

Research Article

Conditions for Oscillation of a Neutral Differential Equation

Weiping Yan and Jurang Yan

School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China

Correspondence should be addressed to Jurang Yan, jryan@sxu.edu.cn

Received 18 November 2009; Revised 2 April 2010; Accepted 13 May 2010

Academic Editor: Josef Diblík

Copyright © 2010 W. Yan and J. Yan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

For a neutral differential equation with positive and changeable sign coefficients $[x(t) - a(t)x(\delta(t))] + p(t)F(x(\tau(t))) - q(t)G(x(\sigma(t))) = 0$, oscillation criteria are established, where $q(t)$ is not required as nonnegative. Several new results are obtained.

1. Introduction

Consider the delay neutral differential equation with positive and changeable sign coefficients

$$[x(t) - a(t)x(\delta(t))] + p(t)F(x(\tau(t))) - q(t)G(x(\sigma(t))) = 0, \quad t \geq t_0, \quad (1.1)$$

where the following conditions are assumed to hold throughout this paper:

- (A₁) $p, q, a \in C([t_0, \infty), R)$, $p \geq 0$, $a \geq 0$;
- (A₁) $\delta \in C([t_0, \infty), R)$, $\tau, \sigma \in C^1([t_0, \infty), R)$, and $\tau'(t) \geq 0$, $\sigma'(t) \geq 0$, $\tau(t) \leq \sigma(t) \leq t$, $\delta(t) \leq t$, $\lim_{t \rightarrow \infty} \tau(t) = \infty$, and $\lim_{t \rightarrow \infty} \delta(t) = \infty$;
- (A₃) $F, G \in C(R, R)$ and $xF(x) > 0$, $xG(x) > 0$, $|F(x)| \geq |x|$, $|G(x)| \leq |x|$ for all $x \neq 0$.

Recently, oscillation of first-order differential equations and difference equations with positive and negative coefficients has been investigated by many authors. Several interesting results have been obtained. We refer to [1–12] and the references cited therein. However, to the best of our knowledge, up to now, there are not works on oscillation of solutions of (1.1) with $q(t)$ able to change sign. The purpose of this paper is to study oscillation properties of

(1.1) by some new technique. Our results improve and extend several known results in the literature. In particular, our results can be applied to linear neutral differential equation

$$[x(t) - a(t)x(\delta(t))] + p(t)x(\tau(t)) - q(t)x(\sigma(t)) = 0. \quad (1.1)'$$

By a solution of (1.1) we mean a function $x(t) \in C([\bar{t}_0, \infty), R)$ for some $\bar{t}_0 \geq t_0$ such that $x(t) - a(t)x(\delta(t))$ is continuously differentiable on $[\bar{t}_0, \infty)$ and satisfies (1.1) for $t \geq \bar{t}$, where $\bar{t} = \inf_{t \geq \bar{t}_0} \{\delta(t), \tau(t)\}$. As is customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

In the sequel, unless otherwise specified, when we write a functional inequality on t it will hold for all sufficiently large t .

First, we establish the following lemma. It extends and improves in [3, Lemma 3.7.1], [4, Lemma 2.6.1], [7, Lemma 2.1], and [9, Lemma 1].

Lemma 1.1. *Assume that*

$$p(t) - \frac{q_+(\sigma^{-1}(\tau(t)))\tau'(t)}{\sigma'(\sigma^{-1}(\tau(t)))} \geq 0 \text{ is not identically zero,} \quad (1.2)$$

$$a(t) + \int_{\tau(t)}^{\sigma(t)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))} ds \leq 1, \quad a(t) \neq 1, \quad (1.3)$$

where $q_+(t) = \max\{q(t), 0\}$, and $\sigma^{-1}(t)$ is the inverse function of $\sigma(t)$. Let $x(t)$ be a nonoscillatory solution of (1.1) and

$$y(t) = x(t) - a(t)x(\delta(t)) - \int_{\tau(t)}^{\sigma(t)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))} x(s) ds. \quad (1.4)$$

