Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2015, Article ID 641481, 6 pages
http://dx.doi.org/10.1155/2015/641481

Research Article

On Stability of Basis Property of Root Vectors System of the Sturm-Liouville Operator with an Integral Perturbation of Conditions in Nonstrongly Regular Samarskii-Ionkin Type Problems

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Received 19 July 2015; Revised 19 November 2015; Accepted 3 December 2015

Academic Editor: Patricia J. Y. Wong

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We study a question on stability and instability of basis property of system of eigenfunctions and associated functions of the double differentiation operator with an integral perturbation of Samarskii-Ionkin type boundary conditions.

1. Introduction

In space $L_2(0,1)$ we consider an operator L_0, generated by the following ordinary differential expression:

$$L_0(u) \equiv -u''(x) + q(x)u(x), \quad q(x) \in C[0,1], \ 0 < x < 1 \tag{1}$$

and the boundary value conditions of the general form:

$$U_j(u) = a_{j1}u'(0) + a_{j2}u'(1) + a_{j3}u(0) + a_{j4}u(1) = 0, \quad j = 1,2. \tag{2}$$

In the case when the boundary conditions (2) are strongly regular, the results of Mikhailov [1] and Kesellman [2] provide the Riesz basis property in $L_2(0,1)$ of the eigenfunction and associated functions (E&AF) system of the problem. In the case when the boundary conditions are regular but not strongly regular, the question on basis property of E&AF system is not yet completely resolved.

We introduce the matrix of coefficients of the boundary conditions (2):

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}. \tag{3}$$

By $A(ij)$ we denote the matrix composed of the ith and jth columns of matrix A, $A_{ij} = \det A(ij)$. Let the boundary conditions (2) be regular but not strongly regular. According to [3, page 73], if the following conditions hold:

$$\begin{gathered} A_{12} = 0, \\ A_{14} + A_{23} \neq 0, \\ A_{14} + A_{23} = \mp (A_{13} + A_{24}), \end{gathered} \tag{4}$$

then the boundary conditions (2) are equivalent regular, but not strongly boundary, conditions.

In [4] Makin suggested dividing all regular, but not strongly regular, boundary conditions into four types:

(I) $A_{14} = A_{23}$, $A_{34} = 0$;
(II) $A_{14} = A_{23}$, $A_{34} \neq 0$;

(III) $A_{14} \neq A_{23}$, $A_{34} = 0$;

(IV) $A_{14} \neq A_{23}$, $A_{34} \neq 0$.

For example, boundary conditions with periodical or antiperiodical conditions form the (I)-type and can be determined in the following form:

$$A_{14} = A_{23}, \quad A_{34} = 0. \tag{5}$$

That is, $a_{11} = -a_{12}$, $a_{13} = a_{14} = a_{21} = a_{22} = 0$, and $a_{23} = -a_{24}$. These conditions will be equivalent to boundary value conditions, given by matrix A, where the following options are possible:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \text{ are periodical or} \quad A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \text{ are anti-periodical,} \tag{6}$$

and the boundary conditions with "the lowest coefficients" form the (II)-type. The boundary value conditions defined as $A_{14} \neq A_{23}$, $A_{34} = 0$ form the (III)-type. These conditions are always equivalent to boundary conditions, given by matrix A:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7}$$

This case will be the aim of our research in this paper.

Moreover, Makin [4] allocated the one type of non-strongly regular boundary value conditions, when E&AF system of the spectral problem

$$L_0(u) \equiv -u''(x) + q(x)u(x) = \lambda u(x), \quad q(x) \in C[0,1], \; 0 < x < 1, \tag{8}$$

with boundary conditions of the general form (2) forms Riesz basis at any potentials $q(x)$.

When $q(x) \equiv 0$, the problem about basis property of E&AF system of the problem with general regular boundary conditions has been completely resolved in [5].

In [6, 7] questions on convergence of eigenfunction expansion of the Dirac operator in vector-matrix form and the Hill operator, forming Reitz basis in $L_2(0,1)$, with regular, but not strongly regular, boundary value conditions have been considered.

