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Research Article

Optimal Control of the Ill-Posed Cauchy Elliptic Problem

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We give a characterization of the control for ill-posed problems of oscillating solutions. More precisely, we study the control of Cauchy elliptic problems via a regularization approach which generates incomplete information. We obtain a singular optimality system characterizing the no-regret control for the Cauchy problem.

1. Introduction

Cauchy problems for partial differential equations of elliptic type are present in many physical systems such as plasma physics [1], mechanical engineering [2], or electrocardiography [3]. One of the important examples is the Helmholtz equation and its applications in acoustic, wave propagation and scattering, vibration of the structure, and electromagnetic scattering (see [4–6] and the references therein). Here, we investigate a Cauchy problem for the Laplacian elliptic operator:

$$\Delta z = 0, \tag{1}$$

where $z = z(x)$, on an open set $\Omega \subset \mathbb{R}^3$ of class $\mathscr{C}^2$, of boundary $\partial\Omega = \Gamma$. Dirichlet and Neumann conditions are prescribed on a part of the boundary $\Gamma_0 \subset \Gamma$, $\Gamma_0 \neq \Gamma$:

$$\begin{gathered} z = v_0, \\ \frac{\partial z}{\partial \nu} = v_1 \\ \text{on } \Gamma_0. \end{gathered} \tag{2}$$

The goal is to reconstruct the solution on Ω and its trace on $\Gamma_1 = \Gamma \setminus \Gamma_0$ from a perturbed system, under the assumption that the solution exists for the exact data v_0 and v_1 which here are control variables. This problem is a classical example of ill-posed problems. So, regularization methods may be considered.

Theoretical concepts and also computational implementation related to the Cauchy problem of the elliptic equation have been discussed by many authors, and a lot of methods were provided (see, e.g., Qian et al. [7] for the 4th-order regularization method or Xiong and Fu [8]). Generally, it is assumed that instead of exact data some noisy boundary conditions v_0^ε and v_1^ε are given with the error bound.

But, in this paper, we consider another regularization method of the Laplacian, where we introduce a new data:

$$\begin{gathered} z = g_0, \\ \frac{\partial z}{\partial \nu} = g_1 \\ \text{on } \Gamma_1 \end{gathered} \tag{3}$$

with g_0 and g_1 being unknown functions. Hence, we have a regularized problem but with incomplete data. A special optimal control method of problems of incomplete data should then be applied. We here use a method that we find well adapted: the low-regret control concept, introduced by Lions in the late 80s (see, e.g., [9, 10] and the references therein).

In [11, 12], the control of distributed system with incomplete data is performed. The proof of the existence and characterization of the no-regret control is obtained as the limit of the low-regret control. Here, we admit the possibility of making a choice of controls v *slightly worse than by doing better than* $v = 0$ (i.e., better than a noncontrolled system), with respect to certain criteria (cost function):

$$J(u^\gamma, g) \le J(0, g) + \gamma \|g\|_Y^2 \quad \forall g \text{ in a Hilbert space } Y, \tag{4}$$

where γ is the small positive parameter that tends to 0, with g being the pollution or the incomplete data.

This method is previously introduced by Savage [13] in statistics. Lions was the first to use it to control distributed systems of incomplete data, motivated by a number of applications in economics and ecology. In this paper, we generalize the method to ill-posed problems of elliptic type.

It seems that the control of Cauchy system for elliptic operators is globally an open problem. Lions in [14] proposed a method of approximation by penalization and obtained a singular optimality system, under a supplementary hypothesis of Slater type. In [15], Sougalo and Nakoulima analyzed the Cauchy problem using a regularization method, consisting in viewing a singular problem as a limit of a family of well-posed problems. They have obtained a singular optimality system for the considered control problem, also assuming the Slater condition. Unfortunately, the recent paper by Massengo Mophou and Nakoulima [16] is the same as the one by Sougalo and Nakoulima (1998) using the same old references, and nothing new is brought.

In the present paper, we use another approximation method which consists in considering the elliptic Cauchy problem as a singular limit of sequence of well-posed elliptic problems, where the Slater condition is not used and where we apply the low-regret control notion. The same analysis can be generalized to the Helmholtz equation with no difficulty.

The paper is organized as follows. In Section 2, we present the regularization method. In Section 3, the optimal control of the regularized system is discussed and the approximated optimality system is presented. We pass to the limit in the last section; we show that we obtain a singular optimality system for the low-regret and no-regret controls to the original problem of Laplacian.

2. Existence of Solutions to Cauchy Elliptic Problems

Let Ω be an open bounded subset of R^n, with a boundary Γ of class C^2, $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. The boundaries Γ_0 and Γ_1 are nonempty and are of positive measure. We consider here the problem:

$$\begin{aligned} \Delta z &= 0 \quad \text{in } \Omega, \\ z &= v_0, \\ \frac{\partial z}{\partial \nu} &= v_1 \\ &\quad \text{on } \Gamma_0, \end{aligned} \tag{5}$$

with $z \in L^2(\Omega)$ and $(v_0, v_1) \in L^2(\Gamma_0) \times L^2(\Gamma_0)$. Problem (5) is a Cauchy problem for the Laplacian operator. It is well known that it is ill-posed in the sense that it does not admit a solution in general and that existing solutions (if any) are unstable. This problem is present in many applications, so it is important to control the Cauchy data.

