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Research Article

Numerical Analysis of a Distributed Optimal Control Problem Governed by an Elliptic Variational Inequality

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The objective of this work is to make the numerical analysis, through the finite element method with Lagrange's triangles of type 1, of a continuous optimal control problem governed by an elliptic variational inequality where the control variable is the internal energy g. The existence and uniqueness of this continuous optimal control problem and its associated state system were proved previously. In this paper, we discretize the elliptic variational inequality which defines the state system and the corresponding cost functional, and we prove that there exist a discrete optimal control and its associated discrete state system for each positive h (the parameter of the finite element method approximation). Finally, we show that the discrete optimal control and its associated state system converge to the continuous optimal control and its associated state system when the parameter h goes to zero.

1. Introduction

We consider a bounded domain $\Omega \subset \mathbb{R}^n$ whose regular boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ consists of the union of two disjoint portions Γ_1 and Γ_2 with $meas(\Gamma_1) > 0$. We consider the following free boundary problem (S):

$$u \geq 0; \quad u(-\Delta u - g) = 0; \quad -\Delta u - g \geq 0 \quad \text{in } \Omega; \tag{1}$$

$$u = b \quad \text{on } \Gamma_1; \qquad -\frac{\partial u}{\partial n} = q \quad \text{on } \Gamma_2, \tag{2}$$

where the function g in (1) can be considered as the internal energy in Ω, b is the constant temperature on Γ_1, and q is the heat flux on Γ_2. The variational formulation of the above problem is given as follows: find $u = u_g \in K$ such that $\forall v \in K$

$$a(u, v - u) \geq (g, v - u)_H - \int_{\Gamma_2} q(v - u)\, ds, \tag{3}$$

where

$$\begin{aligned} V &= H^1(\Omega), \\ K &= \{v \in V : v \geq 0 \text{ in } \Omega, v = b \text{ on } \Gamma_1\}, \\ V_0 &= \{v \in V : v = 0 \text{ on } \Gamma_1\}, \\ H &= L^2(\Omega), \\ Q &= L^2(\Gamma_2), \\ (u, v)_Q &= \int_{\Gamma_2} uv\, ds \quad \forall u, v \in Q, \\ (u, v)_H &= \int_\Omega uv\, dx \quad \forall u, v \in H, \\ a(u, v) &= \int_\Omega \nabla u \cdot \nabla v\, dx \quad \forall u, v \in V. \end{aligned} \tag{4}$$

We note that a is bilinear, continuous, and symmetric on V and a coercive form on V_0 [1]; that is to say, there exists a constant $\lambda > 0$ such that

$$a(v, v) \geq \lambda \|v\|_V^2 \quad \forall v \in V_0. \tag{5}$$

In [2], the following continuous distributed optimal control problem associated with (S) or the elliptic variational inequality (3) was considered as follows.

Problem (P). Find the continuous distributed optimal control $g_{\text{op}} \in H$ such that [3–5]

$$J\left(g_{\text{op}}\right) = \min_{g \in H} J\left(g\right), \quad (6)$$

where the quadratic cost functional $J : H \to \mathbb{R}_0^+$ is defined by

$$J\left(g\right) = \frac{1}{2}\left\|u_g\right\|_H^2 + \frac{M}{2}\left\|g\right\|_H^2 \quad (7)$$

with $M > 0$, a given constant, and u_g is the corresponding solution of the elliptic variational inequality (3) associated with the control g.

Several continuous optimal control problems are governed by elliptic variational inequalities, for example, the process of biological waste-water treatment; reorientation of a satellite by propellers; and economics: the problem of consumer regulation of a monopoly and so forth. There exists an abundant literature for optimal control problems governed by elliptic variational equalities or inequalities [6–12], for numerical analysis of variational inequalities or optimal control problems [13–16].

The objective of this work is to make the numerical analysis of the optimal control problem (P) which is governed by the elliptic variational inequality (3) by proving the convergence of a discrete solution to the continuous optimal control problems.

