigrant prey. (c) Population density of the predator.

the prey-predator model is discussed. An error-based fitness function along with initial populations is given as follows:

$$\text{Minimize}\, \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6. \quad (38)$$

where ϵ_1 to ϵ_6 are defined as follows:

$$\begin{cases} \epsilon_1 = \dfrac{1}{N}\displaystyle\sum_{n=1}^{34}\left(\dfrac{dS_n}{dt} - S_n\left((0.12) - \dfrac{(0.12)S_n}{50}\right) - (0.2)S_nX_n + (0.01)S_nY_n\right)^2, \\ \epsilon_2 = \dfrac{1}{N}\displaystyle\sum_{n=35}^{68}\left(\dfrac{dX_n}{dt} - X_n\left((0.2) - \dfrac{(0.2)X_n}{k_2}\right) + (0.1)S_nX_n + \gamma_2 X_nY_n\right)^2, \\ \epsilon_3 = \dfrac{1}{N}\displaystyle\sum_{n=69}^{102}\left(\dfrac{dY_n}{dt} - (0.9)S_nY_n - (0.8)X_nY_n + (0.01)Y_n + (0.01)Y_n^2\right)^2, \\ \epsilon_4 = (S_0 - 0.5)^2, \\ \epsilon_5 = (X_0 - 0.5)^2, \\ \epsilon_6 = (Y_0 - 0.5)^2. \end{cases} \quad (39)$$

Five cases are considered, depending on the value of γ_2.

Case I: $\gamma_2 = 0.7$

Table 30: Statistics of global performance indices for variations in the increasing rate of immigrant prey on the prey-predator model.

Cases	$S(t)$				$X(t)$				$Y(t)$			
	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE
I	$5.58E-07$	$4.70E-05$	$2.36E-05$	$3.69E-05$	$4.72E-07$	$5.09E-05$	$3.59E-05$	$1.10E-05$	$8.00E-07$	$9.48E-05$	$3.40E-05$	$4.71E-06$
II	$1.48E-06$	$5.02E-05$	$2.49E-05$	$5.05E-05$	$4.76E-07$	$5.11E-05$	$3.63E-05$	$1.24E-05$	$7.09E-07$	$9.42E-05$	$3.35E-05$	$5.21E-06$
III	$2.13E-06$	$5.09E-05$	$2.58E-05$	$6.78E-05$	$8.11E-07$	$5.28E-05$	$3.85E-05$	$1.18E-05$	$7.99E-07$	$7.69E-05$	$2.79E-05$	$3.93E-06$

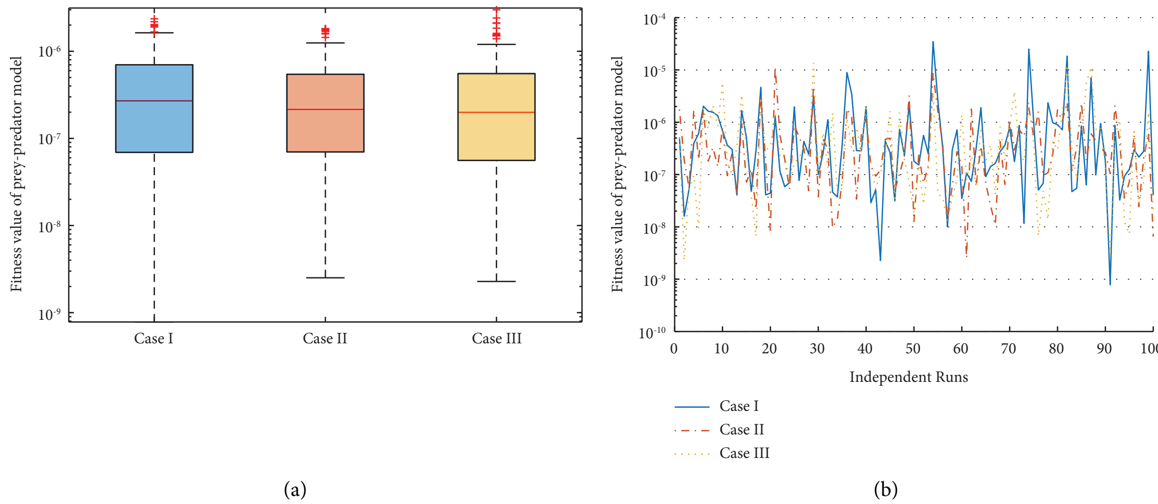


FIGURE 14: Comparison between the box plots and convergence graphs of fitness evaluation over 100 independent runs for the prey-predator model with variations in β_2. (a) Box plots for fitness evaluation of the prey-predator model with variations in β_2. (b) Convergence of the fitness value of the prey-predator model with variations in β_2.

TABLE 31: Convergence analysis of population density of local, immigrant prey, and predator under the influence of variations in β_2.

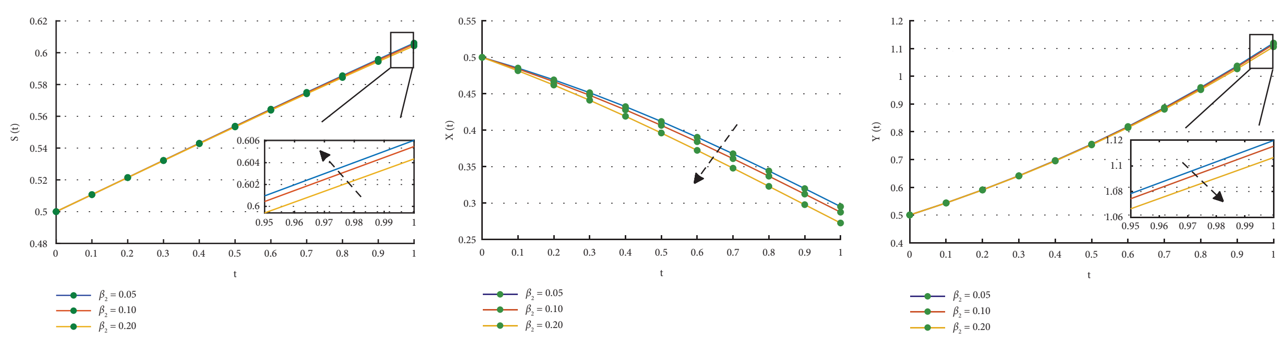
	Cases	FIT				MAD			TIC			ENSE		
		$\leq 10^{-6}$	$\leq 10^{-7}$	$\leq 10^{-8}$	$\leq 10^{-9}$	$\leq 10^{-4}$	$\leq 10^{-5}$	$\leq 10^{-6}$	$\leq 10^{-4}$	$\leq 10^{-5}$	$\leq 10^{-6}$	$\leq 10^{-5}$	$\leq 10^{-6}$	$\leq 10^{-7}$
$S(t)$	I	100	86	32	3	100	92	10	100	100	25	92	44	10
	II	100	63	12	0	100	89	10	100	99	26	89	48	10
	III	96	65	14	2	100	89	15	100	97	37	89	52	15
$X(t)$	I	100	86	45	14	100	88	9	100	94	15	100	68	25
	II	100	91	48	12	100	90	7	100	94	19	97	73	30
	III	99	82	42	15	100	88	12	100	96	20	98	69	27
$Y(t)$	I	99	80	39	5	100	67	2	100	94	24	100	86	48
	II	100	76	39	14	100	71	3	100	93	23	100	88	42
	III	98	86	48	12	100	77	3	100	96	26	100	89	64

TABLE 32: Unknown parameters obtained by the LeNN-WOA-NM algorithm for the prey-predator model under the influence of variations in β_2.