Denote by A the closed subset of $(L^2(\Gamma_0))^2 \times L^2(\Omega)$ defined by

$$A = \left\{ ((v_0, v_1), z) \in \left(L^2(\Gamma_0)\right)^2 \times L^2(\Omega),\ \Delta z = 0 \text{ in } \Omega,\ z|_{\Gamma_0} = v_0,\ \left.\frac{\partial z}{\partial \nu}\right|_{\Gamma_0} = v_1 \right\}, \tag{6}$$

and suppose that $A \neq \emptyset$. We will call any control-state pair $(v_0, v_1, z) \in A$ *admissible couple*.

Let J be a strictly convex cost functional, defined for all admissible control-state couples (v_0, v_1, z) by

$$J(v_0, v_1, z) = \|z - z_d\|_{L^2(\Omega)}^2 + N_0 \|v_0\|_{L^2(\Gamma_0)}^2 + N_1 \|v_1\|_{L^2(\Gamma_0)}^2, \tag{7}$$

where z_d and (N_0, N_1) are, respectively, given in $L^2(\Omega)$ and in $(R_+ \setminus \{0\})^2$. We want to find the couple control-state solution of

$$\inf J(v_0, v_1, z), \quad (v_0, v_1, z) \in A. \tag{8}$$

According to the structure of J, problem (8) admits a unique solution (u_0, u_1, y) that we should characterize. To obtain a singular optimality system (SOS) associated with (u_0, u_1, y), Lions [14] has proposed a method of approximation by penalization. He obtained SOS, under the supplementary hypothesis of Slater type:

The admissible set of controls has a nonempty interior. (9)

Here, we do not consider Slater hypothesis, but instead we consider the no-regret and low-regret techniques.

3. The Low-Regret and No-Regret Control

Due to the ill-posedness of the Cauchy elliptic problem, it is impossible to solve it directly [17]. This requires special techniques as the technique of regularization. Our method consists in regularizing (5) into an elliptic problem of incomplete data. For any $\varepsilon > 0$, we consider the regularized problem:

$$\begin{aligned} &\Delta^2 z_\varepsilon + \varepsilon z_\varepsilon = 0 \quad \text{in } \Omega, \\ &z_\varepsilon - \frac{\partial \Delta z_\varepsilon}{\partial \nu} = v_0, \\ &\frac{\partial z_\varepsilon}{\partial \nu} + \Delta z_\varepsilon = v_1 \\ &\qquad \text{on } \Gamma_0, \\ &\varepsilon z_\varepsilon - \frac{\partial \Delta z_\varepsilon}{\partial \nu} = \varepsilon g_0, \\ &\varepsilon \frac{\partial z_\varepsilon}{\partial \nu} + \Delta z_\varepsilon = \varepsilon g_1 \\ &\qquad \text{on } \Gamma_1. \end{aligned} \tag{10}$$

We denote $v := (v_0, v_1)$ and $g := (g_0, g_1)$ for simplicity.

We begin by an important remark.

Remark 1. For every fixed εg_0 and εg_1 we assume the existence of a unique solution to (10). Indeed, in the following subsections, εg_0 and εg_1 are considered as data perturbations, and the solution of (10) may not exist in the sense of Hadamard [18].

3.1. Back to the Original Problem. We show how to come back to the original Cauchy problem, starting from (10). Indeed, if we put $\varepsilon = 0$ and we do the change of variables $\eta = \Delta z$, we obtain

$$\begin{aligned} &\Delta \eta = 0 \quad \text{in } \Omega, \\ &\frac{\partial \eta}{\partial \nu} = 0, \\ &\eta = 0 \\ &\qquad \text{on } \Gamma_1, \end{aligned} \tag{11}$$

$$\begin{aligned} &z - \frac{\partial \eta}{\partial \nu} = v_0, \\ &\frac{\partial z}{\partial \nu} + \eta = v_1 \\ &\qquad \text{on } \Gamma_0. \end{aligned} \tag{12}$$

Using the uniqueness property of the solution of the Laplace equation and the unique continuation theorem of Mizohata [19], we deduce from (11) that we also have

$$\frac{\partial \eta}{\partial \nu} = \eta = 0 \quad \text{on } \Gamma_0. \tag{13}$$

Hence, conditions (12) become

$$\begin{aligned} &z = v_0, \\ &\frac{\partial z}{\partial \nu} = v_1 \\ &\qquad \text{on } \Gamma_0, \end{aligned} \tag{14}$$

that is, the same conditions of the original problem (5).

3.2. Cost Function and Low-Regret Control. Consider the cost functional

$$\begin{aligned} J_\varepsilon (v, g) = {} & \| z_\varepsilon (v, g) - z_d \|^2_{L^2(\Omega)} + N_0 \| v_0 \|^2_{L^2(\Gamma_0)} \\ & + N_1 \| v_1 \|^2_{L^2(\Gamma_0)} \end{aligned} \tag{15}$$

that we want to minimize in the context of no-regret control due to the presence of the incomplete data g.

Definition 2. We say that $u \in (L^2(\Gamma_0))^2$ is a no-regret control for (5)–(15), if u is a solution to the following problem:

$$\inf_{v \in (L^2(\Gamma_0))^2} \left(\sup_{g \in (L^2(\Gamma_1))^2} \left(J_\varepsilon (v, g) - J_\varepsilon (0, g) \right) \right). \tag{16}$$

As seen in [11], the no-regret control is difficult to characterize directly. Below, we define the low-regret control which tends to the no-regret control when the parameter of penalization tends to zero. In the case of no pollution g, the no-regret control and the classical control are the same.