Then

$$x(t)y(t) > 0, \quad y(t)y'(t) \leq 0. \quad (1.5)$$

Proof. Let $x(t)$ be an eventually positive solution. The case when $x(t)$ is an eventually negative solution is similar and its proof is omitted. Thus we have

$$x(\tau(t)) > 0, \quad x(\sigma(t)) > 0, \quad x(\delta(t)) > 0. \quad (1.6)$$

By (A₃), (1.4), and (1.2), we obtain

$$\begin{aligned} y'(t) &= -p(t)F(x(\tau(t))) + q(t)G(x(\sigma(t))) - q_+(t)x(\sigma(t)) + \frac{q_+(\sigma^{-1}(\tau(t)))\tau'(t)}{\sigma'(\sigma^{-1}(\tau(t)))} x(\tau(t)) \\ &\leq - \left[p(t) - \frac{q_+(\sigma^{-1}(\tau(t)))\tau'(t)}{\sigma'(\sigma^{-1}(\tau(t)))} \right] x(\tau(t)), \end{aligned} \quad (1.7)$$

which implies that $y(t)$ is decreasing. Hence, if (1.5) does not hold, then eventually $y(t) < 0$, and there exist $T \geq t_0$ and positive constant α such that $y(t) \leq -\alpha < 0$ for all $t \geq T$, that is,

$$x(t) \leq -\alpha + a(t)x(\delta(t)) + \int_{\tau(t)}^{\sigma(t)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))} x(s) \, ds, \quad t \geq T. \quad (1.8)$$

We consider the following two possible cases.

The first case. $x(t)$ is unbounded, that is, $\limsup_{t \rightarrow \infty} x(t) = \infty$. Thus there exists a sequence of points $\{s_n\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} x(s_n) = \infty$ and $x(s_n) = \max\{x(t) : T \leq t \leq s_n, n = 1, 2, \dots\}$. From (1.8) we have

$$\begin{aligned} x(s_n) &\leq -\alpha + a(s_n)x(\delta(s_n)) + \int_{\tau(s_n)}^{\sigma(s_n)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))} x(s) \, ds \\ &\leq -\alpha + x(s_n) \left[a(s_n) + \int_{\tau(s_n)}^{\sigma(s_n)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))} \, ds \right] \\ &\leq -\alpha + x(s_n). \end{aligned} \quad (1.9)$$

This is a contradiction.

The second case. $x(t)$ is bounded, that is, $\limsup_{t \rightarrow \infty} x(t) < \infty$. Choose a sequence of points $\{\bar{s}_n\}_{n=1}^{\infty}$ such that $\bar{s}_n \rightarrow \infty$ and $x(\bar{s}_n) \rightarrow l$ as $n \rightarrow \infty$. Let $\xi(t) = \min\{\delta(t), \tau(t)\}$, $\eta(t) = \max\{\delta(t), \sigma(t)\}$, and $x(t_n) = \max\{x(s) : \xi(\bar{s}_n) \leq s \leq \eta(\bar{s}_n), t_n \in [\xi(\bar{s}_n), \eta(\bar{s}_n)], n = 1, 2, \dots\}$. Then $\limsup_{n \rightarrow \infty} x(t_n) \leq l$. Thus, in view of (1.8) we obtain

$$x(\bar{s}_n) \leq -\alpha + x(t_n) \left[a(\bar{s}_n) + \int_{\tau(\bar{s}_n)}^{\sigma(\bar{s}_n)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))} \, ds \right] \leq -\alpha + x(t_n). \quad (1.10)$$

Therefore $l \leq -\alpha + l$, which is also a contradiction. Hence (1.5) holds. The proof of Lemma 1.1 is complete. \square

