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Research Article

Fréchet Differentiability for a Damped Kirchhoff-Type Equation and Its Application to Bilinear Minimax Optimal Control Problems

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We consider a damped Kirchhoff-type equation with Dirichlet boundary conditions. The objective is to show the Fréchet differentiability of a nonlinear solution map from a bilinear control input to the solution of a Kirchhoff-type equation. We use this result to formulate the minimax optimal control problem. We show the existence of optimal pairs and find their necessary optimality conditions.

1. Introduction

Let Ω be an open bounded set of $\mathbf{R}^n (n \leq 3)$ with a smooth boundary Γ. We set $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$ for $T > 0$. We consider a strongly damped Kirchhoff-type equation described by the following Dirichlet boundary value problem:

$$\begin{gathered} y'' - \left(1 + \int_\Omega |\nabla y|^2 \, dx\right) \Delta y - \mu \Delta y' = \mathbf{U} y + f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \\ y'(0,x) = y_1(x) \\ \text{in } \Omega, \end{gathered} \tag{1}$$

where $' = \partial/\partial t$, y is the displacement of a string (or membrane), $\mu > 0$, f is a forcing function, and $\mathbf{U}$ is a bilinear forcing term, which is usually a bilinear control variable that acts as a multiplier of the displacement term. $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^n$. As is well known by Kirchhoff [1], the nonlinear part of (1) represents an extension effect of a vibrating string (or membrane). Many kinds of Kirchhoff-type equations have been research subject of many researchers (see Arosio [2], Spagnolo [3], Pohozaev [4], Lions [5], Nishihara and Yamada [6], and references therein).

From a physical perspective, the damping of (1) represents an internal friction in an elastic string (or membrane) that makes the vibration smooth. Therefore, we can obtain the well-posedness in the Hadamard sense under sufficiently smooth initial conditions (see [7]). Based on this result, Hwang and Nakagiri [8] set up optimal control problems developed by Lions [9] with (1) using distributed forcing controls. They proved the Gâteaux differentiability of the quasilinear solution map from the control variable to the solution and applied the result to derive the necessary optimality conditions for optimal control in some observation cases.

It is important and challenging to extend the optimal control theory to practical nonlinear partial differential equations. There are several studies on semilinear partial differential equations (see [10]). Indeed, the extension of the theory to quasilinear equations is much more restrictive because the differentiability of a solution map is quite dependent on the model due to the strong nonlinearity. Only a few studies have investigated this topic (see [8, 11, 12]). Thus, the differentiability of a solution map in any sense is important to study optimal control or identification problems. In most cases, Gâteaux differentiability may be enough to solve a quadratic cost optimal control problem as in [8]. However, to study the problem in more general cost function like nonquadratic or nonconvex functions, the Fréchet differentiability of a solution map is more desirable.

In this paper, we show the Fréchet differentiability of the solution map of (1): $\mathbf{U} \longrightarrow y$ from the bilinear control input variables to the solutions of (1). In the author's knowledge, the Fréchet differentiability of a quasilinear solution map is not studied yet. Based on the result, we construct and solve a bilinear minimax optimal control problem on (1). For the study, we refer to the linear results from Belmiloudi [13], in which the author considered some linear parabolic partial differential equations as the state equations for the problem. Minimax control framework has been used by many researchers for various control problems. There are many literatures related to the minimax control problems. We can refer to just a few: Arada and Raymond [14], Lasiecka and Triggiani [15], and Li and Yong [16].

In this paper, the minimax control framework was employed to take into account the undesirable effects of system disturbance (or noise) in control inputs such that a cost function achieves its minimum even when the worst disturbances of the system occur. For this purpose, we replace the bilinear multiplier $\mathbf{U}$ in (1) by $u + v$, where u is a control variable that belongs to the admissible control set $\mathscr{U}_{ad}$ and v is a disturbance (or noise) that belongs to the admissible disturbance set $\mathscr{V}_{ad}$. We introduce the following cost function to be minimized within $\mathscr{U}_{ad}$ and maximized within $\mathscr{V}_{ad}$:

$$J(u,v) = \frac{1}{2}\left\|\mathscr{C}y - Y_d\right\|_M^2 + \frac{\alpha}{2}\|u\|_{L^2(Q)}^2 - \frac{\beta}{2}\|v\|_{L^2(Q)}^2, \quad (2)$$

where y is a solution of (1), M is a Hilbert space of observation variables, $\mathscr{C}$ is an operator from the solution space of (1) to M, $Y_d \in M$ is a desired value, and the positive constants α and β are the relative weights of the second and third terms on the RHS of (2).