	Index	$S(t)$			$X(t)$			$Y(t)$		
		ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
Case I	1	0.27535144	0.31269649	−0.212623	0.35537988	0.47248103	0.10822633	0.52114951	0.185720875	−0.00890278
	2	0.61516838	0.24930343	−0.1133566	−0.2698898	0.23075745	1.45662131	0.00285471	0.54326608	−0.03948103
	3	−0.3331712	0.13683002	0.07415358	−0.007558	−0.1414522	0.00429373	−0.2363309	−0.53264189	−0.12625219
	4	−0.1724685	−0.1569281	−0.1780917	−0.0230189	0.06567477	−0.1619849	−0.07434	0.280494463	0.724308841
	5	0.34235255	0.16601501	0.33779612	0.38214222	0.17206398	0.37567544	0.25394552	−0.20424256	0.497829902
	6	0.00647555	−0.0671007	0.41848498	0.465563	0.17852399	0.25910001	0.62807088	−0.25015769	−0.03607794
	7	0.39917241	−0.2086423	0.27916279	−0.1982513	−0.1003214	0.08612276	0.23112207	−0.11265536	−0.04638131
	8	0.37891578	−0.1471092	0.25516272	0.47437651	−0.0479726	0.3954123	0.23371466	0.155785232	0.111525519
	9	−0.1569822	0.21029864	−0.280122	−0.0498625	0.07898872	0.18498889	−0.1404023	−0.28974674	0.273393228
	10	−0.1405838	−0.0119767	−0.1635125	0.11249313	−0.0654565	0.69188379	−0.0877791	−0.1594545	−0.77121428
	11	0.39296939	0.02457412	−0.0200821	−0.1201447	0.17161889	0.03630063	0.04811392	0.279716864	−0.22688875

Table 32: Continued.

	Index	$S(t)$			$X(t)$			$Y(t)$		
		ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
Case II	1	0.50105252	0.82536339	0.58045483	0.45518044	−0.1160988	−0.3093088	0.14735255	0.576552359	0.192127271
	2	0.26141806	0.21796238	0.31815591	−0.1676624	0.4234933	−0.1614025	0.00975838	0.53917788	0.775847618
	3	−0.1877622	−0.1568166	−0.3859022	0.34192011	−0.1162183	0.78757311	0.40990792	−0.29126952	0.094655869
	4	0.02599785	−0.1764539	0.17491514	0.03149905	0.44911473	−0.1695519	−0.0583248	−0.00447879	−0.06289373
	5	−0.2923416	−0.1065739	0.38764839	0.10732316	−0.003522	0.43009606	0.20268435	0.627584449	0.02333884
	6	−0.3868009	0.44970016	−0.0422409	−0.2581683	−0.4105199	0.01656808	0.00066793	0.075964394	−0.18685644
	7	0.05739953	−0.1193984	−0.2607123	0.17181358	−0.2713628	0.66311474	−0.124996	−0.07981308	0.505938132
	8	−0.0216328	0.8067003	0.31701813	−0.1511588	0.14092788	0.65770582	−0.1918301	−0.00407396	0.121012726
	9	−8.30E − 07	−0.0664668	−0.1168664	0.63180611	−0.1478398	−0.1592677	0.27026843	0.113142772	0.003053122
	10	0.09718671	−0.0187463	−0.0760148	−0.0444081	0.29980599	0.00951493	−0.1584802	−0.01177252	0.821398692
	11	0.19396717	0.18738185	−0.4407164	−0.1400116	−0.1519869	0.03850045	0.32211631	−0.10616995	0.295182432
Case III	1	0.36877614	0.4187033	0.00551721	0.37106378	0.76553227	−0.1135028	0.36766284	0.656543877	0.074012878
	2	0.57123158	0.39029235	0.26925195	0.02897603	−0.0177996	−0.2724976	0.68341531	−0.56659891	0.035271905
	3	−0.2786739	−0.1350866	−0.4079824	0.18172611	−0.1092217	−0.6535241	0.19817369	−0.18104441	0.127718735
	4	0.24896052	0.13322324	0.34245899	0.18535487	0.18336655	0.32712799	−0.1422226	0.152834288	0.025045362
	5	0.29175967	0.21119283	−0.1957816	0.31132875	−0.0532529	−0.3305964	0.04517261	−0.73533086	−0.21051239
	6	0.38609249	0.27590847	0.58209501	−0.535581	−0.1823894	0.31125028	−0.0033038	0.082666375	0.323016755
	7	0.01164667	−0.1421768	0.24328317	0.26480037	0.02173502	0.07848485	0.61775482	0.231893374	0.074346878
	8	0.59381321	0.06375022	−0.3689713	0.35998444	−0.0368666	0.32907219	0.14877358	0.143483247	0.3344067
	9	−0.2877027	−0.0812976	0.58513537	0.11454134	0.04186864	−0.3814356	0.28795352	−0.00041389	−0.13335931
	10	−0.3073994	0.0111124	−0.0068885	0.04196657	−0.127565	0.14629639	−0.4081982	0.0030286	0.33443128
	11	0.06421859	−0.0919722	0.34881249	−0.0684058	−0.1412179	0.497947	0.19874151	0.071110753	0.298703303



(a) (b) (c)

Figure 15: Comparison between solutions obtained by the LeNN-WOA-NM algorithm under the influence of variations in β_2 on population densities of the prey-predator model. (a) Approximate solutions for population density of local prey. (b) Approximate solutions for population density of immigrant prey. (c) Approximate solutions for population density of the predator.

Table 33: Comparison between solutions obtained by the LeNN-WOA-NM algorithm and ode45 for the prey-predator model with $\gamma_1 = 0.015$.

t	$S(t)$	*ode*45	$X(t)$	*ode*45	$Y(t)$	*ode*45
0.0	0.50000029	0.5	0.49999971	0.5	0.50000002	0.5
0.2	0.52117967	0.52117476	0.4667562	0.46676439	0.59043325	0.59042518
0.4	0.54230262	0.54229891	0.42797497	0.42797925	0.69572032	0.695704
0.6	0.5631532	0.56315108	0.38434774	0.38435442	0.81726419	0.81725811
0.8	0.58351051	0.58350611	0.33705551	0.33706036	0.95637538	0.95637013
1.0	0.60315238	0.6031513	0.28769246	0.28769927	1.11420939	1.11419232

TABLE 34: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population densities of local prey, immigrant prey, and predator under the influence of variations in γ_1.

t	$S(t)$				$X(t)$				$Y(t)$			
	$\gamma_1 = 0.005$	$\gamma_1 = 0.015$	$\gamma_1 = 0.020$	$\gamma_1 = 0.025$	$\gamma_1 = 0.005$	$\gamma_1 = 0.015$	$\gamma_1 = 0.020$	$\gamma_1 = 0.025$	$\gamma_1 = 0.005$	$\gamma_1 = 0.015$	$\gamma_1 = 0.020$	$\gamma_1 = 0.025$
0.0	0.49999982	0.50000029	0.50000000	0.49999997	0.49999982	0.49999971	0.49999998	0.50000017	0.500000504	0.500000018	0.499999997	0.500000131
0.2	0.52174700	0.52117967	0.52089269	0.52060661	0.46675174	0.46675620	0.46676007	0.46676615	0.590449582	0.590433254	0.590411317	0.590394050
0.4	0.54359472	0.54230262	0.54165974	0.54101631	0.42795466	0.42797497	0.42797618	0.42799714	0.695846635	0.695720322	0.695630674	0.695558797
0.6	0.56533929	0.56315320	0.56206177	0.56097559	0.38430829	0.38434774	0.38437834	0.38439748	0.817678656	0.817264191	0.817046212	0.816837342
0.8	0.58680848	0.58351051	0.58186269	0.58022442	0.33695644	0.33705551	0.33711236	0.33716085	0.957316487	0.956375378	0.955894440	0.955419606
1.0	0.60782251	0.60315238	0.60083619	0.59852684	0.28751777	0.28769246	0.28777966	0.28787420	1.116062094	1.114209392	1.113259493	1.112332477

Table 35: Absolute errors obtained by the LeNN-WOA-NM algorithm for population densities of local prey, immigrant prey, and predator under the influence of variations in γ_1.