In Section 2, we establish the discrete elliptic variational inequality (10) which is the discrete formulation of the continuous elliptic variational inequality (3), and we obtain that these discrete problems have unique solutions for all positive h. Moreover, on the adequate functional spaces these solutions are convergent when $h \to 0^+$ to the solutions of the continuous elliptic variational inequality (3).

In Section 3, we define the discrete optimal control problem (31) corresponding to continuous optimal control problem (6). We prove the existence of a discrete solution for the optimal control problem (P_h) for each parameter h and we obtain the convergence of this family with its corresponding discrete state system to the continuous optimal control with the corresponding continuous state system of the problem (P).

2. Discretization of the Problem (S)

Let $\Omega \subset \mathbb{R}^n$ be a bounded polygonal domain; b a positive constant; and τ_h a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite elements of class C^0 over Ω, h being the parameter of the finite element approximation which goes to zero [17, 18]. We take h equal to the longest side of the triangles $T \in \tau_h$ and we can approximate the sets V and K by

$$\begin{aligned} V_h &= \left\{v_h \in C^0\left(\overline{\Omega}\right) : v_h \in \mathbb{P}_1\left(T\right) \text{ on } T,\ \forall T \in \tau_h\right\}, \\ V_{h0} &= \left\{v_h \in V_h : v_h = 0 \text{ on } \Gamma_1\right\}, \\ K_h &= \left\{v_h \in V_h : v_h \geq 0,\ v_h = b \text{ on } \Gamma_1\right\}, \end{aligned} \quad (8)$$

where $\mathbb{P}_1(T)$ is the set of the polynomials of degree less than or equal to 1 in the triangle T. Let $\Pi_h : V \to V_h$ be the corresponding linear interpolation operator and $c_0 > 0$ a constant (independent of the parameter h) such that, $\forall v \in H^r(\Omega)$, $1 < r \leq 2$ [17]:

$$\begin{aligned} \left\|v - \Pi_h\left(v\right)\right\|_H &\leq c_0 h^r \left\|v\right\|_r, \\ \left\|v - \Pi_h\left(v\right)\right\|_V &\leq c_0 h^{r-1} \left\|v\right\|_r. \end{aligned} \quad (9)$$

The discrete variational inequality formulation (S_h) of system (S) is defined as follows: find $u_{hg} \in K_h$ such that $\forall v_h \in K_h$

$$a\left(u_{hg}, v_h - u_{hg}\right) \geq \left(g, v_h - u_{hg}\right)_H - \left(q, v_h - u_{hg}\right)_Q. \quad (10)$$

Theorem 1. *Let $g \in H$, $b > 0$, and $q \in Q$; then there exists unique solution of the problem (S_h) given by the elliptic variational inequality (10).*

Proof. It follows from the application of Lax-Milgram Theorem [1]. □

Lemma 2. *Let $g_1, g_2 \in H$ and $u_{hg_1}, u_{hg_2} \in K_h$ be the solutions of (S_h) for g_1 and g_2, respectively; then one has that*

(a) *there exists a constant C independent of h such that*

$$\left\|u_{hg}\right\|_V \leq C, \quad \forall h > 0; \quad (11)$$

(b) $\forall h > 0$

$$\left\|u_{hg_2} - u_{hg_1}\right\|_V \leq \frac{1}{\lambda}\left\|g_2 - g_1\right\|_H; \quad (12)$$

(c) *if $g_n \rightharpoonup g$ in H weak, then $u_{hg_n} \to u_{hg}$ in V strong for each fixed $h > 0$.*

Proof. (a) If we consider $v_h = b \in K_h$ in the discrete elliptic variational inequality (10) we have

$$\begin{aligned} \lambda\left\|u_{hg} - b\right\|_V^2 &\leq a\left(u_{hg} - b, u_{hg} - b\right) \\ &\leq \left(g, u_{hg} - b\right)_H + \left(q, b - u_{hg}\right)_Q \\ &\leq \left(\left\|g\right\|_H + \left\|q\right\|_Q \left\|\gamma_0\right\|\right)\left\|u_{hg} - b\right\|_V, \end{aligned} \quad (13)$$

where γ_0 is the trace operator and therefore (11) holds.