As mentioned, another goal of this paper is to find and characterize the optimal controls of the cost function (2) for the worst disturbances through control input in (1). This leads to the problem of finding the saddle points of the cost function (2). First, we prove the existence of an admissible control $u^* \in \mathscr{U}_{ad}$ and disturbance (or noise) $v^* \in \mathscr{V}_{ad}$ such that (u^*, v^*) is a saddle point of the functional $J(u,v)$ of (2). That is,

$$J(u^*,v) \le J(u^*,v^*) \le J(u,v^*), \qquad \forall (u,v) \in \mathscr{U}_{ad} \times \mathscr{V}_{ad}. \quad (3)$$

Secondly, we derive an optimality condition for (u^*, v^*) in (3). In this paper, we use the terminology *optimal pair* to represent such a saddle point (u^*, v^*) in (3). To prove the existence of an optimal pair (u^*, v^*) satisfying (3), we follow the arguments given by Belmiloudi [13], in which the author employed the minimax theorem in infinite dimensions given by Barbu and Precupanu [17]. Next, we derive the necessary optimal conditions for some observation cases that should be satisfied by the optimal pairs in these observation cases. To derive these conditions, we refer to the studies about bilinear optimal control problems where the state equation is linear partial differential equations such as the reaction diffusion equation or Kirchhoff plate equation (see [13, 18–20] and references therein).

We now explain the content of this paper. In Section 2, we prove the well-posedness of (1) in the Hadamard sense under sufficiently smooth initial conditions, including a stability estimate from the data space to the solution space. In Section 3, we shall show that the solution map of (1): $\mathbf{U} \longrightarrow y$ is Fréchet differentiable. In Section 4, we shall study the minimax optimal control problems: By using the Fréchet differentiability of the solution maps $u \longrightarrow y$ and $v \longrightarrow y$, we prove that the maps $u \longrightarrow J$ and $v \longrightarrow J$ are convex and concave, respectively, under the assumptions that α, β are sufficiently large. And with an assumption on the operator $\mathscr{C}$ in (2), we prove the maps $u \longrightarrow J$ and $v \longrightarrow J$ are lower and upper semicontinuous, respectively. As a result, we can prove the existence of an optimal pair. Next, we derive the necessary optimal conditions for some practical observation cases by employing associate adjoint systems. Especially, we use a first-order Volterra integrodifferential equation as a proper adjoint equation in the velocity's observation case, which is another novelty of this paper.

2. Preliminaries

Throughout this paper, we use C as a generic constant. Let X be a Banach space. We denote its topological dual as X' and the duality pairing between X' and X by $\langle\cdot,\cdot\rangle_{X',X}$. We also introduce the following abbreviations:

$$\begin{aligned} L^p &= L^p(\Omega), \\ H^k &= H^k(\Omega), \\ \|\cdot\|_p &= \|\cdot\|_{L^p}, \end{aligned} \quad (4)$$

where $p \ge 1$. H_0^k is the completions of $C_0^\infty(\Omega)$ in H^k for $k \ge 1$. Let the scalar product on L^2 be $(\cdot,\cdot)_2$. From Poincare's inequality and the regularity theory for elliptic boundary value problems (cf. Temam [21, p. 150]), the scalar products on H_0^1 and $D(\Delta) = H^2 \cap H_0^1$ can be endowed as follows:

$$((\psi,\phi))_{H_0^1} = (\nabla\psi, \nabla\phi)_2, \quad \forall \psi,\phi \in H_0^1; \quad (5)$$

$$((\psi,\phi))_{D(\Delta)} = (\Delta\psi, \Delta\phi)_2, \quad \forall \psi,\phi \in D(\Delta). \quad (6)$$