2. Main Results

In this section, we will prove a comparison theorem on oscillation for (1.1). For convenience of discussions, in the rest of this paper we will use the following notations:

$$\begin{aligned} P(t) &:= p(t) - \frac{q_+(\sigma^{-1}(\tau(t)))\tau'(t)}{\sigma'(\sigma^{-1}(\tau(t)))}, & Q(t) &:= \frac{q_+(\sigma^{-1}(t))}{\sigma'(\sigma^{-1}(t))}, & \delta(t) &\text{ is increasing for } t \geq t_0, \\ \delta^0(t) &= t, & \delta^{k+1}(t) &= \delta(\delta^k(t)), & t &\geq \delta^{-(k+1)}(t_0), \quad k = 1, 2, \dots, \end{aligned} \quad (2.1)$$

where $\delta^{-k}(t)$ is the inverse function of $\delta^k(t)$.

The following comparison theorem is the main result of this paper.

Theorem 2.1. Assume that (1.2) and (1.3) hold and there exists a nonnegative integer m such that all solutions of the following delay differential equation:

$$y'(t) + P(t) \left\{ y(\tau(t)) + a(\tau(t)) \left[y(\delta(\tau(t))) + \sum_{k=0}^m \prod_{i=0}^k a(\delta^{i+1}(\tau(t))) y(\delta^{k+2}(\tau(t))) \right] \right. \\ \left. + \int_{\tau^2(t)}^{\sigma(\tau(t))} Q(s) \left[y(s) + \sum_{k=0}^m \prod_{i=0}^k a(\delta^i(s)) y(\delta^{k+1}(s)) \right] ds \right\} = 0 \quad (2.2)$$

are oscillatory. Then all solutions of (1.1) are also oscillatory.

Proof. Suppose that $x(t)$ is an eventually positive solution of (1.1). The proof of the case where $x(t)$ is eventually negative is similar and will be omitted. By Lemma 1.1, we have

$$y(t) > 0, \quad y'(t) \leq 0, \quad (2.3)$$

where $y(t)$ is given by (1.4). Thus

$$x(t) \geq y(t) + a(t)x(\delta(t)) \geq y(t) + a(t) \left[y(\delta(t)) + a(\delta(t))x(\delta^2(t)) \right]. \quad (2.4)$$

By induction, we see that

$$x(t) \geq y(t) + \sum_{k=0}^m \prod_{i=0}^k a(\delta^i(t)) y(\delta^{k+1}(t)). \quad (2.5)$$

From (1.4), (2.3), and (2.5), we obtain

$$x(t) = y(t) + a(t)x(\delta(t)) + \int_{\tau(t)}^{\sigma(t)} Q(s)x(s)ds \\ \geq y(t) + a(t) \left[y(\delta(t)) + \sum_{k=0}^m \prod_{i=0}^k a(\delta^{i+1}(t)) y(\delta^{k+2}(t)) \right] \\ + \int_{\tau(t)}^{\sigma(t)} Q(s) \left[y(s) + \sum_{k=0}^m \prod_{i=0}^k a(\delta^i(s)) y(\delta^{k+1}(s)) \right] ds. \quad (2.6)$$

It follows from (1.7) that

$$y'(t) \leq -P(t) \left\{ y(\tau(t)) + a(\tau(t)) \left[y(\delta(\tau(t))) + \sum_{k=0}^m \prod_{i=0}^k a(\delta^{i+1}(\tau(t))) y(\delta^{k+2}(\tau(t))) \right] \right. \\ \left. + \int_{\tau^2(t)}^{\sigma(\tau(t))} Q(s) \left[y(s) + \sum_{k=0}^m \prod_{i=0}^k a(\delta^i(s)) y(\delta^{k+1}(s)) \right] ds \right\}. \quad (2.7)$$

By a well-known result (see, e.g., [4, Corollary 3.2.2]) we can conclude that (2.2) has also an eventually positive solution. This is a contradiction. The proof of Theorem 2.1 is complete. \square

Following the proof of Theorem 2.1 and taking into account in (2.7) the positivity of the functions P and Q and the properties of the delay functions δ and τ , we state as corollary the following claim.