3.2.1. The Low-Regret Control. As in [20], we define the low-regret control as the solution to the following MinMax problem:

$$\inf_{v \in (L^2(\Gamma_0))^2} \left(\sup_{g \in (L^2(\Gamma_1))^2} \left[J_\varepsilon (v, g) - J_\varepsilon (0, g) - \gamma \| g_0 \|^2_{L^2(\Gamma_1)} - \gamma \| g_1 \|^2_{L^2(\Gamma_1)} \right] \right), \tag{17}$$

where γ is a strictly positive parameter. The solution to problem (17), if it exists, will be the low-regret control.

Now, we introduce $\xi_\varepsilon := \xi_\varepsilon(v, 0)$ solution to the adjoint problem

$$\begin{aligned}
&\Delta^2\xi_\varepsilon + \varepsilon\xi = z_\varepsilon(v,0) \quad \text{in } \Omega,\\
&\xi_\varepsilon - \frac{\partial}{\partial\nu}(\Delta\xi_\varepsilon) = 0,\\
&\frac{\partial\xi_\varepsilon}{\partial\nu} + \Delta\xi_\varepsilon = 0\\
&\qquad\qquad \text{on } \Gamma_0,\\
&\varepsilon\xi_\varepsilon - \frac{\partial}{\partial\nu}(\Delta\xi_\varepsilon) = 0,\\
&\varepsilon\frac{\partial\xi_\varepsilon}{\partial\nu} + \Delta\xi_\varepsilon = 0\\
&\qquad\qquad \text{on } \Gamma_1.
\end{aligned} \tag{18}$$

Then we have the following result.

Proposition 3. *The low-regret control is the solution to the classical optimal control problem*

$$\inf_{v\in(L^2(\Gamma_0))^2} \mathcal{J}_\varepsilon^\gamma(v) \tag{19}$$

with

$$\mathcal{J}_\varepsilon^\gamma(v) := J_\varepsilon(v,0) - J_\varepsilon(0,0) + \frac{\varepsilon^2}{\gamma}\left(\|\xi_\varepsilon\|^2_{L^2(\Gamma_1)} + \left\|\frac{\partial\xi_\varepsilon}{\partial\nu}\right\|^2_{L^2(\Gamma_1)}\right). \tag{20}$$

Proof. After simple computations we have

$$J_\varepsilon(v,g) - J_\varepsilon(0,g) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + 2\left\langle z_\varepsilon(v,0), z_\varepsilon(0,g)\right\rangle, \tag{21}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\Omega)$. To estimate the integral $\langle z_\varepsilon(v,0), z_\varepsilon(0,g)\rangle$ in (21), we use the Green formula:

$$\begin{aligned}
\left\langle\Delta^2 z,\psi\right\rangle &= \left\langle z,\Delta^2\psi\right\rangle + \left\langle\frac{\partial}{\partial\nu}(\Delta z),\psi\right\rangle_\Gamma\\
&\quad - \left\langle\Delta z,\frac{\partial\psi}{\partial\nu}\right\rangle_\Gamma + \left\langle\frac{\partial z}{\partial\nu},\Delta\psi\right\rangle_\Gamma\\
&\quad - \left\langle z,\frac{\partial}{\partial\nu}(\Delta\psi)\right\rangle_\Gamma,
\end{aligned} \tag{22}$$

together with (18). We have

$$\begin{aligned}
\left\langle z_\varepsilon(v,0), z_\varepsilon(0,g)\right\rangle &= \left\langle\Delta^2\xi_\varepsilon + \varepsilon\xi_\varepsilon, z_\varepsilon(0,g)\right\rangle\\
&= 0 + \left\langle\frac{\partial}{\partial\nu}(\Delta\xi_\varepsilon), z_\varepsilon\right\rangle_\Gamma\\
&\quad - \left\langle\Delta\xi_\varepsilon,\frac{\partial z_\varepsilon}{\partial\nu}\right\rangle_\Gamma\\
&\quad + \left\langle\frac{\partial\xi_\varepsilon}{\partial\nu},\Delta z_\varepsilon\right\rangle_\Gamma\\
&\quad - \left\langle\xi_\varepsilon,\frac{\partial}{\partial\nu}(\Delta z_\varepsilon)\right\rangle_\Gamma\\
&= -\varepsilon\left\langle\xi_\varepsilon, z_\varepsilon\right\rangle_{\Gamma_1}\\
&\quad - \left\langle\xi_\varepsilon,\frac{\partial}{\partial\nu}(\Delta z_\varepsilon)\right\rangle_{\Gamma_1}\\
&\quad + \left\langle\varepsilon\frac{\partial\xi_\varepsilon}{\partial\nu},\frac{\partial z_\varepsilon}{\partial\nu}\right\rangle_{\Gamma_1}\\
&\quad + \left\langle\frac{\partial\xi_\varepsilon}{\partial\nu},\Delta z_\varepsilon\right\rangle_{\Gamma_1}\\
&= \left\langle\xi_\varepsilon,\varepsilon g_0\right\rangle_{\Gamma_1} + \left\langle\frac{\partial\xi_\varepsilon}{\partial\nu},\varepsilon g_1\right\rangle_{\Gamma_1},
\end{aligned} \tag{23}$$