(b) As u_{hg_1} and u_{hg_2} are, respectively, the solutions of discrete elliptic variational inequalities (10) for g_1 and g_2, we have

$$\begin{aligned} a\left(u_{hg_i}, v_h - u_{hg_i}\right) &\geq \left(g_i, v_h - u_{hg_i}\right)_H \\ &\quad - \left(q, v_h - u_{hg_i}\right)_Q, \quad \forall v_h \in K_h \end{aligned} \quad (14)$$

for $i = 1, 2$. By coerciveness of a we deduce

$$\begin{aligned}\lambda \left\|u_{hg_2} - u_{hg_1}\right\|_V^2 &\le a\left(u_{hg_2} - u_{hg_1}, u_{hg_2} - u_{hg_1}\right)\\ &\le \left(g_2 - g_1, u_{hg_2} - u_{hg_1}\right)_H\\ &\le \left\|g_2 - g_1\right\|_H \left\|u_{hg_2} - u_{hg_1}\right\|_V \qquad \forall h > 0;\end{aligned} \tag{15}$$

thus (12) holds.

(c) Let $h > 0$. From item (a) we have that $\|u_{hg_n}\| \le C \ \forall n$; then there exist $\eta \in V$ such that $u_{hg_n} \rightharpoonup \eta$ in V weak (in H strong). If we consider the discrete elliptic inequality (10) we have

$$a\left(u_{hg_n}, v_h - u_{hg_n}\right) \ge \left(g_n, v_h - u_{hg_n}\right)_H - \left(q, v_h - u_{hg_n}\right)_Q \tag{16}$$

and using the fact that a is a lower weak semicontinuous application then, when n goes to infinity, we obtain that

$$a\left(\eta, v_h - \eta\right) \ge \left(g, v_h - \eta\right)_H - \left(q, v_h - \eta\right)_Q \tag{17}$$

and from uniqueness of the solution of problem (S_h), we deduce that $\eta = u_{hg} \in K_h$.

Now, it is easily to see that

$$a\left(u_{hg_n} - u_{hg}, u_{hg_n} - u_{hg}\right) \le -\left(g - g_n, u_{hg_n} - u_{hg}\right)_H \tag{18}$$

and from the coerciveness of a we obtain

$$\lambda \left\|u_{hg_n} - u_{hg}\right\|_V^2 \le \left(g - g_n, u_{hg_n} - u_{hg}\right)_H. \tag{19}$$

As $u_{hg_n} \to u_{hg}$ in H and $g_n \rightharpoonup g$ in H, by passing to the limit when $n \to \infty$ in the previous inequality, we obtain

$$\lim_{n\to\infty} \left\|u_{hg_n} - u_g\right\|_V = 0. \tag{20}$$

□

Henceforth we will consider the following definitions [2]: given $\mu \in [0,1]$ and $g_1, g_2 \in H$, we have the convex combinations of two data items

$$g_3(\mu) = \mu g_1 + (1-\mu) g_2 \in H, \tag{21}$$

the convex combination of two discrete solutions

$$u_{h3}(\mu) = \mu u_{hg_1} + (1-\mu) u_{hg_2} \in K_h, \tag{22}$$

and we define $u_{h4}(\mu)$ as the associated state system which is the solution of the discrete elliptic variational inequality (10) for the control $g_3(\mu)$.

Then, we have the following properties.