Corollary 2.2. *Assume that (1.2) and (1.3) hold and there exists a nonnegative integer m such that all solutions of the delay differential equation*

$$y'(t) + P(t) \left\{ 1 + a(\tau(t)) \left[1 + \sum_{k=0}^m \prod_{i=0}^k a(\delta^{i+1}(\tau(t))) \right] + \int_{\tau^2(t)}^{\sigma(\tau(t))} Q(s) \left[1 + \sum_{k=0}^m \prod_{i=0}^k a(\delta^i(s)) \right] ds \right\} y(\tau(t)) = 0. \quad (2.8)$$

are oscillatory. Then all solutions of (1.1) are also oscillatory.

Corollary 2.3. *Consider (1.1) with $q(t) \equiv 0$. Assume that there exists a nonnegative integer m such that all solutions of the delay differential equation*

$$y'(t) + P(t) \left\{ 1 + a(\tau(t)) \left[1 + \sum_{k=0}^m \prod_{i=0}^k a(\delta^{i+1}(\tau(t))) \right] \right\} y(\tau(t)) = 0 \quad (2.9)$$

are oscillatory. Then all solutions of (1.1) are also oscillatory.

Remark 2.4. Corollaries 2.2 and 2.3 extend and improve [3, Theorems 3.7.1 and 3.2.1].

3. Explicit Oscillation Conditions

In this section, we will give several explicit oscillation conditions for (1.1). Let

$$R_m(t) := P(t) \left\{ 1 + a(\tau(t)) \left[1 + \sum_{k=0}^m \prod_{i=0}^k a(\delta^{i+1}(\tau(t))) \right] + \int_{\tau(t)}^{\sigma(t)} Q(s) \left[1 + \sum_{k=0}^m \prod_{i=0}^k a(\delta^i(s)) \right] ds \right\},$$

$$I_n(R_m, \tau(t)) := \int_{\tau(t)}^t R_m(s_1) \int_{\tau(s_1)}^{s_1} R_m(s_2) \int_{\tau(s_{n-1})}^{s_{n-1}} R_m(s_n) ds_n ds_{n-1} \cdots ds_2 ds_1, \quad (3.1)$$

where $n \geq 1$ and $s_0 := t$.

By [11, Corollary 2.1 and Theorem 1] with a well-known oscillation criterion for first-order linear delay differential equations, we have the following result.

Theorem 3.1. Assume that (1.2) and (1.3) hold and there exists a nonnegative integer m and $n \geq 1$ such that

$$\liminf_{t \rightarrow \infty} I_n(R_m, \tau(t)) > \frac{1}{e^n}, \quad (3.2)$$

or there exists a nonnegative integer m such that

$$\limsup_{t \rightarrow \infty} \int_{\tau(t)}^t R_m(s) ds > 1. \quad (3.3)$$

Then all solutions of (1.1) are oscillatory.

Remark 3.2. Theorem 3.1 extends and improves [4, Theorem 2.6.1], [2, Theorem 3], and the relative results in [12].

Consider the autonomous neutral differential equation

$$(x(t) - ax(t - \delta))' + px(t - \tau) - qx(t - \sigma) = 0, \quad t \geq 0, \quad (3.4)$$

where p, δ, τ , and σ are positive constants, a and q are real number, and

$$\tau > \sigma, \quad 0 \leq a + q_+(\tau - \sigma) \leq 1, \quad p - q_+ > 0, \quad q_+ = \max\{q, 0\}. \quad (3.5)$$

Theorem 3.3. Assume that (3.5) holds and there exists a nonnegative integer m satisfying

$$(p - q_+) [1 + a + q_+(\tau - \sigma)] \left(1 + \sum_{k=0}^m a^{k+1} \right) \tau > \frac{1}{e}. \quad (3.6)$$

Then all solutions of (3.4) are oscillatory.