where $z_\varepsilon = z_\varepsilon(0,g)$. Then

$$\begin{aligned}
&\sup_{g\in(L^2(\Gamma_1))^2}\left(2\left\langle z_\varepsilon(v,0), z_\varepsilon(0,g)\right\rangle - \gamma\|g_0\|^2_{L^2(\Gamma_1)}\right.\\
&\quad\left. - \gamma\|g_1\|^2_{L^2(\Gamma_1)}\right) = \sup_{g\in(L^2(\Gamma_1))^2}\left(\left\langle\xi_\varepsilon,\varepsilon g_0\right\rangle_{\Gamma_1}\right.\\
&\quad\left. + \left\langle\frac{\partial\xi_\varepsilon}{\partial\nu},\varepsilon g_1\right\rangle_{\Gamma_1} - \gamma\|g_0\|^2_{L^2(\Gamma_1)} - \gamma\|g_1\|^2_{L^2(\Gamma_1)}\right)\\
&\quad = \frac{\varepsilon^2}{\gamma}\|\xi_\varepsilon\|^2_{L^2(\Gamma_1)} + \frac{\varepsilon^2}{\gamma}\left\|\frac{\partial\xi_\varepsilon}{\partial\nu}\right\|^2_{L^2(\Gamma_1)}
\end{aligned} \tag{24}$$

thanks to the conjugate formula. Combining (17) and (21), we obtain the desired result. □

Remark 4. The no-regret control is obtained by the passage to the limit in the positive parameter γ: it is the weak convergence of the control-state variables of the perturbed system, which corresponds to the limit of the standard low-regret control sequence.

3.3. Approached Optimality System. For the general theory of the characterization of the low-regret optimal control see [10–12]. In this paper, we generalize to the ill-posed problems of elliptic type (5).

Proposition 5. *Problem (19)-(20) admits a unique solution u_ε^γ called the low-regret control.*

Proof. We have $\mathcal{J}_\varepsilon^\gamma(v) \geq -J_\varepsilon(0,0)$, $\forall v\in(L^2(\Gamma_0))^2$. Then

$$d_\varepsilon^\gamma = \inf_{v\in(L^2(\Gamma_0))^2}\mathcal{J}_\varepsilon^\gamma(v) \tag{25}$$

exists. Let $v_n = v_n(\varepsilon,\gamma)$ be a minimizing sequence such that $d_\varepsilon^\gamma = \lim_{n\to\infty}\mathcal{J}_\varepsilon^\gamma(v_n)$. Then we have

$$\begin{aligned}
-J_\varepsilon(0,0) &\leq J_\varepsilon(v_n,0) - J_\varepsilon(0,0)\\
&\quad + \frac{\varepsilon^2}{\gamma}\left(\|\xi_\varepsilon(v_n)\|^2_{L^2(\Gamma_1)} + \left\|\frac{\partial\xi_\varepsilon}{\partial\nu}(v_n)\right\|^2_{L^2(\Gamma_1)}\right)\\
&\leq d_\varepsilon^\gamma + 1.
\end{aligned} \tag{26}$$

And we deduce the bounds

$$\begin{aligned}
\|v_n\|_{(L^2(\Gamma_0))^2} &\le c_\varepsilon^\gamma, \\
\|z_\varepsilon(v_n,0)-z_d\|_{L^2(\Omega)} &\le c_\varepsilon^\gamma, \\
\frac{\varepsilon}{\sqrt{\gamma}}\|\xi_\varepsilon(v_n)\|_{L^2(\Gamma_1)} &\le c_\varepsilon^\gamma, \\
\frac{\varepsilon}{\sqrt{\gamma}}\left\|\frac{\partial\xi_\varepsilon}{\partial\nu}(v_n)\right\|_{L^2(\Gamma_1)} &\le c_\varepsilon^\gamma,
\end{aligned} \tag{27}$$

where the constant c_ε^γ (independent of n) is not the same each time.

Hence, there exists $u_\varepsilon^\gamma \in (L^2(\Gamma_0))^2$ such that $v_n(\varepsilon,\gamma) \rightharpoonup u_\varepsilon^\gamma$ weakly in the Hilbert space $(L^2(\Gamma_0))^2$. Also, $z_\varepsilon(v_n,0) \to z_\varepsilon(u_\varepsilon^\gamma,0)$ (continuity with respect to the data). We also deduce from the strict convexity of the cost function $\mathcal{J}_\varepsilon^\gamma$ that u_ε^γ is unique. □

Now we give the optimality system for the approximate low-regret control u_ε^γ. We denote $y_\varepsilon^\gamma := z_\varepsilon(u_\varepsilon^\gamma,0)$. Then, we proceed as in [20]. We first have the following.

Proposition 6. *The approached low-regret control $u_\varepsilon^\gamma := (u_{0\varepsilon}^\gamma, u_{1\varepsilon}^\gamma)$ solution to (19)-(20) is characterized by the unique solution $\{y_\varepsilon^\gamma, \xi_\varepsilon^\gamma, \rho_\varepsilon^\gamma, p_\varepsilon^\gamma\}$ of the optimality system*

$$\begin{aligned}
\Delta^2 y_\varepsilon^\gamma + \varepsilon y_\varepsilon^\gamma &= 0, \\
\Delta^2 \xi_\varepsilon^\gamma + \varepsilon \xi_\varepsilon^\gamma &= y_\varepsilon^\gamma, \\
\Delta^2 \rho_\varepsilon^\gamma + \varepsilon \rho_\varepsilon^\gamma &= 0, \\
\Delta^2 p_\varepsilon^\gamma + \varepsilon p_\varepsilon^\gamma &= y_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma \\
& \qquad \textit{in } \Omega,
\end{aligned}$$