Questions on basisness of eigenfunctions of the differential operators with involution have been studied in [8–10].

2. Statement of the Problem and Main Results

The spectral problem (8)-(2) with boundary conditions of the (III)-type when $q(x) \equiv 0$ is a non-self-adjoint problem in $L_2(0,1)$. For the case of non-self-adjoint initial operator the question about preservation of the basis properties with some (weak in a certain sense) perturbation was studied in [11]. Riesz basis property of eigenfunctions and associated functions of periodic and antiperiodic Sturm-Liouville problems was considered in [12]. We obtain asymptotic formulas for eigenvalues and eigenfunctions of periodic and antiperiodic Sturm-Liouville problems with boundary conditions, which are not strongly regular, when $q(x)$ is a complex-valued absolutely continuous function, and $q(0) \neq q(1)$. Moreover, using these asymptotic formulas, we prove that the root functions of these operators form a Riesz basis in the space $L_2(0,1)$ [13, 14].

In [15, 16] questions on stability of basis properties of the periodic problem for (8) were investigated with integral perturbation of the boundary conditions (2), when $j = 2$, of the (I)-type, that is, at the conditions $A_{14} = A_{23}$, $A_{34} = 0$. Moreover, in [17], the similar questions have been studied when $q(x) \equiv 0$. In this paper we consider the spectral problem close to research of [17] when $q(x) \equiv 0$, with integral perturbation of the boundary conditions (2) when $j = 2$, belonging to the (III)-type:

$$L_1(u) \equiv -u''(x) = \lambda u(x), \quad 0 < x < 1, \tag{9}$$

$$U_1(u) \equiv u'(0) - u'(1) = 0, \tag{10}$$

$$U_2(u) \equiv u(0) = \int_0^1 \overline{p(x)} u(x)\, dx, \quad p(x) \in L_2(0,1). \tag{11}$$

If $p(x) \equiv 0$, then problem (9)–(11) is called Samarskii-Ionkin problem [11].

From [18] it follows that the E&AF system of problem (9)–(11) is complete and minimal in $L_2(0,1)$. Moreover, the E&AF system at any $p(x)$ forms Riesz basis with brackets. Our aim is to show that the basis property in $L_2(0,1)$ of the E&AF system of problem (9)–(11) is not stable at small changes of kernel $p(x)$ of integral perturbation.

In [19] the construction method of the characteristic determinant of the spectral problem with integral perturbation of the boundary conditions has been suggested. The spectral properties of nonlocal problems have been considered in [20].

The basis properties in $L_p(-1,1)$ of root functions of the nonlocal problem for the equations with involution have been studied in [21]. Instability of basis properties of root functions of the Schrodinger operator with nonlocal perturbation of the boundary condition has been investigated in [22]. In [23, 24] they extended some spectral properties of regular Sturm-Liouville problems to the special type discontinuous boundary value problem, consisting of the Sturm-Liouville equation together with eigenparameter that depended on boundary conditions and two supplementary transmission conditions; we construct the resolvent operator and prove theorems on expansions in terms of eigenfunctions in modified Hilbert space $L_2(a,b)$.

3. Characteristic Determinant of the Problem

In this section we use the method of our paper [19] to construct the characteristic determinant of the problem with integral perturbation of the boundary condition.

One aspect of this problem is the fact that an adjoint problem to (9)–(11) is the spectral problem for the loaded differential equation [16]:

$$L_1^*(v) = -v''(x) + p(x)v'(0) = \overline{\lambda}v(x),$$
$$V_1(v) \equiv v'(1) = 0, \tag{12}$$
$$V_2(v) \equiv v(0) - v(1) = 0.$$