Then we know that

$$\begin{aligned} \|\psi\|_{H_0^1} &= \|\nabla\psi\|_2, \quad \forall \psi \in H_0^1, \\ \|\psi\|_{D(\Delta)} &= \|\Delta\psi\|_2, \quad \forall \psi \in D(\Delta). \end{aligned} \quad (7)$$

The duality pairing between H_0^1 and H^{-1} is denoted by $\langle\phi,\psi\rangle_{1,-1}$. It is clear that

$$D(\Delta) \hookrightarrow H_0^1 \hookrightarrow L^2 \hookrightarrow H^{-1}. \quad (8)$$

Each space is dense in the following one, and the injections are continuous and compact. According to Adams [22], we know that the embeddings

$$H_0^1 \hookrightarrow L^p, \quad \left(i.e.,\ \|\psi\|_p \le C\|\nabla\psi\|_2,\ \forall\psi \in H_0^1\right),\ (1 \le p < 6), \tag{9}$$

$$D(\Delta) \hookrightarrow C^0\left(\overline{\Omega}\right), \quad \left(i.e.,\ \|\phi\|_{C^0(\overline{\Omega})} \le C\|\Delta\phi\|_2,\ \forall\phi \in D(\Delta)\right) \tag{10}$$

are compact when $n \le 3$.

The solution space $S(0,T)$ of (1) of strong solutions is defined by

$$S(0,T) = \left\{g \mid g \in L^2(0,T;D(\Delta)),\ g' \in L^2(0,T;D(\Delta)),\ g'' \in L^2(Q)\right\} \tag{11}$$

which is endowed with the norm

$$\|g\|_{S(0,T)} = \left(\|g\|^2_{L^2(0,T;D(\Delta))} + \|g'\|^2_{L^2(0,T;D(\Delta))} + \|g''\|^2_{L^2(Q)}\right)^{1/2}, \tag{12}$$

where g' and g'' denote the first and second order distributional derivatives of g.

Definition 1. A function y is said to be a strong solution of (1) if $y \in S(0,T)$ and y satisfies

$$\begin{aligned} &y''(t) - \left(1 + \|\nabla y(t)\|_2^2\right)\Delta y(t) - \mu\Delta y'(t) \\ &\quad = \mathbf{U}(t)\,y(t) + f(t), \quad a.e.\ t \in [0,T], \\ &y(0) = y_0, \\ &y'(0) = y_1. \end{aligned} \tag{13}$$

From Dautray and Lions [23, p.480] and Lions and Magnes [24], we remark that

$$S(0,T) \hookrightarrow C([0,T];D(\Delta)) \cap C^1\left([0,T];H_0^1\right). \tag{14}$$

The following variational formulation is used to define the *weak* solution of (1).

Definition 2. A function y is said to be a weak solution of (1) if $y \in W(0,T) \equiv \{g \mid g \in L^2(0,T;H_0^1),\ g' \in L^2(0,T;H_0^1),\ g'' \in L^2(0,T;H^{-1})\}$ and y satisfies

$$\begin{aligned} &\left\langle y''(\cdot),\phi\right\rangle_{-1,1} + \left(1 + \|\nabla y(\cdot)\|_2^2\right)(\nabla y(\cdot),\nabla\phi)_2 \\ &\quad + \mu\left(\nabla y'(\cdot),\nabla\phi\right)_2 = \langle \mathbf{U}(\cdot)\,y(\cdot) + f(\cdot),\phi\rangle_{-1,1} \\ &\qquad \forall\phi \in H_0^1 \text{ in the sense of } \mathscr{D}'(0,T), \\ &y(0) = y_0, \\ &y'(0) = y_1. \end{aligned} \tag{15}$$

The following is the well-known Gronwall inequality.

Lemma 3. *Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies the following differentiable inequality for a.e. $t \in [0,T]$:*

$$\eta'(t) \le \phi(t)\,\eta(t) + \psi(t), \tag{16}$$

where ϕ and ψ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) \le e^{\int_0^t \phi(s)ds}\left(\eta(0) + \int_0^t \psi(s)\,ds\right). \tag{17}$$

Proof. See Evans [25, p.624]. □

Throughout this paper, we will omit writing the integral variables in the definite integral without any confusion. Referring to [7] and the previous result of [8], we can obtain the following theorem on existence, uniqueness, and regularity of a solution of (1).