t	$S(t)$				$X(t)$				$Y(t)$			
	$\gamma_1=0.005$	$\gamma_1=0.015$	$\gamma_1=0.020$	$\gamma_1=0.025$	$\gamma_1=0.005$	$\gamma_1=0.015$	$\gamma_1=0.020$	$\gamma_1=0.025$	$\gamma_1=0.005$	$\gamma_1=0.015$	$\gamma_1=0.020$	$\gamma_1=0.025$
0.0	2.74E−09	8.47E−10	3.60E−10	7.80E−11	5.74E−09	2.00E−09	4.47E−09	4.19E−12	8.54E−10	6.23E−09	9.79E−12	5.10E−13
0.2	2.76E−09	4.69E−10	2.92E−10	1.04E−11	3.59E−09	8.16E−11	4.78E−09	5.77E−10	5.53E−10	6.28E−09	1.38E−11	8.72E−11
0.4	5.70E−10	4.68E−10	4.62E−13	3.63E−10	2.51E−09	3.02E−10	4.26E−10	2.89E−10	5.49E−10	8.44E−10	1.36E−12	2.06E−10
0.6	1.89E−09	1.24E−10	1.54E−10	2.49E−12	4.04E−11	3.74E−10	4.26E−09	2.55E−11	1.47E−12	3.54E−09	9.17E−11	1.14E−09
0.8	2.36E−09	1.80E−11	2.01E−12	8.31E−10	1.28E−09	4.93E−10	3.67E−09	5.03E−10	3.59E−10	3.49E−09	9.80E−11	4.43E−10
1.0	2.82E−09	1.85E−10	4.54E−12	1.17E−09	2.58E−10	1.22E−15	4.68E−09	1.08E−10	2.82E−10	4.48E−09	3.19E−10	1.19E−09

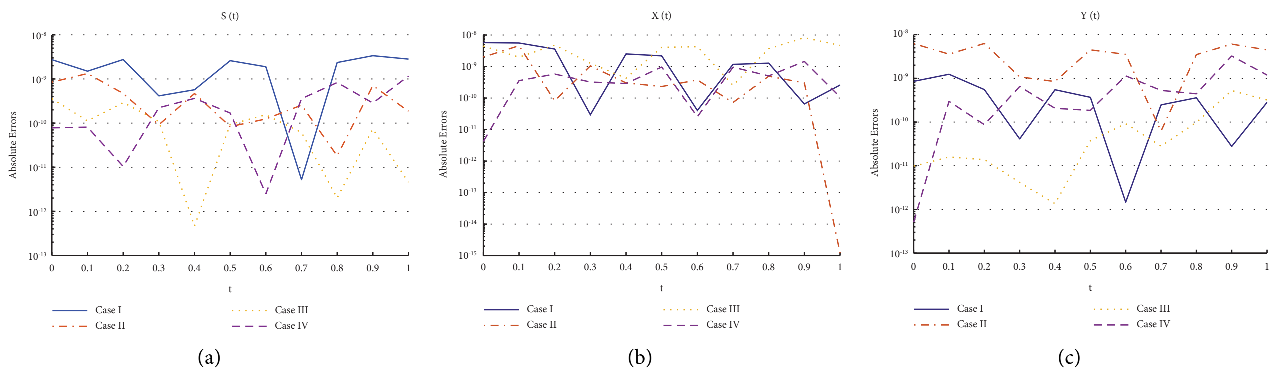


Figure 16: Absolute errors obtained by the LeNN-WOA-NM algorithm under the influence of variations in γ_1 on population densities of the prey-predator model. (a) Population density of local prey. (b) Population density of immigrant prey. (c) Population density of the predator.

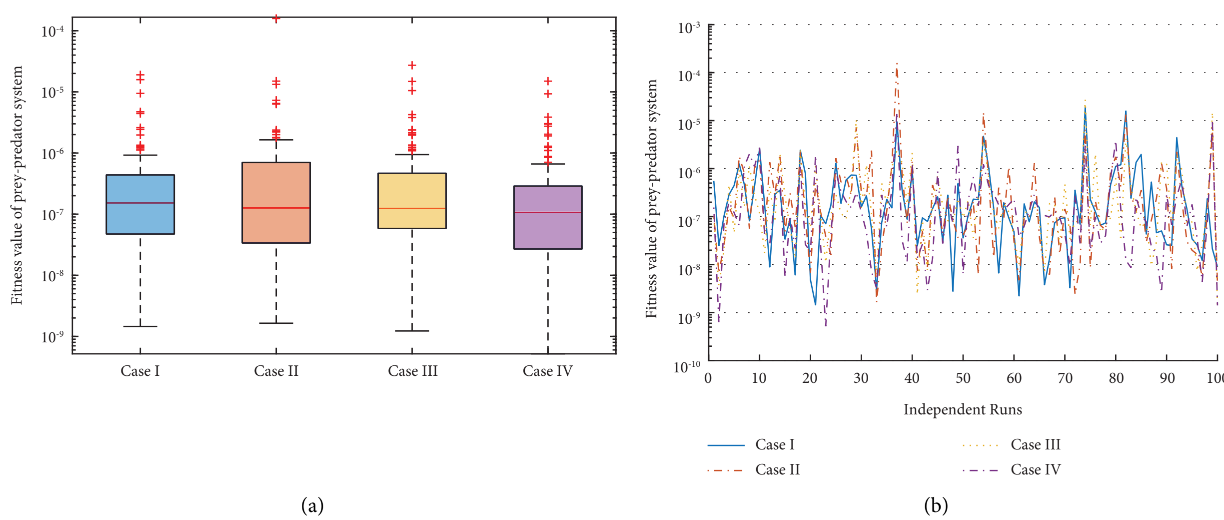


Figure 17: Comparison between the box plots and the convergence graph of fitness evaluation over 100 independent runs for the prey-predator model with variations in the catching rate of local prey. (a) Box plots for fitness evaluation of the prey-predator model with variations in γ_1. (b) Convergence of the fitness value of the prey-predator model with variations in γ_1.

Case II: $\gamma_2 = 0.8$

Case III: $\gamma_2 = 1.0$

Case IV: $\gamma_2 = 1.1$

The LeNN-WOA-NM algorithm is applied to prey-predator model equation (39) to study the influence of variations in the catching rate of immigrant prey. The results calculated by the designed algorithm are compared with those of the Runge–Kutta method using ode45 in MATLAB as shown in Table 39. Approximate solutions for population densities of local prey, immigrant prey, and the predator are given in Table 40. Absolute errors are presented in Table 41 and graphically shown in Figure 19. The minimum fitness values in Figure 20 reflect the accuracy of the solutions by the proposed algorithm. The convergence of our numerical approach is assessed by fitness values and statistical analysis of the performance indicators (MAD, TIC, and ENSE), see Table 42. Table 43 shows the

Table 36: Statistics of global performance indices for variations in the catching rate of local prey on the prey-predator model.

Cases	$S(t)$				$X(t)$				$Y(t)$			
	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE
I	$2.50E-06$	$5.21E-05$	$2.69E-05$	$2.50E-06$	$9.20E-07$	$4.63E-05$	$3.33E-05$	$1.51E-05$	$9.19E-07$	$8.88E-05$	$3.10E-05$	$8.98E-06$
II	$7.32E-06$	$6.20E-05$	$3.14E-05$	$5.63E-05$	$2.43E-06$	$6.86E-05$	$4.92E-05$	$7.58E-05$	$1.44E-06$	$1.06E-04$	$3.80E-05$	$1.70E-05$
III	$2.86E-06$	$4.87E-05$	$2.48E-05$	$9.34E-05$	$7.60E-07$	$5.39E-05$	$3.89E-05$	$1.86E-05$	$1.39E-06$	$8.53E-05$	$3.09E-05$	$5.88E-06$
IV	$1.78E-06$	$4.62E-05$	$2.38E-05$	$7.52E-05$	$5.18E-07$	$4.32E-05$	$3.10E-05$	$1.46E-05$	$6.42E-07$	$8.01E-05$	$2.83E-05$	$9.08E-06$

Table 37: Convergence analysis of population density of local prey, immigrant prey, and predator under the influence of variations in γ_1.