Firstly, we construct the characteristic determinant of the spectral problem. Representing the general solution of (9) by the following formula when $\lambda \neq 0$,

$$u(x,\lambda) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x, \tag{13}$$

and satisfying it by the boundary conditions (10) and (11), we obtain the linear system concerning the coefficients C_k:

$$\sqrt{\lambda}C_1 \sin\sqrt{\lambda} + \sqrt{\lambda}C_2\left(1-\cos\sqrt{\lambda}\right) = 0,$$
$$C_1\left[1 - \int_0^1 \overline{p(x)}\cos\sqrt{\lambda}x\,dx\right] \tag{14}$$
$$- C_2\int_0^1 \overline{p(x)}\sin\sqrt{\lambda}x\,dx = 0.$$

Its determinant is a characteristic determinant of problem (9)–(11):

$$\Delta_1(\lambda) = \begin{vmatrix} \sqrt{\lambda}\sin\sqrt{\lambda} & \sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right) \\ 1 - \int_0^1 \overline{p(x)}\cos\sqrt{\lambda}x\,dx & -\int_0^1 \overline{p(x)}\sin\sqrt{\lambda}x\,dx \end{vmatrix}. \tag{15}$$

When $p(x) = 0$, we get the characteristic determinant of the unperturbed Samarskii-Ionkin problem. Denote it by $\Delta_0(\lambda) = 1 - \cos\sqrt{\lambda}$. The number $\lambda_0^0 = 0$ is simple eigenvalue of the unperturbed Samarskii-Ionkin problem, and $u_0(x) = +\sqrt{3}x$ is the corresponding eigenfunction. Other eigenvalues of the unperturbed problem (9)–(11) are double: $\lambda_k^0 = (2k\pi)^2$, $k = 1,2,3,\ldots$, and $u_{k0}^0 = \sqrt{2}\sin 2k\pi x$ is the eigenfunction corresponding to them; $u_{k1}^0 = (\sqrt{2}/2)x\cos 2k\pi x$ is the associated function. Due to the biorthogonal property $(u_{k1}^0, v_{k1}^0) = 1$, we have the eigenfunction $v_{k1}^0 = 4\sqrt{2}\cos 2k\pi x$ and the associated function $v_{k0}^0 = 2\sqrt{2}(1-x)\sin 2k\pi x$ of the problem adjoint to the Samarskii-Ionkin problem.

We represent function $p(x)$ as the biorthogonal expansion to the Fourier series by system $\{v_{k0}^0, v_{k1}^0\}$:

$$p(x) = \sum_{k=1}^{\infty} a_{k0}v_{k0}^0 + \sum_{k=0}^{\infty} a_{k1}v_{k1}^0$$
$$= \sum_{k=1}^{\infty} a_{k0}2\sqrt{2}(1-x)\sin 2k\pi x \tag{16}$$
$$+ \sum_{k=0}^{\infty} a_{k1}4\sqrt{2}\cos 2k\pi x.$$

Using (16), we find more convenient representation of determinant $\Delta_1(\lambda)$. To do it, first, we evaluate integrals in (15). Simple calculations show that

$$\int_0^1 \overline{p(x)}\cos\sqrt{\lambda}x\,dx$$
$$= 4\sqrt{2\lambda}\sin\sqrt{\lambda}\sum_{k=0}^{\infty}\frac{\overline{a_{k1}}}{\lambda-(2k\pi)^2}$$
$$+ 2\sqrt{2}\sum_{k=1}^{\infty}\left(\frac{2k\pi\overline{a_{k0}}}{\lambda-(2k\pi)^2}\left[1-\frac{2\sqrt{\lambda}\sin\sqrt{\lambda}}{\lambda-(2k\pi)^2}\right]\right),$$
$$\int_0^1 \overline{p(x)}\sin\sqrt{\lambda}x\,dx \tag{17}$$
$$= -2\sqrt{2\lambda}\sin\sqrt{\lambda}\sum_{k=0}^{\infty}\frac{\overline{a_{k0}}}{\lambda-(2k\pi)^2}$$
$$- 2\sqrt{2}\sin\sqrt{\lambda}\sum_{k=1}^{\infty}\frac{2k\pi\overline{a_{k0}}}{\lambda-(2k\pi)^2}$$
$$+ 4\sqrt{2\lambda}\left(1-\cos\sqrt{\lambda}\right)\sum_{k=0}^{\infty}\frac{\overline{a_{k1}}}{\lambda-(2\pi k)^2}.$$