Theorem 4. *Assume that $(y_0, y_1, f) \in D(\Delta)\times H_0^1\times L^2(Q)$, and $\mathbf{U} \in L^\infty(Q)$. Then (1) has a unique strong solution $y \in S(0,T)$. Moreover, the solution mapping $p = (y_0, y_1, f, \mathbf{U}) \longrightarrow y(p)$ of $\mathscr{P} \equiv D(\Delta) \times H_0^1 \times L^2(Q) \times L^\infty(Q)$ into $S(0,T)$ is locally Lipschitz continuous. Let $p_1 = (y_0^1, y_1^1, f_1, \mathbf{U_1}) \in \mathscr{P}$ and $p_2 = (y_0^2, y_1^2, f_2, \mathbf{U_2}) \in \mathscr{P}$. The following is satisfied:*

$$\begin{aligned} \|y(p_1) - y(p_2)\|_{S(0,T)} &\le C\left(\left\|\Delta\left(y_0^1 - y_0^2\right)\right\|_2^2 \right. \\ &\quad + \left\|\nabla\left(y_1^1 - y_1^2\right)\right\|_2^2 + \|f_1 - f_2\|^2_{L^2(Q)} \\ &\quad \left. + \|\mathbf{U_1} - \mathbf{U_2}\|^2_{L^\infty(Q)}\right)^{1/2} \equiv C\|p_1 - p_2\|_{\mathscr{P}}, \end{aligned} \tag{18}$$

where $C > 0$ is a constant depending on the data.

Proof. From [7], for each fixed $\mathbf{U} \in L^\infty(Q)$ in (1), we can infer that (1) admits a unique strong solution $y \in S(0,T)$ under the data condition $(y_0, y_1, f) \in D(\Delta) \times H_0^1 \times L^2(Q)$.

Based on this result, for each $p_1 = (y_0^1, y_1^1, f_1, \mathbf{U}_1) \in P$ and $p_2 = (y_0^2, y_1^2, f_2, \mathbf{U}_2) \in P$, we prove the inequality (18). For that purpose, we denote $y_1 - y_2 \equiv y(p_1) - y(p_2)$ by ψ. Then, from (1), we can know that ψ satisfies the following:

$$\begin{aligned} &\psi'' - \left(1 + \|\nabla y_1\|_2^2\right)\Delta\psi - \mu\Delta\psi' \\ &\quad = \epsilon(\psi) + \mathbf{U}_1\psi + (\mathbf{U}_1 - \mathbf{U}_2)\,y_2 + f_1 - f_2 \quad \text{in } Q, \\ &\psi = 0 \quad \text{on } \Sigma, \\ &\psi(0) = y_0^1 - y_0^2, \\ &\psi'(0) = y_1^1 - y_1^2 \qquad \text{in } \Omega, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \epsilon(\psi) &= \left(\|\nabla y_1\|_2^2 - \|\nabla y_2\|_2^2\right)\Delta y_2 \\ &= (\nabla\psi, \nabla y_1 + \nabla y_2)_2\,\Delta y_2. \end{aligned} \tag{20}$$

In estimating ψ in (19), we can refer to the previous results [8, Theorem 2.1] to obtain the following inequality:

$$\begin{aligned}
&\left\|\nabla\psi'(t)\right\|_2^2 + \left\|\Delta\psi(t)\right\|_2^2 + \int_0^t \left\|\Delta\psi'\right\|_2^2 ds \\
&\le C\Big(\left\|\Delta\left(y_0^1 - y_0^2\right)\right\|_2^2 + \left\|\nabla\left(y_1^1 - y_1^2\right)\right\|_2^2 \\
&+ \left\|\left(\mathbf{U}_1 - \mathbf{U}_2\right) y_2 + f_1 - f_2\right\|_{L^2(Q)}^2\Big).
\end{aligned} \tag{21}$$