Lemma 3. *Given the controls $g_1, g_2 \in H$, one has that*

(a)

$$\begin{aligned}\left\|u_{h3}\right\|_H^2 &= \mu \left\|u_{hg_1}\right\|_H^2 + (1-\mu)\left\|u_{hg_2}\right\|_H^2\\ &\quad - \mu(1-\mu)\left\|u_{hg_2} - u_{hg_1}\right\|_H^2,\end{aligned} \tag{23}$$

(b)

$$\begin{aligned}\left\|g_3(\mu)\right\|_H^2 &= \mu \left\|g_1\right\|_H^2 + (1-\mu)\left\|g_2\right\|_H^2\\ &\quad - \mu(1-\mu)\left\|g_2 - g_1\right\|_H^2.\end{aligned} \tag{24}$$

Proof. (a) From definition (22) we get

$$\begin{aligned}\left\|u_{h3}\right\|_H^2 &= \mu^2 \left\|u_{hg_1}\right\|_H^2 + (1-\mu)^2\left\|u_{hg_2}\right\|_H^2\\ &\quad + 2\mu(1-\mu)\left(u_{hg_1}, u_{hg_2}\right)_H,\\ \left\|u_{hg_2} - u_{hg_1}\right\|_H^2 &= \left\|u_{hg_2}\right\|_H^2 + \left\|u_{hg_1}\right\|_H^2\\ &\quad - 2\left(u_{hg_1}, u_{hg_2}\right)_H,\end{aligned} \tag{25}$$

and then we conclude (23).

(b) It follows from a similar method to part (a). □

Theorem 4. *If u_g and u_{hg} are the solutions of the elliptic variational inequalities (3) and (10), respectively, for the control $g \in H$, then $u_{hg} \to u_g$ in V strong when $h \to 0^+$.*

Proof. From Lemma 2 we have that there exists a constant $C > 0$ independent of h such that $\|u_{hg}\|_V \le C \ \forall h > 0$, and then we conclude that there exists $\eta \in V$ so that $u_{hg} \rightharpoonup \eta$ in V weak as $h \to 0^+$ and $\eta \in K$. On the other hand, given $v \in K$ there exist v_h^* such that $v_h^* \in K_h$ for each h and $v_h^* \to v$ in V strong when h goes to zero. Now, by considering $v_h^* \in K_h$ in the discrete elliptic variational inequality (10) we get

$$\begin{aligned}a\left(u_{hg}, u_{hg}\right) &\le a\left(u_{hg}, v_h^*\right) - \left(g, v_h^* - u_{hg}\right)\\ &\quad + \left(q, v_h^* - u_{hg}\right)_Q\end{aligned} \tag{26}$$

and when we pass to the limit as $h \to 0^+$ in (26) by using the fact that bilinear form a is lower weak semicontinuous in V we obtain

$$a(\eta, \eta) \le a(\eta, v) - (g, v - \eta) + (q, v - \eta)_Q; \tag{27}$$

that it is to say,

$$a(\eta, v - \eta) \ge (g, v - \eta) - (q, v - \eta)_Q \quad \forall v \in K \tag{28}$$

and, from the uniqueness of the solution of the discrete elliptic variational inequality (3), we obtain that $\eta = u_g$.

Now, we will prove the strong convergence. If we consider $v = u_{hg} \in K_h \subset K$ in the elliptic variational inequality (3) and $v_h = \Pi_h(u_g) \in K_h$ in (10), from the coerciveness of a and by some mathematical computation, we obtain that

$$\begin{aligned}\lambda \left\|u_{hg} - u_g\right\|_V^2 &\le a\left(u_{hg} - u_g, u_{hg} - u_g\right)\\ &\le a\left(u_{hg}, \Pi_h\left(u_g\right) - u_g\right)\\ &\quad - \left(g, \Pi_h\left(u_g\right) - u_g\right)\\ &\quad + \left(q, \Pi_h\left(u_g\right) - u_g\right)_Q;\end{aligned} \tag{29}$$

then by passing to the limit when $h \to 0^+$ it results in $\lim_{h\to 0^+} \|u_{hg} - u_g\|_V = 0$. □