Using the obtained results, determinant (15) is reduced to the following form by the standard conversions:

$$\Delta_1(\lambda) = \Delta_0(\lambda)\cdot A(\lambda),$$
$$A(\lambda) = \left[1 + 4\sqrt{2}\pi\sum_{k=1}^{\infty}\overline{a_{k0}}\frac{k}{\lambda-(2k\pi)^2}\right]. \tag{18}$$

$\Delta_0(\lambda) = 0$ implies that $\lambda_k^{(1)} = \lambda_k^0 = (2k\pi)^2$. Quantities $\lambda_k^{(2)} = [2k\pi + \overline{a_{k0}}(\sqrt{2} + \underline{\underline{O}}(1/\sqrt{k}))]^2$ are roots of equation $A(\lambda) = 0$.

These roots are eigenvalues of the perturbed spectral problems (9)–(11).

Therefore, we prove the following.

Theorem 1. *Characteristic determinant of the spectral problem with perturbed boundary value conditions (9)–(11) can be represented as (18), where $\Delta_0(\lambda)$ is the characteristic determinant of the unperturbed Samarskii-Ionkin spectral problem, a_{k0} are Fourier coefficients of the biorthogonal expansion (16) of the function $p(x)$ by the E&AF system of adjoint unperturbed Samarskii-Ionkin spectral problem.*

Function $A(\lambda)$ from (18) has a pole of the first order at points $\lambda = \lambda_k^0$, and function $\Delta_0(\lambda)$ has zeroes of the second order at these points. Hence, function $\Delta_1(\lambda)$ represented by formula (18) is an entire analytical function of variable λ.

The characteristic determinant, which is an entire analytical function, related to the problem on eigenvalues of differential operator of the third order with nonlocal boundary conditions has been studied in [25].

4. Partial Cases of the Characteristic Determinant

If coefficients $a_{j0} = 0$ of (16) for all indexes j, then $\lambda_j^1 = \lambda_j^0$ is a double eigenvalue of the perturbed problem (9)–(11).

More simple characteristic determinant (18) is in the case when $p(x)$ is represented as (16) with the finite first sum. That is, when there exists a number N such that $a_{k0} = 0$ for all $k > N$. In this case formula (18) takes the following form:

$$\Delta_1(\lambda) = \Delta_0(\lambda)\left[1 + 4\sqrt{2}\pi\sum_{k=1}^{N}\overline{a_{k0}}\frac{k}{\lambda - (2k\pi)^2}\right]. \quad (19)$$

From this partial case (18) it is easy to establish the following.

Corollary 2. *For any numbers given in advance, that is, complex $\widehat{\lambda}$ and positive integer $\widehat{m}$, there always exists function $p(x)$ such that $\widehat{\lambda}$ will be eigenvalue of problem (9)–(11) of $\widehat{m}$ multiplicity.*

From analysis of (19) it is also easy to see that $\Delta_1(\lambda_k^0) = 0$ for all $k > N$. That is, all eigenvalues λ_k^0, $k > N$, of the Samarskii-Ionkin problem are eigenvalues of the perturbed spectral problem (9)–(11). Moreover, it is not hard to see that multiplicity of the eigenvalues λ_k^0, $k > N$, is also preserved.

Furthermore, from the orthogonal condition $p(x) \perp u_{j0}^0$, $p(x) \perp u_{j1}^0$ at all $j > N$, it follows that, in this case,

$$\int_0^1 \overline{p(x)}u_{j0}^0(x)\,dx = \int_0^1 \overline{p(x)}u_{j1}^0(x)\,dx = 0. \quad (20)$$

Therefore, eigenfunctions $u_{j0}^0(x)$ and associated functions $u_{j1}^0(x)$ of the unperturbed Samarskii-Ionkin problem for all $j > N$ satisfy the boundary conditions (10), (11), and perturbed Samarskii-Ionkin spectral problem consequently. Hence, the functions are eigenfunctions and associated functions of the perturbed problem (9)–(11). Accordingly, in this case, E&AF system of the perturbed problem (9)–(11) and the E&AF system of the unperturbed Samarskii-Ionkin problem (forming Riesz basis) differ from each other only in a finite number of the first members. Hence, the E&AF system of the perturbed system (9)–(11) also forms Riesz basis in $L_2(0,1)$.