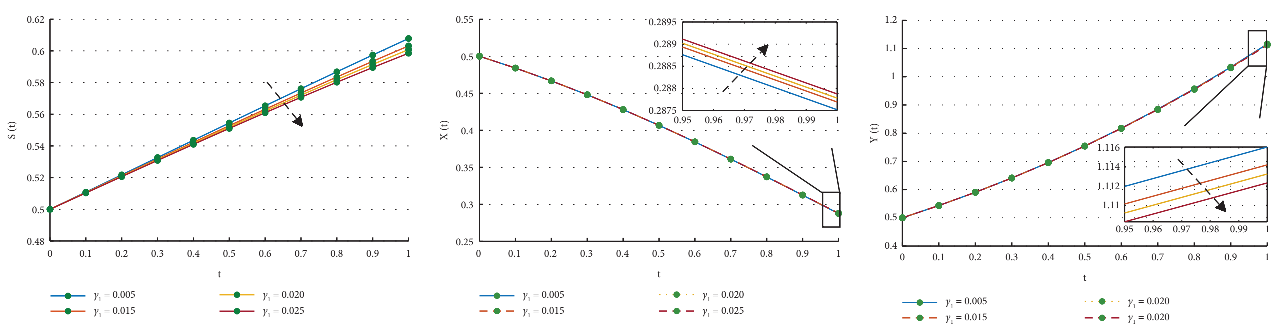
	Cases	FIT				MAD			TIC			ENSE		
		$\le 10^{-6}$	$\le 10^{-7}$	$\le 10^{-8}$	$\le 10^{-9}$	$\le 10^{-4}$	$\le 10^{-5}$	$\le 10^{-6}$	$\le 10^{-4}$	$\le 10^{-5}$	$\le 10^{-6}$	$\le 10^{-5}$	$\le 10^{-6}$	$\le 10^{-7}$
$S(t)$	I	95	70	20	5	100	85	21	100	94	37	85	56	21
	II	94	67	25	2	99	81	23	100	94	43	83	57	21
	III	95	71	17	2	100	90	19	100	96	42	91	55	19
	IV	97	78	29	6	100	91	26	100	97	42	90	57	26
$X(t)$	I	97	90	59	22	100	91	15	100	94	27	97	8	38
	II	97	86	56	24	99	84	16	99	91	28	96	78	40
	III	99	85	64	17	100	88	15	100	89	21	94	81	33
	IV	99	93	57	22	100	93	20	100	96	31	98	81	46
$Y(t)$	I	99	87	51	15	99	77	7	100	95	25	99	91	59
	II	96	78	48	15	99	73	8	100	93	30	98	87	62
	III	98	82	49	15	100	75	10	100	97	30	99	88	61
	IV	98	90	60	17	100	93	37	100	93	37	99	93	64

Table 38: Unknown parameters obtained by the LeNN-WOA-NM algorithm for the prey-predator model under the influence of variations in γ_1.

	Index	$S(t)$			$X(t)$			$Y(t)$		
		ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
Case I	1	0.10433411	−0.9093816	−0.0551384	0.10814164	−0.0975817	−0.4024759	0.11057901	0.030277215	−0.07683093
	2	−0.1015821	−0.2339063	0.11183609	−0.0079649	0.12409457	−0.0066366	0.03120447	0.065984151	0.169815479
	3	−0.1732219	0.05666544	−0.3418627	−0.1576219	0.20511634	0.23158339	−0.3370419	−0.15418238	0.194154313
	4	0.2870666	0.04587599	−0.075017	0.0317597	−0.1567915	−0.2960607	0.12661418	0.176984003	−0.08307132
	5	0.18918051	−0.2634862	0.09296959	0.17712714	0.06690293	0.05596124	0.06180474	−0.14442377	−0.98944725
	6	0.08011991	−0.0241843	0.04384194	0.09323427	−0.1029541	0.30404503	−0.1508149	−0.05051503	0.191092976
	7	−0.4734878	−0.1888967	0.13971948	0.24925106	0.00225215	0.00501846	−0.9616715	−0.19702187	0.011340631
	8	0.19753354	−0.0151743	−0.0358307	−0.2646482	−0.4318811	−0.2034545	−0.0870373	−0.08872014	0.079363347
	9	0.07352321	0.04374259	0.17641082	−0.004594	−0.0709633	−0.188571	0.01228411	0.051586673	−0.1707669
	10	0.05966701	−0.009466	0.11306597	−1.0101745	−0.1195556	0.30440396	−0.16551	0.072728886	0.05362364
	11	0.24419278	0.21523627	0.12501506	0.10887851	0.12112026	−0.1899001	−0.157371	−0.15906121	0.128530508
Case II	1	0.42562436	0.22405135	0.02503956	0.35262391	−0.2253664	0.26259455	0.27111686	0.328796691	0.228927872
	2	0.41666988	0.5078151	−0.1889093	0.12171391	0.20998105	0.3804858	0.30390747	0.389116532	0.460701796
	3	0.10577695	0.06002846	0.21768786	−0.5393398	0.39027559	0.13902226	0.47350419	0.465419864	0.661970007
	4	0.46517407	0.09698879	−0.1633478	0.11014297	0.19739907	0.6654931	0.05604469	−0.00392697	0.278777394
	5	0.60100029	0.01932057	0.56830498	0.2354314	0.41404057	0.20437334	−0.1267241	−0.13128894	0.293844939
	6	0.08127736	0.10628146	0.38610942	0.23539962	0.17187529	0.38579085	0.35225137	0.067649428	0.188489281
	7	0.35498411	0.04490212	0.04900664	0.06476587	0.35238074	0.50235806	0.36253639	0.203702979	−0.0792048
	8	0.38497077	0.01823575	0.34815044	−0.0755594	0.49478279	0.38035495	0.16102938	−0.12482814	0.340540011
	9	−0.0561765	0.73291315	−0.1890215	0.0053794	−0.4438896	0.46641096	0.2889986	−0.01721494	0.103497157
	10	$-8.42E-05$	−0.0191478	0.33057513	0.01138671	0.21012745	0.28013928	0.41790512	0.011022434	0.111373073
	11	0.03758884	−0.0090962	0.29841949	0.06181435	0.28211233	0.1044331	0.76029441	−0.05305741	0.400559476
Case III	1	0.34210927	−0.1686215	−0.0637152	0.23427117	0.25285975	0.39214738	0.26900094	0.565520288	0.485240258
	2	−0.0413096	−0.0081301	0.10614537	0.60106519	−0.0765896	0.42188566	0.34146251	0.575583706	0.081232416
	3	0.29904262	−0.075813	−0.1967563	0.45939119	0.15935586	−0.0737513	−0.0371428	0.395780195	0.526025075
	4	0.31966978	−0.0296846	0.13492819	0.74449918	0.13823099	0.14733057	0.31296573	0.170356341	0.449030416
	5	0.38511171	0.02742915	0.19245868	0.17480946	0.26067733	0.19256129	−0.1966744	0.026652715	0.648556737
	6	0.15405406	−0.085595	0.03900298	0.20316026	−0.0939363	0.36999635	0.14116303	0.18349354	0.157284014
	7	0.01160236	0.19042424	−0.0099165	−0.1955586	−0.0145249	0.5115481	0.09234283	0.050183053	0.529576694
	8	−0.0119424	−0.0017222	0.19894821	−0.0459019	0.16995456	0.09912122	0.44277162	0.268197932	0.360452198
	9	−0.013509	0.02826738	0.13366696	−0.0986121	0.12555621	0.29387726	−0.0351511	0.15868406	−0.02656018
	10	0.36233153	−0.0504481	0.46422188	0.33154131	0.04488349	0.44423777	0.18233457	0.002699819	0.25017744
	11	0.20780852	0.06187582	0.19012834	0.47309423	−0.0001455	0.28544612	0.32655747	0.232204493	0.038376682

Table 38: Continued.

	Index	$S(t)$			$X(t)$			$Y(t)$		
		ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
Case IV	1	0.08743197	−0.0274586	0.13838454	0.10216597	−0.6205394	−1.1625473	0.08890874	−0.46821489	0.031906641
	2	0.85138918	−0.2596089	0.27697631	−0.0395679	−0.3190484	0.47807164	−1.0766899	0.194522468	0.051473955
	3	0.32484395	−0.0320128	0.04910259	0.59721285	−0.7402654	−0.5159275	−0.5457082	0.584514565	0.334208342
	4	1.26539529	0.0637656	−0.3298542	0.00746795	0.12675579	−0.2119938	0.0689466	0.078060063	0.341243622
	5	0.27124475	−0.0145706	−0.8795409	−0.2258574	−1.0489144	0.18138843	−0.7172783	−0.44654588	0.288352353
	6	−0.0615221	−0.0357181	0.05167451	−0.0093528	−0.9076303	0.00529929	0.11164685	0.084161905	0.021304977
	7	−0.9330419	0.00587016	0.08238699	−0.0038462	0.5318456	−0.0029243	−0.5624983	0.07126723	−0.07194593
	8	−0.2222851	0.15964489	0.15929347	0.02677628	−0.0859581	0.71257834	−0.1564259	0.011627909	−0.01141564
	9	−0.0746732	0.03595603	−0.1688295	0.91853869	0.0432707	−0.1996406	0.9708232	−0.20718687	0.491048316
	10	0.11105985	0.03339826	−0.0412419	−0.9111904	−0.0757299	0.38318296	0.14046109	0.068233399	0.447903219
	11	−0.9956577	−0.0325751	0.95862042	0.01858614	−0.0328286	0.2342197	−0.8047726	−0.03547963	0.018056695



(a) (b) (c)

Figure 18: Comparison between solutions obtained by the LeNN-WOA-NM algorithm under the influence of variations in γ_1 on population densities of the prey-predator model. (a) Approximate solutions for population density of local prey. (b) Approximate solutions for population density of immigrant prey. (c) Approximate solutions for population density of the predator.