3. Discretization of the Optimal Control Problem

Now, we consider the continuous optimal control problem which was established in (6). The associated discrete cost functional $J_h : H \to \mathbb{R}_0^+$ is defined by the following expression:

$$J_h(g) = \frac{1}{2}\|u_{hg}\|_H^2 + \frac{M}{2}\|g\|_H^2 \tag{30}$$

and we establish the discrete optimal control problem (P_h) as follows: find $g_{\mathrm{op}_h} \in H$ such that

$$J_h(g_{\mathrm{op}_h}) = \min_{g\in H} J_h(g), \tag{31}$$

where u_{hg} is the associated state system solution of the problem (S_h) which was described for the discrete elliptic variational inequality (10) for a given control $g \in H$.

Theorem 5. *Given the control $g \in H$, one has*

(a)

$$\lim_{\|g\|_H \to \infty} J_h(g) = \infty; \tag{32}$$

(b) $J_h(g) \geq (M/2)\|g\|_H^2 - C\|g\|_H$ *for some constant C independent of h;*

(c) *the functional J_h is a lower weakly semicontinuous application in H;*

(d) *there exists a solution of the discrete optimal control problem (31) for all $h > 0$.*

Proof. (a) From the definition of $J_h(g)$ we obtain (a) and (b).

(c) Let $g_n \rightharpoonup g$ in H weak; then by using the equality $\|g_n\|_H^2 = \|g_n - g\|_H^2 - \|g\|_H^2 + 2(g_n, g)_H$ we obtain that $\|g\|_H \leq \liminf_{n\to\infty}\|g_n\|_H^2$. Therefore, we have

$$\liminf_{n\to\infty} J_h(g_n) \geq \frac{1}{2}\|u_{hg}\|_H^2 + \frac{M}{2}\|g\|_H^2 = J_h(g). \tag{33}$$

(d) It follows from [4]. □

Lemma 6. *If the continuous state system has the regularity $u_g \in H^r(\Omega)$ $(1 < r \leq 2)$ then one has the following estimations $\forall g \in H$:*

(a)

$$\|u_{hg} - u_g\|_V \leq Ch^{(r-1)/2}, \tag{34}$$

(b)

$$|J_h(g) - J(g)| \leq Ch^{(r-1)/2}, \tag{35}$$

where C's are constants independent of h.

Proof. (a) As $u_g \in K$, we have that $\Pi_h(u_g) \in K_h \subset K$. If we consider $v_h = \Pi_h(u_g)$ in (10), by using the inequalities (29), we obtain

$$\begin{aligned}\lambda\|u_{hg} - u_g\|_V^2 &\leq a(u_{hg} - u_g, u_{hg} - u_g)\\ &\leq a(u_{hg}, \Pi_h(u_g) - u_g)\\ &\quad - (g, \Pi_h(u_g) - u_g)\\ &\quad + (q, \Pi_h(u_g) - u_g)_Q\\ &\leq C\|\Pi_h(u_g) - u_g\|_V \leq C\|u_g\|_r h^{r-1}\\ &\leq Ch^{r-1},\end{aligned} \tag{36}$$

and then (34) holds.

(b) From the definitions of J and J_h, it results in

$$\begin{aligned}J_h(g) - J(g) &= \frac{1}{2}\left(\|u_{hg}\|_H^2 - \|u_g\|_H^2\right)\\ &= \frac{1}{2}\left[\|u_{hg} - u_g\|_H^2 + (u_g, u_{hg} - u_g)\right]\end{aligned} \tag{37}$$

and therefore

$$\begin{aligned}&|J_h(g) - J(g)|\\ &\quad\leq \left(\frac{1}{2}\|u_{hg} - u_g\|_H + \|u_g\|_H\right)\|u_{hg} - u_g\|_H\\ &\quad\leq Ch^{(r-1)/2}.\end{aligned} \tag{38}$$

□

Following the idea given in [2] we define an open problem: given the controls $g_1, g_2 \in H$ and $\forall\mu \in [0,1]$, $\forall h > 0$

$$0 \leq u_{h4}(\mu) \leq u_{h3}(\mu) \quad \text{in } \Omega, \tag{39}$$

$$\|u_{h4}(\mu)\|_H \leq \|u_{h3}(\mu)\|_H. \tag{40}$$

Remark 7. We have that (39)⇒(40).