A set of the functions $p(x)$, represented as the finite series (16), is dense in $L_2(0,1)$. Therefore, we prove the following.

Theorem 3. *Let $A_{14} \neq A_{23}$, $A_{34} = 0$; that is, the boundary conditions (10), (11) are equivalent to the type-(III) with the integral perturbation. Then the set of the functions $p(x) \in L_2(0,1)$ such that the E&AF system of the perturbed Samarskii-Ionkin problem (9)–(11) forms Riesz basis in $L_2(0,1)$ is dense in $L_2(0,1)$.*

5. Instability of Basis Property

Now we prove that the basis property of E&AF system of the perturbed Samarskii-Ionkin problem (9)–(11) is instable at arbitrarily small integral perturbation of the boundary condition (11).

Theorem 4. *If $A_{14} \neq A_{23}$, $A_{34} = 0$, that is, the boundary conditions (10), (11) belong to the type-(III), then the set of the functions $p(x) \in L_2(0,1)$ such that E&AF system of the perturbed Samarskii-Ionkin problem (9)–(11) does not form even a simple basis in $L_2(0,1)$ is also dense in $L_2(0,1)$.*

Proof. It is obvious that the set of the functions $p(x) \in L_2(0,1)$ represented as (16), coefficients of which asymptotically (i.e., beginning with some number) have the property $a_{k0} \neq 0$, $a_{k1} = 0$, is dense in $L_2(0,1)$. Therefore, to prove the theorem, it is enough to show that for these functions $p(x)$ the E&AF system of the problem does not form a simple basis.

Let j be a large enough number such that $a_{j0} \neq 0$, $a_{j1} = 0$. Then from (18) it is not hard to see that $\lambda_j^0 = (2j\pi)^2$ is a simple eigenvalue of problem (9)–(11). By the direct calculation it is easy to get that the corresponding eigenfunction to this value of the adjoint problem (12) is $v_j^1(x) = \sqrt{2}\cos(2j\pi x)$ and $\|v_j^1(x)\|^2 = 1$.

We find an eigenfunction of problem (9)–(11). For large enough $\lambda = \lambda_j^0 = (2j\pi)^2$ the first equation of the system from Section 3 becomes an identity, and the second equation is transformed into the following form:

$$C_1\left[1 - \frac{\sqrt{2}}{4\pi}\frac{\overline{a_{j0}}}{j} + \frac{\sqrt{2}}{\pi}\sum_{k=1,k\neq j}^{\infty}\overline{a_{k0}}\frac{k}{j^2 - k^2}\right] - C_2\frac{\overline{a_{j0}}}{\sqrt{2}} = 0. \quad (21)$$

Since $a_{j0} \neq 0$, then we write C_2 by C_1. Therefore, eigenfunction of problem (9)–(11) has the following form:

$$u_j^1(x) = C_1\left\{\cos(2j\pi x) + \frac{\sqrt{2}}{\overline{a_{j0}}}\left[1 - \frac{\sqrt{2}}{4\pi}\frac{\overline{a_{j0}}}{j} + \frac{\sqrt{2}}{\pi}\sum_{k=1,k\neq j}^{\infty}\overline{a_{k0}}\frac{k}{j^2 - k^2}\right] \cdot \sin(2j\pi x)\right\}. \quad (22)$$