$$\begin{aligned}
y_\varepsilon^\gamma - \frac{\partial}{\partial\nu}(\Delta y_\varepsilon^\gamma) &= u_{0\varepsilon}^\gamma, \\
\frac{\partial y_\varepsilon^\gamma}{\partial\nu} + \Delta y_\varepsilon^\gamma &= u_{1\varepsilon}^\gamma, \\
\xi_\varepsilon - \frac{\partial}{\partial\nu}(\Delta\xi_\varepsilon) &= 0, \\
\frac{\partial\xi_\varepsilon}{\partial\nu} + \Delta\xi_\varepsilon &= 0, \\
\rho_\varepsilon^\gamma - \frac{\partial}{\partial\nu}(\Delta\rho_\varepsilon^\gamma) &= 0, \\
\frac{\partial\rho_\varepsilon^\gamma}{\partial\nu} + \Delta\rho_\varepsilon^\gamma &= 0, \\
p_\varepsilon^\gamma - \frac{\partial}{\partial\nu}(\Delta p_\varepsilon^\gamma) &= 0, \\
\frac{\partial p_\varepsilon^\gamma}{\partial\nu} + \Delta p_\varepsilon^\gamma &= 0 \\
& \qquad \textit{on } \Gamma_0,
\end{aligned}$$

$$\begin{aligned}
\varepsilon y_\varepsilon^\gamma - \frac{\partial}{\partial\nu}(\Delta y_\varepsilon^\gamma) &= 0, \\
\varepsilon\frac{\partial y_\varepsilon^\gamma}{\partial\nu} + \Delta y_\varepsilon^\gamma &= 0, \\
\varepsilon\xi_\varepsilon - \frac{\partial}{\partial\nu}(\Delta\xi_\varepsilon) &= 0, \\
\varepsilon\frac{\partial\xi_\varepsilon}{\partial\nu} + \Delta\xi_\varepsilon &= 0, \\
\varepsilon\rho_\varepsilon^\gamma - \frac{\partial}{\partial\nu}(\Delta\rho_\varepsilon^\gamma) &= \frac{\varepsilon^2}{\gamma}\xi_\varepsilon, \\
\varepsilon\frac{\partial\rho_\varepsilon^\gamma}{\partial\nu} + \Delta\rho_\varepsilon^\gamma &= -\frac{\varepsilon^2}{\gamma}\frac{\partial\xi_\varepsilon}{\partial\nu} \\
\varepsilon p_\varepsilon^\gamma - \frac{\partial}{\partial\nu}(\Delta p_\varepsilon^\gamma) &= 0, \\
\varepsilon\frac{\partial p_\varepsilon^\gamma}{\partial\nu} + \Delta p_\varepsilon^\gamma &= 0 \\
& \qquad \textit{on } \Gamma_1,
\end{aligned} \tag{28}$$

with the adjoint equation

$$p_\varepsilon^\gamma + N_0 u_{0\varepsilon}^\gamma + N_1 u_{1\varepsilon}^\gamma = 0 \quad \textit{in } L^2(\Gamma_0). \tag{29}$$

Proof. Let u_ε^γ be the solution of (19)-(20) on $L^2(\Gamma_0)$. The Euler-Lagrange necessary condition gives for every $w := (w_0, w_1) \in (L^2(\Gamma_0))^2$

$$\begin{aligned}
&\langle y_\varepsilon^\gamma - z_d, z_\varepsilon(w,0)\rangle + N_0\langle u_{0\varepsilon}^\gamma, w_0\rangle_{\Gamma_0} + N_1\langle u_{1\varepsilon}^\gamma, w_1\rangle_{\Gamma_0} \\
&\quad + \left\langle \frac{\varepsilon^2}{\gamma}\xi_\varepsilon^\gamma, \xi_\varepsilon(w)\right\rangle_{\Gamma_1} + \left\langle \frac{\varepsilon^2}{\gamma}\frac{\partial\xi_\varepsilon^\gamma}{\partial\nu}, \frac{\partial\xi_\varepsilon}{\partial\nu}(w)\right\rangle_{\Gamma_1} \\
&= 0,
\end{aligned} \tag{30}$$

where $\xi_\varepsilon^\gamma := \xi_\varepsilon(u_\varepsilon^\gamma, 0)$. Denoting $\rho_\varepsilon^\gamma = \rho(u_\varepsilon^\gamma, 0)$ as the unique solution to

$$\begin{aligned}
\Delta^2\rho_\varepsilon^\gamma + \varepsilon\rho_\varepsilon^\gamma &= 0 \quad \text{in } \Omega, \\
\rho_\varepsilon^\gamma - \frac{\partial}{\partial\nu}(\Delta\rho_\varepsilon^\gamma) &= 0, \\
\frac{\partial\rho_\varepsilon^\gamma}{\partial\nu} + \Delta\rho_\varepsilon^\gamma &= 0 \\
& \qquad \text{on } \Gamma_0,
\end{aligned} \tag{31}$$

$$\begin{aligned}
\varepsilon\rho_\varepsilon^\gamma - \frac{\partial}{\partial\nu}(\Delta\rho_\varepsilon^\gamma) &= \frac{\varepsilon^2}{\gamma}\xi_\varepsilon^\gamma, \\
\varepsilon\frac{\partial\rho_\varepsilon^\gamma}{\partial\nu} + \Delta\rho_\varepsilon^\gamma &= -\frac{\varepsilon^2}{\gamma}\frac{\partial\xi_\varepsilon^\gamma}{\partial\nu} \\
& \qquad \text{on } \Gamma_1,
\end{aligned}$$