Remark 8. The equivalent inequality (39) for the continuous optimal control problem (P) is true; that is [2], for all $g_1, g_2 \in H$, and $\forall\mu \in [0,1]$,

$$0 \leq u_4(\mu) \leq u_3(\mu) \quad \text{in } \Omega, \tag{41}$$

where $u_3(\mu) = \mu u_{g_1} + (1-\mu)u_{g_2} \in K$, u_{g_i} $(i = 1, 2)$ is the unique solution of the elliptic variational inequality (3) when we consider g_i instead of g and $u_4(\mu)$ is the unique solution of the elliptic variational inequality (3) when we consider $g_3(\mu)$ instead of g.

Remark 9. If (40) (or (39)) is true, then the functional J_h is H-elliptic and a strictly convex application because we have

$$\begin{aligned}\mu J_h(g_1) &+ (1-\mu) J_h(g_2) - J_h(g_3(\mu)) \\ &= \frac{\mu(1-\mu)}{2}\left[\left\|u_{hg_2} - u_{hg_1}\right\|_H^2 + M\left\|g_2 - g_1\right\|_H^2\right] \\ &\quad + \frac{1}{2}\left[\left\|u_{h3}\right\|_H^2 - \left\|u_{h4}\right\|_H^2\right] \\ &\geq \frac{\mu(1-\mu)}{2}\left[\left\|u_{hg_2} - u_{hg_1}\right\|_H^2 + M\left\|g_2 - g_1\right\|_H^2\right] > 0\end{aligned} \tag{42}$$

and therefore, the uniqueness for the discrete optimal control problem (P_h) holds in Theorem 5.

Now, we will show the convergence result for optimal control problems governed by elliptic variational inequalities in order to generalize the result for optimal control problems governed by elliptic variational equalities [19]. We remark that there exist a few numbers of papers for the numerical analysis of optimal control problems governed by elliptic variational inequalities, for example [20–22].

Theorem 10. *Let $u_{g_{op}} \in K$ be the continuous state system associated with the optimal control $g_{op} \in H$ which is the solution of the continuous distributed optimal control problem (6). If, for each $h > 0$, one chooses an optimal control $g_{op_h} \in H$ which is the solution of the discrete distributed optimal control problem (31) and its corresponding discrete state system $u_{hg_{op_h}} \in K_h$, one obtains that*

$$\begin{aligned} &u_{hg_{op_h}} \longrightarrow u_{g_{op}} \quad \text{on } V \text{ strong}, \\ &g_{op_h} \longrightarrow g_{op} \quad \text{on } H \text{ strong when } h \longrightarrow 0^+. \end{aligned} \tag{43}$$

Proof. Let $h > 0$ and let g_{op_h} be a solution of (31), and let $u_{hg_{\text{op}_h}}$ be its associated discrete optimal state system which is the solution of the discrete elliptic variational inequality (10) for each $h > 0$. From (30) we have that for all $g \in H$

$$\begin{aligned} J_h\left(g_{\text{op}_h}\right) &= \frac{1}{2}\left\|u_{hg_{\text{op}_h}}\right\|_H^2 + \frac{M}{2}\left\|g_{\text{op}_h}\right\|_H^2 \\ &\leq \frac{1}{2}\left\|u_{hg}\right\|_H^2 + \frac{M}{2}\left\|g\right\|_H^2. \end{aligned} \tag{44}$$

Then, if we consider $g = 0$ and u_{h0} its corresponding associated state system, it results in the following:

$$J_h\left(g_{\text{op}_h}\right) = \frac{1}{2}\left\|u_{hg_{\text{op}_h}}\right\|_H^2 + \frac{M}{2}\left\|g_{\text{op}_h}\right\|_H^2 \leq \frac{1}{2}\left\|u_{h0}\right\|_H^2. \tag{45}$$