Proof. By (3.2), we find that for (3.4)

$$I_n(R_m, \tau) = (p - q_+)^n \left[1 + a \left(1 + \sum_{k=0}^m a^{k+1} \right) + q_+(\tau - \sigma) \left(1 + \sum_{k=0}^m a^{k+1} \right) \right]^n \tau^n > \frac{1}{e^n}, \quad (3.7)$$

which implies that (3.6) holds. By Theorem 3.1, (3.6) leads to that all solutions of (3.4) are oscillatory. The proof of Theorem 3.3 is complete. \square

Corollary 3.4. Assume that (3.5) with $0 \leq a < 1$ holds and

$$(p - q_+) [1 + a + q_+(\tau - \sigma)] \frac{\tau}{1 - a} > \frac{1}{e}. \quad (3.8)$$

Then all solutions of (3.4) are oscillatory.

Remark 3.5. When $a = 0$, (3.8) reduces to

$$(p - q_+) [1 + q_+(\tau - \sigma)] \tau > \frac{1}{e}. \quad (3.9)$$

Equation (3.9) improves conditions (2) and (4) in [1, Corollary 2.3] where the following two oscillation criteria for the solutions of (3.4) with $a = 0$ and $p > q > 0$ are obtained:

$$\begin{aligned} (p - q) [\tau + \sigma q(\tau - \sigma)] &> \frac{1}{e}, \\ (p - q) [1 + q(\tau - \sigma)] \sigma &> \frac{1}{e}. \end{aligned} \quad (3.10)$$

Obviously, since $\tau > \sigma$, it follows that (3.9) is respectively better than (3.10).

Acknowledgments

The authors would like to thank the editor and the reviewers for their constructive comments and suggestions to improve the paper. This work was supported by Tianyuan Mathematics Fund of China (no. 10826080) and Youth Science Foundation of Shanxi Province (no. 2009021001-1).

References

- [1] L. Bereznansky, Y. Domshlak, and E. Braverman, "On oscillation properties of delay differential equations with positive and negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 1, pp. 81–101, 2002.
- [2] Q. Chuanxi and G. Ladas, "Oscillations of neutral differential equations with variable coefficients," *Applicable Analysis*, vol. 32, no. 3-4, pp. 215–228, 1989.
- [3] L. H. Erbe, Q. Kong, and B. G. Zhang, *Oscillation Theory for Functional-Differential Equations*, vol. 190 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1995.
- [4] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford Mathematical Monographs, Clarendon Press, Oxford, UK, 1991.
- [5] H. A. El-Morshedy, "New oscillation criteria for second order linear difference equations with positive and negative coefficients," *Computers & Mathematics with Applications*, vol. 58, no. 10, pp. 1988–1997, 2009.
- [6] Ö. Öcalan, "Oscillation of neutral differential equation with positive and negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 644–654, 2007.
- [7] J. H. Shen and L. Debnath, "Oscillations of solutions of neutral differential equations with positive and negative coefficients," *Applied Mathematics Letters*, vol. 14, no. 6, pp. 775–781, 2001.
- [8] J. S. Yu, "Neutral time-delay differential equations with positive and negative coefficients," *Acta Mathematica Sinica*, vol. 34, no. 4, pp. 517–523, 1991 (Chinese).
- [9] J. S. Yu and J. Yan, "Oscillation in first order neutral differential equations with "integrally small" coefficients," *Journal of Mathematical Analysis and Applications*, vol. 187, no. 2, pp. 361–370, 1994.
- [10] X. Zhang and J. Yan, "Oscillation criteria for first order neutral differential equations with positive and negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 253, no. 1, pp. 204–214, 2001.
- [11] B. Li, "Multiple integral average conditions for oscillation of delay differential equations," *Journal of Mathematical Analysis and Applications*, vol. 219, no. 1, pp. 165–178, 1998.
- [12] Y. G. Sficas and I. P. Stavroulakis, "Oscillation criteria for first-order delay equations," *The Bulletin of the London Mathematical Society*, vol. 35, no. 2, pp. 239–246, 2003.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