we have by the Green formula

$$\begin{aligned} 0 &= \left\langle \Delta^2 \rho_\varepsilon^\gamma + \varepsilon \rho_\varepsilon^\gamma, \xi_\varepsilon(w,0) \right\rangle \\ &= \langle \rho_\varepsilon^\gamma, z_\varepsilon(w,0) \rangle + \left\langle \frac{\varepsilon^2}{\gamma} \xi_\varepsilon^\gamma, \xi_\varepsilon(w,0) \right\rangle_{\Gamma_1} \\ &\quad + \left\langle \frac{\varepsilon^2}{\gamma} \frac{\partial \xi_\varepsilon^\gamma}{\partial \nu}, \frac{\partial \xi_\varepsilon}{\partial \nu}(w,0) \right\rangle_{\Gamma_1}. \end{aligned} \tag{32}$$

And as it is classical, we introduce the adjoint state $p_\varepsilon^\gamma := p(u_\varepsilon^\gamma, 0)$ defined by

$$\begin{aligned} &\Delta^2 p_\varepsilon^\gamma + \varepsilon p_\varepsilon^\gamma = y_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma \quad \text{in } \Omega, \\ &p_\varepsilon^\gamma - \frac{\partial}{\partial \nu}(\Delta p_\varepsilon^\gamma) = 0, \\ &\frac{\partial p_\varepsilon^\gamma}{\partial \nu} + \Delta p_\varepsilon^\gamma = 0 \\ &\qquad \text{on } \Gamma_0, \\ &\varepsilon p_\varepsilon^\gamma - \frac{\partial}{\partial \nu}(\Delta p_\varepsilon^\gamma) = 0, \\ &\varepsilon \frac{\partial p_\varepsilon^\gamma}{\partial \nu} + \Delta p_\varepsilon^\gamma = 0 \\ &\qquad \text{on } \Gamma_1, \end{aligned} \tag{33}$$

and, using again the Green formula, we obtain

$$\begin{aligned} \langle y_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma, z_\varepsilon(w,0) \rangle &= \langle p_\varepsilon^\gamma, w \rangle_{\Gamma_0}, \\ &\qquad \forall w \in \left(L^2(\Gamma_0)\right)^2. \end{aligned} \tag{34}$$

And then (30) becomes

$$\langle p_\varepsilon^\gamma + N_0 u_{0\varepsilon}^\gamma + N_1 u_{1\varepsilon}^\gamma, w \rangle_{\Gamma_0} = 0, \quad \forall w \in \left(L^2(\Gamma_0)\right)^2 \tag{35}$$

which is (29). □

4. Singular Optimality System (SOS)

In this section, we give the SOS for the low-regret control for the Cauchy problem (5). We first show the following estimates.

Lemma 7. *There is a positive constant C such that*

$$\begin{aligned} &\|u_{\varepsilon 0}^\gamma\|_{L^2(\Gamma_0)} \le C, \\ &\|u_{\varepsilon 1}^\gamma\|_{L^2(\Gamma_0)} \le C, \\ &\|y_\varepsilon^\gamma\|_{L^2(\Omega)} \le C, \\ &\frac{\varepsilon}{\sqrt{\gamma}} \|\xi_\varepsilon^\gamma\|_{L^2(\Gamma_1)} \le C, \\ &\frac{\varepsilon}{\sqrt{\gamma}} \left\| \frac{\partial \xi_\varepsilon^\gamma}{\partial \nu} \right\|_{L^2(\Gamma_1)} \le C. \end{aligned} \tag{36}$$

Proof. Since u_ε^γ is the approximate low-regret control, we have

$$J_\varepsilon^\gamma(u_\varepsilon^\gamma) \le J_\varepsilon^\gamma(v), \quad \forall v \in \left(L^2(\Gamma_0)\right)^2. \tag{37}$$

In the particular case where $v = 0$, we obtain

$$\begin{aligned} &\|y_\varepsilon^\gamma - z_d\|_{L^2(\Omega)}^2 + N_0 \|u_{\varepsilon 0}^\gamma\|_{L^2(\Gamma_0)}^2 + N_1 \|u_{\varepsilon 1}^\gamma\|_{L^2(\Gamma_0)}^2 \\ &\quad + \frac{\varepsilon^2}{\gamma} \left(\|\xi_\varepsilon^\gamma\|_{L^2(\Gamma_1)}^2 + \left\| \frac{\partial \xi_\varepsilon^\gamma}{\partial \nu} \right\|_{L^2(\Gamma_1)}^2 \right) \\ &\le \|z_\varepsilon(0,0) - z_d\|_{L^2(\Omega)}^2 \\ &\quad + \frac{\varepsilon^2}{\gamma} \left(\|\xi_\varepsilon(0,0)\|_{L^2(\Gamma_1)}^2 + \left\| \frac{\partial \xi_\varepsilon}{\partial \nu}(0,0) \right\|_{L^2(\Gamma_1)}^2 \right). \end{aligned} \tag{38}$$

But,

$$\begin{aligned} &z_\varepsilon(0,0) = 0 \quad \text{in } \Omega, \\ &\xi_\varepsilon(0,0) = \frac{\partial \xi_\varepsilon}{\partial \nu}(0,0) = 0 \quad \text{on } \Gamma_1. \end{aligned} \tag{39}$$

Then

$$\begin{aligned} &\|y_\varepsilon^\gamma - z_d\|_{L^2(\Omega)}^2 + N_0 \|u_{\varepsilon 0}^\gamma\|_{L^2(\Gamma_0)}^2 + N_1 \|u_{\varepsilon 1}^\gamma\|_{L^2(\Gamma_0)}^2 \\ &\quad + \frac{\varepsilon^2}{\gamma} \left(\|\xi_\varepsilon^\gamma\|_{L^2(\Gamma_1)}^2 + \left\| \frac{\partial \xi_\varepsilon^\gamma}{\partial \nu} \right\|_{L^2(\Gamma_1)}^2 \right) \le \|z_d\|_{L^2(\Omega)}^2 = C. \end{aligned} \tag{40}$$

□

Remark 8. From Section 3.1, we showed how to come back from the bi-Laplacian problem to the original one. We will use these techniques in this last part.