From Lemma 2 we have that $\|u_{h0}\|_H \leq C \ \forall h$; then we can obtain

$$\begin{aligned} &\left\|u_{hg_{\text{op}_h}}\right\|_H \leq C \quad \forall h > 0, \\ &\left\|g_{\text{op}_h}\right\|_H \leq \frac{1}{\sqrt{M}}\left\|u_{h0}\right\|_H \leq \frac{C}{\sqrt{M}} \quad \forall h > 0. \end{aligned} \tag{46}$$

If we consider $v_h = b \in K_h$ in inequality (10) for g_{op_h}, we obtain

$$\begin{aligned} a\left(u_{hg_{\text{op}_h}}, b - u_{hg_{\text{op}_h}}\right) &\geq \left(g_{\text{op}_h}, b - u_{hg_{\text{op}_h}}\right) \\ &\quad - \left(q, b - u_{hg_{\text{op}_h}}\right)_Q; \end{aligned} \tag{47}$$

therefore

$$\begin{aligned} a&\left(u_{hg_{\text{op}_h}} - b, u_{hg_{\text{op}_h}} - b\right) \\ &\leq \left(g_{\text{op}_h}, u_{hg_{\text{op}_h}} - b\right) - \left(q, u_{hg_{\text{op}_h}} - b\right)_Q, \end{aligned} \tag{48}$$

and from the coerciveness of the application a we have that $\|u_{hg_{\text{op}_h}} - b\|_V \leq C$ and in consequence $\|u_{hg_{\text{op}_h}}\|_V \leq C$.

Now we can say that there exist $\eta \in V$ and $f \in H$ such that $u_{hg_{\text{op}_h}} \rightharpoonup \eta$ in V weak (in H strong), and $g_{\text{op}_h} \rightharpoonup f$ in H weak when $h \to 0^+$. Then, $\eta/\Gamma_1 = b$ and $\eta \geq 0$ in Ω; that is, $\eta \in K$.

Letting $v \in K$, there exist $v_h \in K_h$ such that $v_h \to v$ in V strong when $h \to 0^+$. Then, if we consider the variational elliptic inequality (10) for $g = g_{\text{op}_h}$ we have

$$\begin{aligned} a\left(u_{hg_{\text{op}_h}}, v_h\right) &\geq a\left(u_{hg_{\text{op}_h}}, u_{hg_{\text{op}_h}}\right) + \left(g_{\text{op}_h}, v_h - u_{hg_{\text{op}_h}}\right) \\ &\quad - \left(q, v_h - u_{hg_{\text{op}_h}}\right)_Q. \end{aligned} \tag{49}$$

Taking into account that the application a is a lower weak semicontinuous application in V and by passing to the limit when h goes to zero in (49) we obtain that

$$a(\eta, v - \eta) \geq (f, v - \eta) - (q, v - \eta)_Q, \quad \forall v \in K \tag{50}$$

and by the uniqueness of the solution of the problem given by the elliptic variational inequality (3), we deduce that $\eta = u_f$.

Finally, the norm on H is a lower semicontinuous application in the weak topology; then we can prove that

$$\begin{aligned} J(f) &= \frac{1}{2}\left\|u_f\right\|_H^2 + \frac{M}{2}\|f\|_H^2 \leq \liminf_{h\to 0^+} J_h\left(g_{\text{op}_h}\right) \\ &\leq \liminf_{h\to 0^+} J_h(g) = \frac{1}{2}\lim_{h\to 0^+}\left\|u_{hg}\right\|_H^2 + \frac{M}{2}\|g\|_H^2 \\ &= \frac{1}{2}\left\|u_g\right\|_H^2 + \frac{M}{2}\|g\|_H^2 = J(g), \quad \forall g \in H \end{aligned} \tag{51}$$

and because of the uniqueness of the optimal problem (6), it results in $f = g_{\text{op}}$ and $\eta = u_{g_{\text{op}}}$.