Choose constant C_1 from the biorthogonal condition $(u_j^1(x), v_j^1(x)) = 1$. It is easy to see that $C_1 = \sqrt{2}$. Finally, we find the eigenfunction of problem (9)–(11):

$$u_j^1(x) = \sqrt{2}\cos(2j\pi x) - \left[\frac{1}{\sqrt{2}j\pi} - \frac{2}{\overline{a_{j0}}}\left(1 - \frac{\sqrt{2}}{\pi}\sum_{k=1,k\neq j}^{\infty}\overline{a_{k0}}\frac{k}{j^2 - k^2}\right)\right] \cdot \sin(2j\pi x). \quad (23)$$

By the direct calculation we find its norm in $L_2(0,1)$:

$$\left\|u_j^1(x)\right\|^2 = 1 + \frac{1}{2}\left|\frac{1}{\sqrt{2}j\pi} - \frac{2}{a_{j0}}\left(1 - \frac{\sqrt{2}}{\pi}\sum_{k=1,k\neq j}^{\infty}\overline{a_{k0}}\frac{k}{j^2-k^2}\right)\right|^2. \tag{24}$$

From the Young theorem [26, Theorem 276, page 240] it follows that

$$\lim_{j\to+\infty}\sum_{k=1,k\neq j}^{\infty}\overline{a_{k0}}\frac{k}{j^2-k^2} = 0. \tag{25}$$

Therefore,

$$\lim_{j\to+\infty}\left\|u_j^1(x)\right\|^2 = 1 + 2\lim_{j\to+\infty}\left|\frac{1}{a_{j0}}\right|^2 = +\infty. \tag{26}$$

Consequently, $\lim_{j\to\infty}\|u_j^1\|\cdot\|\vartheta_j^1\| = \infty$. Thus, necessary condition of basis property does not hold (see [11] and references in it) and, therefore, it does not form even a simple basis in $L_2(0,1)$.

Theorem 4 is proved. □

Since adjoint operators at the same time have the Riesz basis property of root functions, therefore, we obtain the following.

Corollary 5. *Let $A_{14} \neq A_{23}$, $A_{34} = 0$; that is, the boundary conditions (10), (11) belong to the type-(III). Then the set* $\mathbf{P}$ *of the functions $p(x) \in L_2(0,1)$ such that the system of eigenfunctions of the problem (12) for the loaded differential equation forms Riesz basis in $L_2(0,1)$ is everywhere dense in $L_2(0,1)$. The set $L_2(0,1)\backslash\mathbf{P}$ is also everywhere dense in $L_2(0,1)$.*

The results of the paper, in contrast to [18], show instability of basis property of root functions of the problem with an integral perturbation of boundary conditions of the type-(III), which are regular, but not strongly regular.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant no. 0825/GF4).

References

[1] V. P. Mikhailov, "On Riesz basis in L_2 (0,1)," *Doklady Mathematics*, vol. 144, no. 5, pp. 981–984, 1962 (Russian).

[2] G. M. Kesellman, "About unconditional convergence of expansion by eigenfunctions of some differential operators," *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, no. 2, pp. 82–93, 1964 (Russian).

[3] M. A. Naimark, *Linear Differential Operators*, Nauka, Moscow, Russia, 1969 (Russian).

[4] A. S. Makin, "On spectral expansions, responsible to non self-adjoint Sturm-Liouville operator," *Doklady Mathematics*, vol. 406, no. 1, pp. 21–24, 2006 (Russian).

[5] P. Lang and J. Locker, "Spectral theory of two-point differential operators determined by $-D^2$. I. Spectral properties," *Journal of Mathematical Analysis and Applications*, vol. 141, no. 2, pp. 538–558, 1989.

[6] P. Djakov and B. Mityagin, "Unconditional convergence of spectral decompositions of 1D Dirac operators with regular boundary conditions," *Indiana University Mathematics Journal*, vol. 61, no. 1, pp. 359–398, 2012.

[7] P. Djakov and B. Mityagin, "Convergence of spectral decompositions of Hill operators with trigonometric polynomial potentials," *Mathematische Annalen*, vol. 351, no. 3, pp. 509–540, 2011.

[8] A. M. Sarsenbi and A. A. Tengaeva, "On the basis properties of the root functions of two generalized spectral problems," *Differential Equations*, vol. 48, no. 2, pp. 306–308, 2012.