Table 39: Comparison between solutions obtained by the LeNN-WOA-NM algorithm and ode45 for the prey-predator model with $\gamma_2 = 0.8$.

t	$S(t)$	*ode*45	$X(t)$	*ode*45	$Y(t)$	*ode*45
0.0	0.50000006	0.5	0.5000005	0.5	0.49999982	0.5
0.2	0.52151097	0.52117476	0.47186632	0.46676439	0.59068568	0.59042518
0.4	0.54316182	0.54229891	0.438185	0.42797925	0.69694028	0.695704
0.6	0.56475603	0.56315108	0.39935526	0.38435442	0.82053699	0.81725811
0.8	0.58608332	0.58350611	0.35617116	0.33706036	0.96319165	0.95637013
1.0	0.60693367	0.6031513	0.30986509	0.28769927	1.12649634	1.11419232

Table 40: Approximate solutions obtained by the LeNN-WOA-NM algorithm for population densities of local prey, immigrant prey, and predator under the influence of γ_2.

t	$S(t)$				$X(t)$				$Y(t)$			
	$\gamma_2 = 0.7$	$\gamma_2 = 0.8$	$\gamma_2 = 1.0$	$\gamma_2 = 1.1$	$\gamma_2 = 0.7$	$\gamma_2 = 0.8$	$\gamma_2 = 1.0$	$\gamma_2 = 1.1$	$\gamma_2 = 0.7$	$\gamma_2 = 0.8$	$\gamma_2 = 1.0$	$\gamma_2 = 1.1$
0.0	0.50000038	0.50000006	0.49999999	0.49999984	0.50000048	0.5000005	0.50000004	0.49999999	0.499999779	0.499999818	0.499999984	0.499999697
0.2	0.5215693	0.52151097	0.52141035	0.52135099	0.47702117	0.47186632	0.46172579	0.45673788	0.590933443	0.590685682	0.590202752	0.589971805
0.4	0.5433938	0.54316182	0.542727	0.54250628	0.44865641	0.438185	0.418006	0.40828236	0.69811051	0.696940282	0.694649876	0.693532481
0.6	0.56528285	0.56475603	0.56374521	0.56325551	0.41500305	0.39935526	0.3699049	0.35605878	0.823696938	0.820536986	0.814493466	0.811599233
0.8	0.58704862	0.58608332	0.584264	0.58339627	0.37651173	0.35617116	0.31896825	0.30195215	0.969820246	0.963191645	0.950766655	0.944938366
1.0	0.60847136	0.60693367	0.60409654	0.60277219	0.3340243	0.30986509	0.26708179	0.24814624	1.138532264	1.126496337	1.104411161	1.094260679

Table 41: Absolute errors obtained by the LeNN-WOA-NM algorithm for population densities of local prey, immigrant prey, and predator under the influence of γ_2.

t	$S(t)$				$X(t)$				$Y(t)$			
	$\gamma_2 = 0.7$	$\gamma_2 = 0.8$	$\gamma_2 = 1.0$	$\gamma_2 = 1.1$	$\gamma_2 = 0.7$	$\gamma_2 = 0.8$	$\gamma_2 = 1.0$	$\gamma_2 = 1.1$	$\gamma_2 = 0.7$	$\gamma_2 = 0.8$	$\gamma_2 = 1.0$	$\gamma_2 = 1.1$
0.0	1.26E − 09	1.32E − 10	4.79E − 10	5.02E − 10	1.48E − 11	1.87E − 10	7.79E − 10	5.53E − 10	9.67E − 10	1.58E − 09	5.89E − 10	1.90E − 09
0.2	1.31E − 09	4.54E − 11	1.43E − 10	5.38E − 10	1.04E − 10	4.83E − 12	8.26E − 10	2.09E − 10	3.34E − 11	1.58E − 09	6.20E − 10	2.77E − 10
0.4	7.22E − 10	3.13E − 12	4.65E − 10	7.47E − 11	7.86E − 10	3.53E − 10	2.65E − 09	7.30E − 10	1.16E − 09	3.29E − 10	7.51E − 11	1.87E − 09
0.6	9.27E − 10	2.41E − 12	1.09E − 10	1.51E − 10	3.37E − 12	6.92E − 10	1.12E − 09	2.57E − 10	1.29E − 09	1.11E − 09	5.35E − 10	1.29E − 09
0.8	2.37E − 09	5.90E − 11	2.34E − 11	2.37E − 10	9.11E − 10	1.53E − 10	1.69E − 10	3.50E − 11	3.98E − 10	1.50E − 09	5.10E − 10	1.19E − 10
1.0	2.91E − 09	1.44E − 10	1.20E − 12	2.88E − 10	9.82E − 10	1.61E − 10	1.58E − 09	4.63E − 11	5.13E − 10	1.68E − 09	7.98E − 10	4.24E − 10

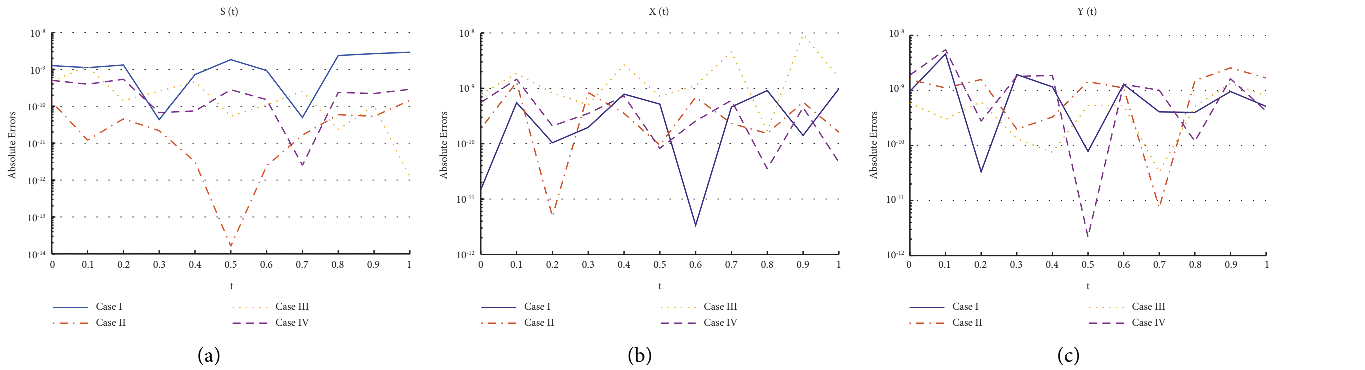


FIGURE 19: Absolute errors obtained by the LeNN-WOA-NM algorithm under the influence of variations in γ_2 on population densities of the prey-predator model. (a) Population density of local prey. (b) Population density of immigrant prey. (c) Population density of the predator.