Theorem 9. *The low-regret control u^γ for problem (5) is characterized by the unique solution $\{y^\gamma, \xi^\gamma, \rho^\gamma, p^\gamma\}$ of the optimality system:*

$$\begin{aligned} &\Delta y^\gamma = 0, \\ &\Delta \xi^\gamma = 0, \\ &\Delta \rho^\gamma = 0, \\ &\Delta p^\gamma = y^\gamma - z_d + \rho^\gamma \\ &\qquad \textit{in } \Omega, \\ &y^\gamma = u_0^\gamma, \\ &\xi^\gamma = 0, \\ &\rho^\gamma = 0, \\ &p^\gamma = 0, \\ &\frac{\partial y^\gamma}{\partial \nu} = u_1^\gamma, \\ &\frac{\partial \xi^\gamma}{\partial \nu} = 0, \end{aligned}$$

$$\frac{\partial \rho^\gamma}{\partial \nu} = 0, \quad \frac{\partial p^\gamma}{\partial \nu} = 0 \quad \textit{on } \Gamma_0, \tag{41}$$

with the adjoint equation

$$p^\gamma + N_0 u_0^\gamma + N_1 u_1^\gamma = 0 \quad \textit{in } L^2(\Gamma_0). \tag{42}$$

Proof. From the optimality system (28) in Proposition 6, we deduce that y_ε^γ is solution of the system

$$\begin{gathered}\Delta^2 y_\varepsilon^\gamma + \varepsilon y_\varepsilon^\gamma = 0, \quad \text{in } \Omega,\\ y_\varepsilon^\gamma - \frac{\partial}{\partial \nu}(\Delta y_\varepsilon^\gamma) = u_{0\varepsilon}^\gamma, \quad \frac{\partial y_\varepsilon^\gamma}{\partial \nu} + \Delta y_\varepsilon^\gamma = u_{1\varepsilon}^\gamma \quad \text{on } \Gamma_0,\\ \varepsilon y_\varepsilon^\gamma - \frac{\partial}{\partial \nu}(\Delta y_\varepsilon^\gamma) = 0, \quad \varepsilon \frac{\partial y_\varepsilon^\gamma}{\partial \nu} + \Delta y_\varepsilon^\gamma = 0, \quad \text{on } \Gamma_1.\end{gathered} \tag{43}$$

As in Section 3.1, we denote

$$\eta_\varepsilon^\gamma = \Delta y_\varepsilon^\gamma. \tag{44}$$

Then system (43) is written as

$$\begin{gathered}\Delta \eta_\varepsilon^\gamma + \varepsilon y_\varepsilon^\gamma = 0, \quad \text{in } \Omega,\\ y_\varepsilon^\gamma - \frac{\partial \eta_\varepsilon^\gamma}{\partial \nu} = u_{0\varepsilon}^\gamma, \quad \frac{\partial y_\varepsilon^\gamma}{\partial \nu} + \eta_\varepsilon^\gamma = u_{1\varepsilon}^\gamma \quad \text{on } \Gamma_0,\\ \varepsilon y_\varepsilon^\gamma - \frac{\partial \eta_\varepsilon^\gamma}{\partial \nu} = 0, \quad \varepsilon \frac{\partial y_\varepsilon^\gamma}{\partial \nu} + \eta_\varepsilon^\gamma = 0, \quad \text{on } \Gamma_1.\end{gathered} \tag{45}$$

From estimates (36) of Lemma 7, sequence (y_ε^γ) is bounded in $L^2(\Omega)$ by constant M. Hence, there exists a subsequence still denoted by (y_ε^γ), such that

$$y_\varepsilon^\gamma \rightharpoonup y^\gamma \quad \text{weakly in } L^2(\Omega) \tag{46}$$

as $\varepsilon \to 0$. We deduce from the first equation in (45) that

$$\|\Delta \eta_\varepsilon^\gamma\| \le \varepsilon \|y_\varepsilon^\gamma\| \le \varepsilon M \longrightarrow 0. \tag{47}$$

That is,

$$\Delta \eta_\varepsilon^\gamma \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega). \tag{48}$$

Using the same arguments and from the last two equations in (45) we have

$$\frac{\partial \eta_\varepsilon^\gamma}{\partial \nu} \rightharpoonup 0, \quad \eta_\varepsilon^\gamma \rightharpoonup 0 \quad \text{in } L^2(\Gamma_1) \tag{49}$$

when $\varepsilon \to 0$. We resume by

$$\Delta \eta^\gamma = 0 \quad \text{in } \Omega, \quad \frac{\partial \eta^\gamma}{\partial \nu} = 0, \quad \eta^\gamma = 0 \quad \text{on } \Gamma_1. \tag{50}$$