[9] M. A. Sadybekov and A. M. Sarsenbi, "Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution," *Differential Equations*, vol. 48, no. 8, pp. 1112–1118, 2012.

[10] A. A. Kopzhassarova, A. L. Lukashov, and A. M. Sarsenbi, "Spectral properties of non-self-adjoint perturbations for a spectral problem with involution," *Abstract and Applied Analysis*, vol. 2012, Article ID 590781, 5 pages, 2012.

[11] V. A. Il'in and L. V. Kritskov, "Properties of spectral expansions corresponding to non-self-adjoint differential operators," *Journal of Mathematical Sciences*, vol. 116, no. 5, pp. 3489–3550, 2003.

[12] O. A. Veliev and A. A. Shkalikov, "On the Riesz basis property of eigen- and associated functions of periodic and anti-periodic Sturm-Liouville problems," *Matematicheskie Zametki*, vol. 85, no. 5, pp. 671–686, 2009 (Russian).

[13] A. A. Kirac, "Riesz basis property of the root functions of non-selfadjoint operators with regular boundary conditions," *International Journal of Mathematical Analysis*, vol. 3, no. 21–24, pp. 1101–1109, 2009.

[14] A. A. Kıraç, "On the Riesz basisness of systems composed of root functions of periodic boundary value problems," *Abstract and Applied Analysis*, vol. 2015, Article ID 945049, 7 pages, 2015.

[15] M. A. Sadybekov and N. S. Imanbaev, "On the basis property of root functions of a periodic problem with an integral perturbation of the boundary condition," *Differential Equations*, vol. 48, no. 6, pp. 896–900, 2012.

[16] N. S. Imanbaev and M. A. Sadybekov, "On spectral properties of a periodic problem with an integral perturbation of the boundary condition," *Eurasian Mathematical Journal*, vol. 4, no. 3, pp. 53–62, 2013.

[17] A. S. Makin, "About non-local perturbation of periodic problem on eigenvalues," *Differential Equations*, vol. 42, no. 4, pp. 560–562, 2006.

[18] A. A. Shkalikov, "On basis property of eigenfunctions of ordinary differential operators with integral boundary value conditions," *Vestnik MGU. Mathematics and Mechanics*, vol. 6, pp. 12–21, 1982.

[19] N. S. Imanbaev and M. A. Sadybekov, "Characteristic determinant of the spectral problem for the ordinary differential operator with the boundary load," *AIP Conference Proceedings*, vol. 1611, pp. 261–265, 2014, International Conference on Analysis and Applied Mathematics (ICAAM '14).

[20] T. S. Kal'menov, M. A. Sadybekov, and A. M. Sarsenbi, "Biorthogonal expansions in root functions of differential operators," *Differential Equations*, vol. 47, no. 1, pp. 144–148, 2011.

[21] L. V. Kritskov and A. M. Sarsenbi, "Spectral properties of a nonlocal problem for a second-order differential equation with an involution," *Differential Equations*, vol. 51, no. 8, pp. 984–990, 2015.