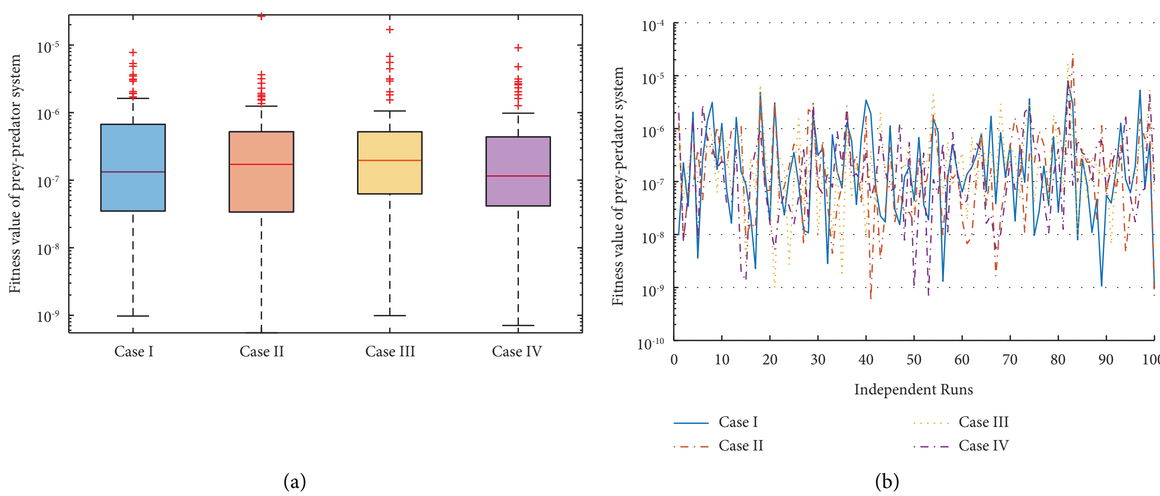


FIGURE 20: Comparison between the box plots and convergence graphs of fitness evaluation over 100 independent runs for the prey-predator model with variations in the catching rate of immigrant prey. (a) Box plots for fitness evaluation of the prey-predator model with variations in γ_2. (b) Convergence of the fitness value of the prey-predator model with variations in γ_2.

TABLE 42: Convergence analysis of population density of local prey, immigrant prey, and predator under the influence of variations in γ_2.

	Cases	FIT				MAD			TIC			ENSE		
		$\leq 10^{-6}$	$\leq 10^{-7}$	$\leq 10^{-8}$	$\leq 10^{-9}$	$\leq 10^{-4}$	$\leq 10^{-5}$	$\leq 10^{-6}$	$\leq 10^{-4}$	$\leq 10^{-5}$	$\leq 10^{-6}$	$\leq 10^{-5}$	$\leq 10^{-6}$	$\leq 10^{-7}$
$S(t)$	I	95	65	24	5	100	88	20	100	95	41	89	59	20
	II	98	64	24	4	100	92	18	100	99	45	93	60	18
	III	86	63	12	4	100	90	16	100	95	32	90	55	16
	IV	98	67	20	4	100	92	21	100	97	44	92	62	21
$X(t)$	I	99	85	52	22	100	85	20	100	92	30	94	69	32
	II	100	87	57	22	100	87	23	100	91	27	98	73	33
	III	98	83	46	13	100	82	10	100	93	16	98	71	28
	IV	99	90	54	19	100	94	17	100	97	23	99	82	44
$Y(t)$	I	98	84	44	17	100	77	11	100	92	28	100	87	57
	II	99	81	47	18	99	75	13	100	97	31	99	87	55
	III	99	84	41	15	100	75	6	100	94	26	99	89	56
	IV	99	85	50	14	100	72	9	100	94	34	100	87	55

Table 43: Statistics of global performance indices for variations in the catching rate of local prey on the prey-predator model.

Cases	$S(t)$				$X(t)$				$Y(t)$			
	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE	GFIT	GMAD	GTIC	GENSE
I	$1.98E-06$	$4.80E-05$	$2.43E-05$	$5.78E-05$	$5.99E-07$	$5.04E-05$	$3.39E-05$	$2.68E-05$	$9.28E-07$	$9.13E-05$	$3.24E-05$	$5.44E-06$
II	$2.17E-06$	$3.83E-05$	$1.95E-05$	$2.91E-05$	$5.21E-07$	$4.70E-05$	$3.30E-05$	$1.44E-05$	$1.36E-06$	$9.17E-05$	$3.25E-05$	$1.01E-05$
III	$2.16E-06$	$4.95E-05$	$2.52E-05$	$5.55E-05$	$1.01E-06$	$5.58E-05$	$4.16E-05$	$1.53E-05$	$7.10E-07$	$8.79E-05$	$3.17E-05$	$6.81E-06$
IV	$1.55E-06$	$4.01E-05$	$2.00E-05$	$4.85E-05$	$5.75E-07$	$4.49E-05$	$3.40E-05$	$7.86E-06$	$7.28E-07$	$8.59E-05$	$3.08E-05$	$5.50E-06$

Table 44: Unknown parameters obtained by the LeNN-WOA-NM algorithm for the prey-predator model under the influence of variations in γ_1.

	Index	$S(t)$			$X(t)$			$Y(t)$		
		ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
Case I	1	0.27328231	−0.0407082	−0.0036568	0.19095405	0.19619913	0.25334058	0.15448032	0.339199456	0.646260329
	2	0.00203693	0.00761269	0.24655049	0.67156041	−0.1202185	0.41029823	0.35456658	0.167129547	−0.02220108
	3	0.22854565	−0.1965182	−0.0799706	0.423423	0.03898304	−0.1160494	−0.1932211	0.327298457	0.589634109
	4	0.43068766	−0.0047135	−0.0067749	0.7614451	0.24436812	0.10386	0.45804558	−0.10504105	0.096632479
	5	0.42381358	−0.1502652	0.51827442	0.10757996	0.24696908	0.11302049	−0.0490828	0.054497193	0.541464355
	6	0.03064345	0.387909	−0.0908338	0.20021796	−0.1093334	0.49803749	0.08502235	0.292312181	0.168951799
	7	−0.0256272	0.18134944	0.04903339	−0.1439181	0.01678089	0.01522287	0.12228645	0.040610745	0.504241595
	8	−0.0029234	0.05369165	0.16118629	0.09238371	0.35203171	−0.0742458	0.58296355	0.326394622	0.311232469
	9	0.01433551	−0.0062826	0.11461784	0.00070281	0.08170655	0.27442616	−0.0624975	0.046728012	−0.00253833
	10	0.52132655	−0.0745824	0.32373133	0.31625889	0.06469584	0.35518716	0.43265994	0.001675421	−0.07508196
	11	0.04700892	−0.1431138	0.19096034	0.61086683	0.05879562	0.07145247	0.31602732	0.294568546	0.005492958
Case II	1	0.22888098	0.15662727	0.72118398	0.66396301	0.48400554	−0.0014987	0.2930431	−0.36246252	−0.19827556
	2	0.32039169	0.21349716	0.26632024	−0.1848203	−0.0673308	−0.0648246	0.13125299	0.041657028	0.029954374
	3	−0.0829388	0.03094832	0.00303522	0.62951247	−0.1925438	0.24736655	−0.1963121	0.431531435	0.708095767
	4	−0.1658907	−0.1447463	−0.6955688	−0.232498	−0.2616095	0.27025227	0.22782911	−0.28937041	−0.12994496
	5	−0.0174758	0.23215182	0.04290016	0.12334482	−0.2128439	−0.0360696	−0.1459274	0.143383282	0.127654212
	6	−0.0712296	−0.3626918	−0.0348668	0.2600438	0.13168403	0.45131095	0.73463526	−0.09109894	0.100058657
	7	−0.0244551	−0.0112202	−0.1246274	−0.3445732	0.31169855	−0.0680798	0.14809738	0.029526212	0.176039899
	8	−0.2963948	0.27684365	0.02435043	−0.1570692	−0.1058024	0.01048308	−0.0287022	0.052204737	0.500290252
	9	−0.0250215	−0.1024261	0.05651485	0.61926963	−0.205578	0.43411127	−0.3748423	0.146360873	−0.41978655
	10	−0.2847311	0.13336075	−0.0903463	−0.1464593	−0.3554768	−0.0475072	0.16657978	−0.04184543	−0.04410262
	11	−0.0012914	0.02947063	−0.2027891	−0.0119255	0.47238101	0.09448735	0.42979806	−0.05005385	−0.00793862
Case III	1	0.17136911	−0.6220119	−0.0659451	0.16001338	0.06604026	−0.6259462	0.41743442	0.34745146	0.123402726
	2	0.13248093	0.22407198	0.16846536	−0.0174004	0.46048206	0.16172196	0.6363567	0.027892663	0.349217115
	3	−0.029666	0.0085174	−0.4187633	−0.0573402	0.29673852	0.13113638	−0.7934564	−0.18170045	0.087257639
	4	0.50916399	0.04634821	−0.1437019	0.08923057	−0.2567021	−0.2934746	0.26333479	−0.21308607	−0.02505351
	5	0.28783823	−0.2662163	0.0766028	0.09914485	0.57665811	0.04300083	0.20662283	−0.48985711	−1.09740743
	6	0.08917337	−0.1050009	0.23818256	0.02035607	−0.1072192	0.2659477	−0.0029764	−0.13148311	0.313665366
	7	−0.7503685	−0.1704471	0.18719634	−0.0233035	−0.0088938	0.26343414	−1.0737247	−0.19018699	0.280100524
	8	0.34230089	0.00558946	−0.1427919	−0.2957063	−0.6490481	0.10546459	0.27322547	0.041606539	−0.10278157
	9	0.25879736	0.21953927	0.11052919	0.00206003	−0.0680416	−0.2189914	0.0378666	0.055450179	−0.26693807
	10	0.11482869	0.04256621	0.24740165	−1.0210288	0.10826852	0.1081171	0.28972162	0.107963625	−0.01416421
	11	0.21645891	0.26405307	−0.2682646	−0.0251217	0.2673504	−0.2407415	−0.3201356	−0.12644907	0.363928738
Case IV	1	0.16321044	1.00101739	0.44610851	0.19117524	−0.1869131	0.03284449	0.48709406	0.492839005	0.068811431
	2	0.05034014	0.27448502	0.05595427	0.34759862	−0.0428559	−0.4656632	0.41211291	0.550350162	0.261254457
	3	0.02309873	0.00700133	−0.3300406	−0.032857	−0.2059139	−0.1572132	0.43069278	0.119708938	0.13204548
	4	0.60643185	−0.1677085	0.01351505	0.01269345	0.07414495	−0.2533687	−0.1990881	0.074189084	−0.00874219
	5	0.06424168	−0.0394268	0.04062341	0.10346071	0.24294981	−0.1406991	0.1530591	0.00796601	0.142324569
	6	−0.1607398	−0.0067892	−0.1142057	−0.3542569	−0.317926	0.89646875	0.19029585	−0.0777253	0.138215439
	7	−0.0249375	−0.142179	0.21181449	0.00745531	−0.0807929	−0.0606121	−0.3138917	−0.04179489	−0.08395995
	8	0.01271854	0.07904033	−0.2032421	0.30678256	−0.0970233	−0.0973761	−0.2681465	−0.26023333	−0.05748785
	9	0.15729166	0.02004037	0.08441775	0.50899351	−0.0107112	0.1654774	0.05321702	−0.55038421	−0.01142195
	10	0.45526814	0.30848224	0.03885589	−0.067078	0.03709569	−0.1103419	0.00132508	−0.0601008	−0.00809993
	11	0.00144682	−0.0160469	−0.1485655	0.07678165	−0.0117808	−0.039538	−0.1553724	0.090185082	−0.05537005