Since $\|y_2\|_{S(0,T)} \le C(y_0^2, y_1^2, f_2)$ and $S(0,T) \hookrightarrow L^2(Q)$, we have

$$\begin{aligned}
\left\|\left(\mathbf{U}_1 - \mathbf{U}_2\right) y_2\right\|_{L^2(Q)} &\le \left\|\mathbf{U}_1 - \mathbf{U}_2\right\|_{L^\infty(Q)} \left\|y_2\right\|_{L^2(Q)} \\
&\le C\left\|y_2\right\|_{S(0,T)} \left\|\mathbf{U}_1 - \mathbf{U}_2\right\|_{L^\infty(Q)} \\
&\le C\left\|\mathbf{U}_1 - \mathbf{U}_2\right\|_{L^\infty(Q)}.
\end{aligned} \tag{22}$$

Together with (21) and (22), we can deduce the following:

$$\begin{aligned}
&\left\|\nabla\psi'(t)\right\|_2^2 + \left\|\Delta\psi(t)\right\|_2^2 + \int_0^t \left\|\Delta\psi'\right\|_2^2 ds \\
&\le C\Big(\left\|\Delta\left(y_0^1 - y_0^2\right)\right\|_2^2 + \left\|\nabla\left(y_1^1 - y_1^2\right)\right\|_2^2 \\
&+ \left\|f_1 - f_2\right\|_{L^2(Q)}^2 + \left\|\mathbf{U}_1 - \mathbf{U}_2\right\|_{L^\infty(Q)}^2\Big) \equiv C\left\|p_1 \right.\\
&\left. - p_2\right\|_{\mathscr{P}}^2.
\end{aligned} \tag{23}$$

Applying (23) to (19), we have

$$\left\|\psi''\right\|_{L^2(Q)} \le C\left\|p_1 - p_2\right\|_{\mathscr{P}}. \tag{24}$$

From (23) and (24), we can obtain

$$\left\|\psi\right\|_{S(0,T)} \le C\left\|p_1 - p_2\right\|_{\mathscr{P}}. \tag{25}$$

This completes the proof. □

Corollary 5. *For $p_1 = (y_0, y_1, f, \mathbf{U}_1)$, $p_2 = (y_0, y_1, f, \mathbf{U}_2) \in \mathscr{P}$, the following inequality is satisfied:*

$$\left\|y(p_1) - y(p_2)\right\|_{S(0,T)} \le C\left\|\mathbf{U}_1 - \mathbf{U}_2\right\|_{L^2(Q)}, \tag{26}$$

where $C > 0$ is a constant depending on the data and $y(p_1)$ and $y(p_2)$ are the solutions of (1) corresponding to p_1 and p_2, respectively.

Proof. We denote $y(p_1) - y(p_2)$ by ψ. Then, as in the proof of Theorem 4, we can know that ψ satisfies the following:

$$\begin{aligned}
&\psi'' - \left(1 + \left\|\nabla y_1\right\|_2^2\right)\Delta\psi - \mu\Delta\psi' \\
&\quad = \epsilon(\psi) + \mathbf{U}_1\psi + \left(\mathbf{U}_1 - \mathbf{U}_2\right) y_2 \quad \text{in } Q, \\
&\psi = 0 \quad \text{on } \Sigma, \\
&\psi(0) = 0, \\
&\psi'(0) = 0 \\
&\qquad\qquad\qquad \text{in } \Omega,
\end{aligned} \tag{27}$$

where $\epsilon(\psi)$ is given in (20). Estimating ψ in (27) as in the proof of Theorem 4, we can arrive at

$$\left\|\psi\right\|_{S(0,T)} \le C\left\|\left(\mathbf{U}_1 - \mathbf{U}_2\right) y(p_2)\right\|_{L^2(Q)}. \tag{28}$$

Thanks to the fact that $y(p_2) \in S(0,T) \hookrightarrow C([0,T]; D(\Delta))$ and (10), we can know that $S(0,T) \hookrightarrow C^0(\overline{Q})$. Thus we have

$$\begin{aligned}
\text{RHS of (28)} &\le C\left\|y(p_2)\right\|_{C^0(\overline{Q})} \left\|\mathbf{U}_1 - \mathbf{U}_2\right\|_{L^2(Q)} \\
&\le C\left\|y(p_2)\right\|_{S(0,T)} \left\|\mathbf{U}_1 - \mathbf{U}_2\right\|_{L^2(Q)} \\
&\le C\left\|p_2\right\|_{\mathscr{P}} \left\|\mathbf{U}_1 - \mathbf{U}_2\right\|_{L^2(Q)}.
\end{aligned} \tag{29}$$

Consequently, from (28) and (29), we have (26).