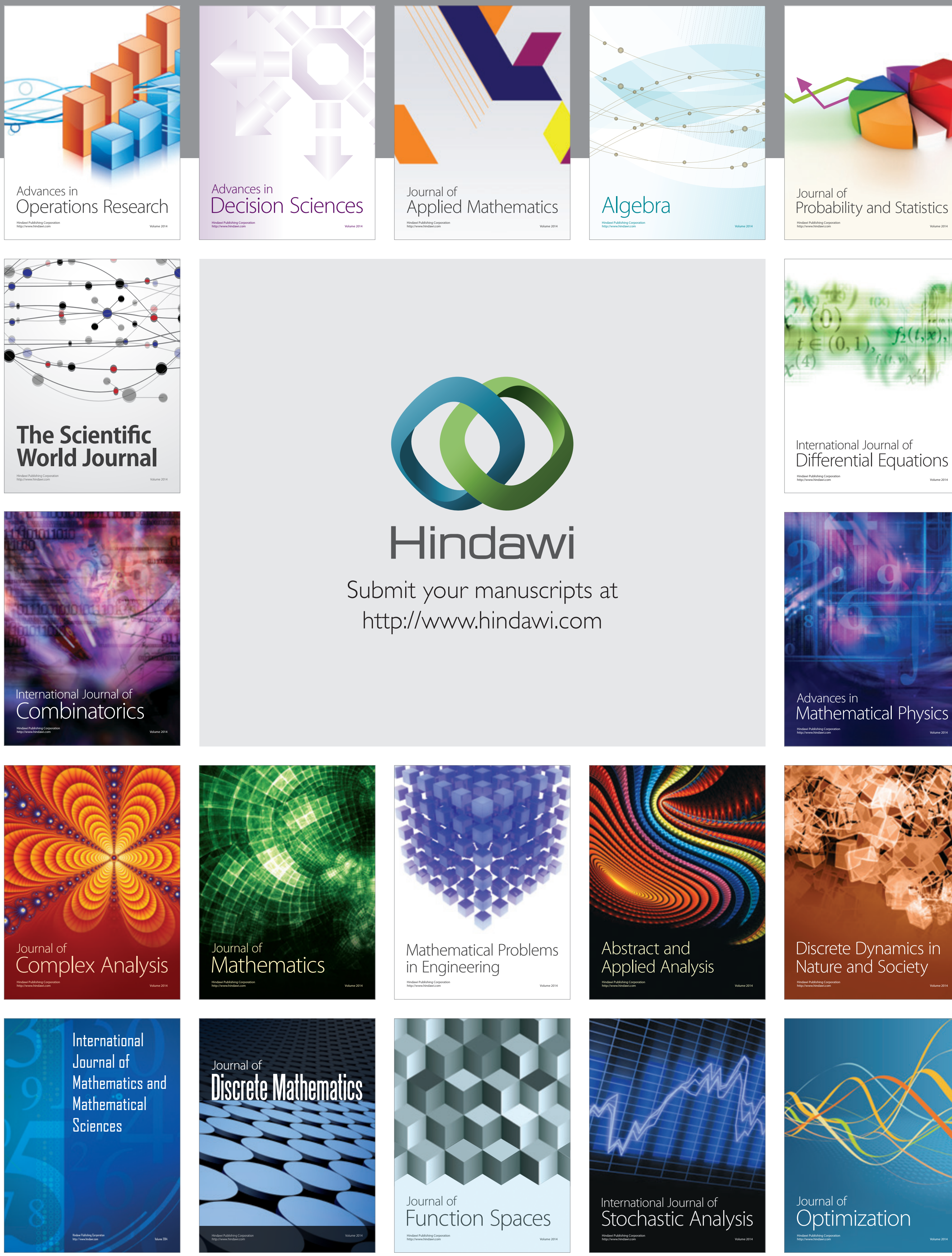
[22] N. S. Imanbayev, B. T. Kalimbetov, and A. M. Sarsenbi, "Study basicity of root functions of the Schrodinger operator with a non-Local perturbation," *International Journal of Pure and Applied Mathematics*, vol. 89, no. 3, pp. 433–438, 2013.

[23] N. Bondarenko, "An inverse spectral problem for the matrix Sturm-Liouville operator with a Bessel-type singularity," *International Journal of Differential Equations*, vol. 2015, Article ID 647396, 4 pages, 2015.

[24] O. S. Mukhtarov and K. Aydemir, "Eigenfunction expansion for Sturm-Liouville problems with transmission conditions at one interior point," *Acta Mathematica Scientia B*, vol. 35, no. 3, pp. 639–649, 2015.

[25] N. S. Imanbaev, B. E. Kanguzhin, and B. T. Kalimbetov, "On zeros of the characteristic determinant of the spectral problem for a third-order differential operator on a segment with nonlocal boundary conditions," *Advances in Difference Equations*, vol. 2013, article 110, 2013.

[26] G. G. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, vol. 456, Moscow, Russia, 1948 (Russian).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