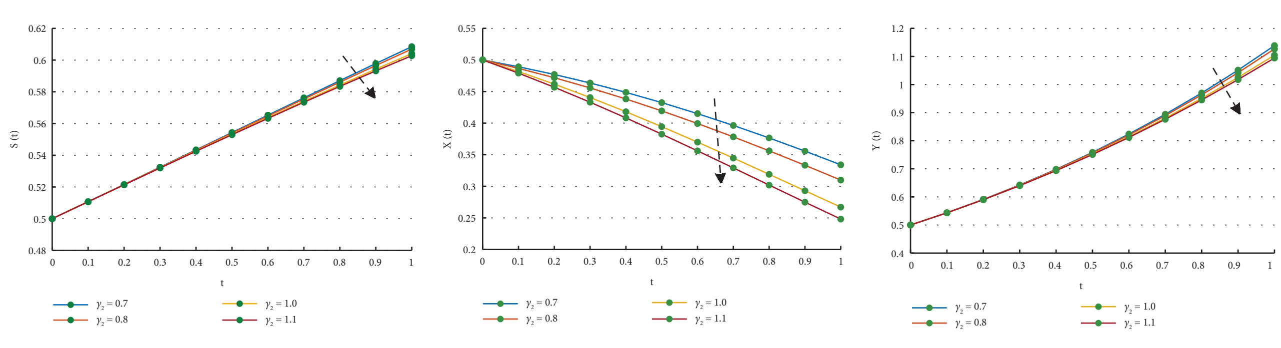


Figure 21: Comparison between solutions obtained by the LeNN-WOA-NM algorithm under the influence of variations in γ_2 on population densities of the prey-predator model. (a) Approximate solutions for population density of local prey. (b) Approximate solutions for population density of immigrant prey. (c) Approximate solutions for population density of the predator.

correctness of the proposed algorithm to tackle real-world problems. Trained neurons in LeNN are shown in Table 44. From Figure 21, the following conclusions can be drawn:

(i) Population densities of local prey, immigrant prey, and the predator vary inversely with variation in the catching rate of immigrant prey γ_2.

8. Conclusion

In this paper, we have analyzed the system of the singular differential equation (SDE) representing the phenomena of the prey-predator model with immigrant prey. Generally, solving SDE is one of the challenging tasks, and therefore, we have designed a novel soft computing technique known as a LeNN-WOA-NM algorithm. Weighted Legendre polynomials are used to model approximate series solutions for the prey-predator model with variations in various coefficients, including the growth rate of local and immigrant prey (α_1, α_2), force interaction between local and immigrant prey (β_1, β_2), and the catching rate of local and immigrant prey γ_1, γ_2. We summarize our findings as follows:

(i) Variations in $\alpha_1, \alpha_2, \beta_1$, and β_2 has a direct impact on population density of local prey $S(t)$, while population density of local prey is inversely affected by variations in γ_1 and γ_2.

(ii) Variations in $\alpha_1, \alpha_2, \beta_2$, and γ_2 has an inverse impact on population density of immigrant prey $X(t)$, while population density of immigrant prey is directly affected by variations in β_1 and γ_2.

(iii) Variations in α_1, α_2 and β_1 has a direct impact on population density of the predator $S(t)$, while population density of the predator is inversely affected by variations in γ_1, β_2, and γ_2.

(iv) Approximate solutions obtained by the LeNN-WOA-NM algorithm are compared with those obtained by the homotopy perturbation method and MATLAB solver ode45. The results show the dominance of the proposed technique.

(v) Lower absolute errors in our solutions and convergence analysis of fitness evaluation, MAD, TIC, and ENSE show the accuracy and robustness of the proposed algorithm for obtaining solutions to real-world problems.

In future, this approach can be utilized to solve the complex nonlinear systems of fractional differential equations characterizing real-world problems, for instance, anomalous diffusion of contaminant from the fracture into the porous rock matrix, microbial survival and growth curves, smoking dynamics, parametric identification of Hammerstein systems with time delay, and asymmetric dead zones.

Abbreviations

LeNN: Legendre neural network
NM: Nelder–Mead
MAD: Mean absolute diviation
TIC: Theil's inequality coefficient
NSE: Nash–Sutcliffe efficiency
ENSE: Error in Nash–Sutcliffe efficiency
ANNs: Artificial neural networks
WOA: Whale optimization algorithm
α_1: Intrinsic growth rate of local prey
α_2: Increasing rate of immigrant prey
β_1, β_2: Force of interaction between local prey and immigrant prey
k_1: Carrying capacity of local prey
k_2: Carrying capacity of immigrant prey
c_1, c_2: Intrinsic growth rate of the predator population
μ_1, μ_2: Suffering loss of the predator population
γ_1: Catching rate of local prey
γ_2: Catching rate of immigrant prey.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding this study.

References

[1] X. Liu and L. Chen, "Complex dynamics of Holling type II Lotka–Volterra predator–prey system with impulsive perturbations on the predator," *Chaos, Solitons & Fractals*, vol. 16, no. 2, pp. 311–320, 2003.
[2] C. S. Holling, "The functional response of invertebrate predators to prey density," *Memoirs of the Entomological Society of Canada*, vol. 98, no. S48, pp. 5–86, 1966.
[3] M. Danca, S. Codreanu, and B. Bako, "Detailed analysis of a nonlinear prey-predator model," *Journal of Biological Physics*, vol. 23, no. 1, pp. 11–20, 1997.
[4] F. J. Solis, "Self-limitation in a discrete predator–prey model," *Mathematical and Computer Modelling*, vol. 48, no. 1-2, pp. 191–196, 2008.
[5] M. Fan and K. Wang, "Periodicity in a delayed ratio-dependent predator–prey system," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 1, pp. 179–190, 2001.
[6] H. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcal Dekker, New York, USA, 1980.
[7] H. Freedman and R. Mathsen, "Persistence in predator-prey systems with ratio-dependent predator influence," *Bulletin of Mathematical Biology*, vol. 55, no. 4, pp. 817–827, 1993.
[8] Y. Cai, M. Banerjee, Y. Kang, and W. Wang, "Spatiotemporal complexity in a predator–prey model with weak allee effects," *Mathematical Biosciences and Engineering*, vol. 11, no. 6, pp. 1247–1274, 2014.
[9] M. Bandyopadhyay and S. Banerjee, "A stage-structured prey–predator model with discrete time delay," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1385–1398, 2006.
[10] Z. Jing and J. Yang, "Bifurcation and chaos in discrete-time predator–prey system," *Chaos, Solitons & Fractals*, vol. 27, no. 1, pp. 259–277, 2006.