Using the unique continuation theorem of Mizohata [19], we deduce from (50) that we also have

$$\eta^\gamma \equiv 0 \quad \text{everywhere on } \overline{\Omega}. \tag{51}$$

Then,

$$\frac{\partial \eta^\gamma}{\partial \nu} = \eta^\gamma = 0 \quad \text{on } \Gamma_0. \tag{52}$$

From another side, estimates (36) also give

$$(u_{\varepsilon 0}^\gamma, u_{\varepsilon 1}^\gamma) \rightharpoonup (u_0^\gamma, u_1^\gamma) \quad \text{weakly in } L^2(\Gamma_0) \times L^2(\Gamma_0). \tag{53}$$

We come back to the notation $\eta^\gamma = \Delta y^\gamma$. System (45) transforms to

$$\Delta y^\gamma = 0, \quad \text{in } \Omega, \quad y^\gamma = u_0^\gamma, \quad \frac{\partial y^\gamma}{\partial \nu} = u_1^\gamma \quad \text{on } \Gamma_0. \tag{54}$$

Again, we use the estimates of Lemma 7 and from (36) we deduce the following limits:

$$\frac{\varepsilon}{\sqrt{\gamma}} \xi_\varepsilon^\gamma \rightharpoonup \lambda_0^\gamma \quad \text{weakly in } L^2(\Gamma_1), \quad \frac{\varepsilon}{\sqrt{\gamma}} \frac{\partial \xi_\varepsilon^\gamma}{\partial \nu} \rightharpoonup \lambda_1^\gamma \quad \text{weakly in } L^2(\Gamma_1). \tag{55}$$

Hence, from the optimality system (28) we deduce that both

$$\frac{\varepsilon^2}{\gamma}\xi_\varepsilon^\gamma,\ \frac{\varepsilon^2}{\gamma}\frac{\partial \xi_\varepsilon^\gamma}{\partial \nu} \text{ tend to } 0 \quad \text{when } \varepsilon \longrightarrow 0. \tag{56}$$

Now, as above we use the same arguments and we obtain $\rho_\varepsilon \rightharpoonup \rho^\gamma$ in $L^2(\Omega)$ solution to (41). Finally we have

$$\begin{gathered} \Delta \xi^\gamma = y^\gamma, \quad \text{in } \Omega, \\ \xi^\gamma = 0, \\ \frac{\partial \xi^\gamma}{\partial \nu} = 0 \\ \text{on } \Gamma_0. \end{gathered} \tag{57}$$

From another side, we deduce from (29)

$$p_\varepsilon^\gamma = -N_0 u_{0\varepsilon}^\gamma - N_1 u_{1\varepsilon}^\gamma \quad \text{in } L^2(\Gamma_0) \tag{58}$$

and finally, from (36) and (53), that

$$p_\varepsilon^\gamma \rightharpoonup p^\gamma = -N_0 u_0^\gamma - N_1 u_1^\gamma \quad \text{weakly in } L^2(\Gamma_0). \tag{59}$$

□

We now easily deduce the singular optimality system of the no-regret control for (5) by the following.

Corollary 10. *The no-regret control u for problem (5) is characterized by the unique solution* $\{y, \xi, \rho, p\}$ *of the optimality system:*

$$\begin{gathered} \Delta y = 0, \\ \Delta \xi = 0, \\ \Delta \rho = 0, \\ \Delta p = y - z_d + \rho \\ \text{in } \Omega, \\ y = u_0, \\ \xi = 0, \\ \rho = 0, \\ p = 0, \\ \frac{\partial y}{\partial \nu} = u_1, \\ \frac{\partial \xi}{\partial \nu} = 0, \\ \frac{\partial \rho}{\partial \nu} = 0, \\ \frac{\partial p}{\partial \nu} = 0 \\ \text{on } \Gamma_0, \end{gathered} \tag{60}$$

with the adjoint equation

$$p + N_0 u_0 + N_1 u_1 = 0 \quad \text{in } L^2(\Gamma_0). \tag{61}$$

Proof. We here pass to the limit in $\gamma \rightarrow 0$. From the equalities in (41) we easily deduce the limits:

$$\begin{gathered} \xi^\gamma \rightharpoonup \xi = 0, \\ \rho^\gamma \rightharpoonup \rho = 0, \\ p^\gamma \rightharpoonup p = 0 \\ \text{on } \Gamma_0. \end{gathered} \tag{62}$$

Then from the adjoint equation (42) we obtain

$$(u_0^\gamma, u_1^\gamma) \rightharpoonup (u_0, u_1) \quad \text{weakly in } L^2(\Gamma_0) \times L^2(\Gamma_0). \tag{63}$$

Hence sequences $(y^\gamma)_\gamma$ and $(\partial y^\gamma/\partial \nu)_\gamma$ are bounded in $L^2(\Gamma_0)$ and we obtain the weak convergence:

$$\begin{gathered} y^\gamma \rightharpoonup y = u_0, \\ \frac{\partial y^\gamma}{\partial \nu} \rightharpoonup \frac{\partial y}{\partial \nu} = u_1 \\ \text{on } \Gamma_0. \end{gathered} \tag{64}$$

The same applies to the other limits. □

5. Conclusion

In this work, we obtain a characterization of the control for the ill-posed Laplacian Cauchy problem, using the no-regret concept. The method consists in considering the elliptic Cauchy problem as a singular limit of sequence of well-posed elliptic problems.

The regularization approach generates incomplete information which implies the use of the low-regret approach. In case of no perturbation, the no-regret optimal control function is the same as the classical control one.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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