Now, if we consider $v = u_{hg_{\text{op}_h}} \in K_h \subset K$ in the elliptic variational inequality (3) for the control g_{op} and we define $z_h = u_{hg_{\text{op}_h}} - u_{g_{\text{op}}}$, we have that

$$\begin{aligned} a(z_h, z_h) &\leq a\left(u_{hg_{\text{op}_h}}, u_{hg_{\text{op}_h}}\right) - a\left(u_{hg_{\text{op}_h}}, u_{g_{\text{op}}}\right) \\ &\quad - \left(g_{\text{op}}, u_{hg_{\text{op}_h}} - u_{g_{\text{op}}}\right) \\ &\quad + \left(q, u_{hg_{\text{op}_h}} - u_{g_{\text{op}}}\right)_Q, \end{aligned} \tag{52}$$

and by considering $v = \Pi_h(u_{g_{op}}) \in K_h$ for $g = g_{op_h}$ in inequality (10) we obtain

$$\begin{aligned} a\left(u_{hg_{op_h}}, u_{hg_{op_h}}\right) \leq & -\left(g_{op_h}, \Pi_h\left(u_{g_{op}}\right) - u_{hg_{op_h}}\right) \\ & + \left(q, \Pi_h\left(u_{g_{op}}\right) - u_{hg_{op_h}}\right)_Q \\ & + a\left(u_{hg_{op_h}}, \Pi_h\left(u_{g_{op}}\right)\right) \end{aligned} \quad (53)$$

and then by the coerciveness of a we get

$$\begin{aligned} \lambda \left\|z_h\right\|_V^2 \leq & \left(q, \Pi_h\left(u_{g_{op}}\right) - u_{g_{op}}\right)_Q \\ & + a\left(u_{hg_{op_h}}, \Pi_h\left(u_{g_{op}}\right) - u_{g_{op}}\right) \\ & + \left(g_{op_h} - g_{op}, u_{hg_{op_h}} - u_{g_{op}}\right) \\ & - \left(g_{op}, \Pi_h\left(u_{g_{op}}\right) - u_{g_{op}}\right). \end{aligned} \quad (54)$$

When we pass to the limit as $h \to 0$ in (54) and by using the strong convergence of $u_{hg_{op_h}}$ to $u_{g_{op}}$ on H and the weak convergence of g_{op_h} to g_{op} on H, we have

$$\lim_{h \to 0^+} \left\|u_{g_{op}} - u_{hg_{op_h}}\right\|_V = 0. \quad (55)$$

The strong convergence of the optimal controls g_{op_h} to g_{op} is obtained by using Theorem 5 and $g_{op_h} \rightharpoonup g_{op}$ weakly on H; that is,

$$\begin{aligned} J\left(g_{op}\right) = & \frac{1}{2}\left\|u_{g_{op}}\right\|_H^2 + \frac{M}{2}\left\|g_{op}\right\|_H^2 \leq \liminf_{h \to 0^+} J_h\left(g_{op_h}\right) \\ & \leq \liminf_{h \to 0^+} J_h\left(g_{op}\right) \\ & = \liminf_{h \to 0^+} \frac{1}{2}\left\|u_{g_{op}}\right\|_H^2 + \frac{M}{2}\left\|g_{op}\right\|_H^2 = J\left(g_{op}\right); \end{aligned} \quad (56)$$

then $\lim_{h \to 0}\|g_{op_h}\|_H = \|g_{op}\|_H$ and therefore $\lim_{h \to 0^+}\|g_{op_h} - g_{op}\|_H = 0$. □

4. Conclusions

We have proved the convergence of a discrete optimal control and its corresponding discrete state system governed by a discrete elliptic variational inequality to the continuous optimal control and its corresponding continuous state system which is also governed by a continuous elliptic variational inequality by using the finite element method with Lagrange's triangles of type 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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