[11] X. Liu and D. Xiao, "Complex dynamic behaviors of a discrete-time predator–prey system," *Chaos, Solitons & Fractals*, vol. 32, no. 1, pp. 80–94, 2007.

[12] Y. Huang, F. Chen, and L. Zhong, "Stability analysis of a prey–predator model with Holling type III response function incorporating a prey refuge," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 672–683, 2006.

[13] Q. Din, "Complexity and chaos control in a discrete-time prey-predator model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 49, pp. 113–134, 2017.

[14] V. Weide, M. C. Varriale, and F. M. Hilker, "Hydra effect and paradox of enrichment in discrete-time predator-prey models," *Mathematical Biosciences*, vol. 310, pp. 120–127, 2019.

[15] Z. Gong, R. Hamazaki, and M. Ueda, "Discrete time-crystalline order in cavity and circuit QED systems," *Physical Review Letters*, vol. 120, no. 4, Article ID 040404, 2018.

[16] T. Huang, H. Zhang, H. Yang, N. Wang, and F. Zhang, "Complex patterns in a space-and time-discrete predator-prey model with Beddington-DeAngelis functional response," *Communications in Nonlinear Science and Numerical Simulation*, vol. 43, pp. 182–199, 2017.

[17] K. P. Hadeler and H. I. Freedman, "Predator-prey populations with parasitic infection," *Journal of Mathematical Biology*, vol. 27, no. 6, pp. 609–631, 1989.

[18] B. Ghanbari and D. Kumar, "Numerical solution of predator-prey model with Beddington-DeAngelis functional response and fractional derivatives with Mittag-Leffler kernel," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 29, no. 6, Article ID 063103, 2019.

[19] M. Perc and A. Szolnoki, "Noise-guided evolution within cyclical interactions," *New Journal of Physics*, vol. 9, no. 8, p. 267, 2007.

[20] G. Q. Sun, Z. Jin, L. Li, and Q. X. Liu, "The role of noise in a predator–prey model with Allee effect," *Journal of Biological Physics*, vol. 35, no. 2, pp. 185–196, 2009.

[21] J. Hofbauer, K. Sigmund, and others, *Evolutionary Games and Population Dynamics*, Cambridge University Press, USA, 1998.

[22] A. Kilbas and J. Trujillo, "Differential equations of fractional order: methods, results and problems. II," *Applicable Analysis*, vol. 81, no. 2, pp. 435–493, 2002.

[23] J. H. He, "A tutorial review on fractal spacetime and fractional calculus," *International Journal of Theoretical Physics*, vol. 53, no. 11, pp. 3698–3718, 2014.

[24] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, and others, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, USA, 2020.

[25] S. Paul, S. P. Mondal, and P. Bhattacharya, "Numerical solution of Lotka Volterra prey predator model by using Runge–Kutta–Fehlberg method and Laplace Adomian decomposition method," *Alexandria Engineering Journal*, vol. 55, no. 1, pp. 613–617, 2016.

[26] S. S. Ray, "A new coupled fractional reduced differential transform method for the numerical solution of fractional predator-prey system," *Computer Modeling in Engineering and Sciences*, vol. 105, pp. 231–249, 2015.

[27] Z. Anastassi and T. Simos, "An optimized Runge–Kutta method for the solution of orbital problems," *Journal of Computational and Applied Mathematics*, vol. 175, pp. 1–9, 2005.

[28] M. R. Garvie, J. Burkardt, and J. Morgan, "Simple finite element methods for approximating predator–prey dynamics in two dimensions using Matlab," *Bulletin of Mathematical Biology*, vol. 77, no. 3, pp. 548–578, 2015.

[29] N. Bildik and S. Deniz, "The use of Sumudu decomposition method for solving predator-prey systems," *Math. Sci. Lett*, vol. 5, no. 3, pp. 285–289, 2016.

[30] I. Gad and P. Novati, "The solution of fractional order epidemic model by implicit Adams methods," *Applied Mathematical Modelling*, vol. 43, pp. 78–84, 2016.

[31] I. Bashkirtseva and L. Ryashko, "Analysis of the noise-induced regimes in Ricker population model with Allee effect via confidence domains technique," *BioMed Research International*, vol. 2014, Article ID 346239, 7 pages, 2014.

[32] J. S. Yu and J. J. Yu, "Homotopy analysis method for a prey-predator system with holling IV functional response," *Applied Mechanics and Materials*, vol. 687-691, pp. 1286–1291, 2014.

[33] O. Dababneh, T. Kipouros, and J. F. Whidborne, "Application of an efficient gradient-based optimization strategy for aircraft wing structures," *Aerospace*, vol. 5, no. 1, p. 3, 2018.

[34] N. A. Khan, M. Sulaiman, A. J. Aljohani, M. A. Bakar, and others, "Mathematical models of CBSC over wireless channels and their analysis by using the LeNN-WOA-NM algorithm," *Engineering Applications of Artificial Intelligence*, vol. 107, Article ID 104537, 2022.

[35] S. B. Alagarsamy and K. Murugan, "Multimodal of ear and face biometric recognition using adaptive approach Runge–Kutta threshold segmentation and classifier with score level fusion," *Wireless Personal Communications*, vol. 124, no. 2, pp. 1061–1080, 2022.

[36] N. A. Khan, M. Sulaiman, C. A. Tavera Romero, and F. K. Alarfaj, "Numerical analysis of electrohydrodynamic flow in a circular cylindrical conduit by using neuro evolutionary technique," *Energies*, vol. 14, no. 22, p. 7774, 2021.

[37] N. A. Khan, F. S. Alshammari, C. A. T. Romero, M. Sulaiman, and G. Laouini, "Mathematical analysis of reaction–diffusion equations modeling the michaelis–menten kinetics in a micro-disk biosensor," *Molecules*, vol. 26, no. 23, p. 7310, 2021.

[38] N. A. Khan, F. S. Alshammari, C. A. Tavera Romero, M. Sulaiman, and S. Mirjalili, "An optimistic solver for the mathematical model of the flow of Johnson segalman fluid on the surface of an infinitely long vertical cylinder," *Materials*, vol. 14, no. 24, p. 7798, 2021.

[39] Y. Cui, Y. Hong, N. A. Khan, and M. Sulaiman, "Application of soft computing paradigm to large deformation analysis of cantilever beam under point load," *Complexity*, vol. 2021, pp. 2021–19, 2021.

[40] S. Mirjalili and A. Lewis, "The whale optimization algorithm," *Advances in Engineering Software*, vol. 95, pp. 51–67, 2016.

[41] S. Singer and J. Nelder, "Nelder-Mead algorithm," *Scholarpedia*, vol. 4, no. 7, p. 2928, 2009.

[42] R. Jovanovic, S. Kais, and F. H. Alharbi, *Cuckoo Search Inspired Hybridization of the Nelder-mead Simplex Algorithm Applied to Optimization of Photovoltaic Cells*, arXiv preprint arXiv:1411.0217, 2014.

[43] N. A. Khan, M. Sulaiman, A. J. Aljohani, P. Kumam, and H. Alrabaiah, *Analysis of Multi-phase Flow through Porous media for Imbibition Phenomena by Using the LeNN-WOA-NM Algorithm*, IEEE Access, USA, 2020.

[44] K. Renganathan, V. Ananthaswamy, and S. Narmatha, "Mathematical analysis of prey predator system with immigrant prey using a new approach to Homotopy perturbation method," *Materials Today Proceedings*, vol. 37, pp. 1183–1189, 2021.