This completes the proof. □

3. Fréchet Differentiability of the Nonlinear Solution Map

In this section, we study the Fréchet differentiability of the nonlinear solution map. The Fréchet differentiability of the solution map plays an important role in many applications. Let $\mathscr{F} = L^\infty(Q)$. We consider the nonlinear solution map from $u \in \mathscr{F}$ to $y(u) \in S(0,T)$, where $y(u)$ is the solution of

$$\begin{aligned}
&y''(u) - \left(1 + \left\|\nabla y(u)\right\|_2^2\right)\Delta y(u) - \mu\Delta y'(u) \\
&\quad = u y(u) + f \quad \text{in } Q, \\
&y(u) = 0 \quad \text{on } \Sigma, \\
&y(u; 0, x) = y_0(x), \\
&y'(u; 0, x) = y_1(x) \\
&\qquad\qquad\qquad \text{in } \Omega.
\end{aligned} \tag{30}$$

Based on Theorem 4, for fixed $(y_0, y_1, f) \in D(\Delta) \times H_0^1 \times L^2(Q)$, we know that the solution map $\mathscr{F} \longrightarrow S(0,T)$, which maps from the term $u \in \mathscr{F}$ of (30) to $y(u) \in S(0,T)$, is well defined and continuous. We define the Fréchet differentiability of the nonlinear solution map as follows.

Definition 6. The solution map $u \longrightarrow y(u)$ of $\mathscr{F}$ into $S(0,T)$ is said to be Fréchet differentiable on $\mathscr{F}$ if for any $u \in \mathscr{F}$ there exists a $T(u) \in \mathscr{L}(\mathscr{F}, S(0,T))$ such that, for any $w \in \mathscr{F}$,

$$\begin{aligned}
&\frac{\left\|y(u+w) - y(u) - T(u)w\right\|_{S(0,T)}}{\|w\|_{\mathscr{F}}} \longrightarrow 0 \\
&\qquad\qquad \text{as } \|w\|_{\mathscr{F}} \longrightarrow 0.
\end{aligned} \tag{31}$$

The operator $T(u)$ is called the Fréchet derivative of y at u, which we denote by $Dy(u)$, and $T(u)w = Dy(u)w \in S(0,T)$ is called the Fréchet derivative of y at u in the direction of $w \in \mathscr{F}$.

Theorem 7. *The solution map $u \longrightarrow y(u)$ of $\mathscr{F}$ to $S(0,T)$ is Fréchet differentiable on $\mathscr{F}$ and the Fréchet derivative of $y(u)$*

at u in the direction $w \in \mathcal{F}$, that is to say $z = Dy(u)w$, is the solution of

$$\begin{aligned}
&z'' - \left(1 + \|\nabla y(u)\|_2^2\right)\Delta z - 2\left(\nabla y(u), \nabla z\right)_2 \Delta y(u) \\
&\quad - \mu \Delta z' = uz + wy(u) \quad \text{in } Q, \\
&z = 0 \quad \text{on } \Sigma, \\
&z(0,x) = 0, \\
&z'(0,x) = 0 \\
&\qquad\qquad\qquad\qquad \text{in } \Omega.
\end{aligned} \tag{32}$$

We prove this theorem by two steps:

(i) For any $w \in \mathcal{F}$, (32) admits a unique solution $z \in S(0,T)$. That is, there exists an operator $T \in \mathcal{L}(\mathcal{F}, S(0,T))$ satisfying $Tw = z(= z(w))$.

(ii) We show that $\|y(u+w) - y(u) - z\|_{S(0,T)} = o(\|w\|_{\mathcal{F}})$ as $\|w\|_{\mathcal{F}} \longrightarrow 0$.

Proof. (i) Let

$$\begin{aligned}
\mathcal{G}(y(u), z) &:= \left(1 + \|\nabla y(u)\|_2^2\right)\Delta z \\
&\quad + 2\left(\nabla y(u), \nabla z\right)_2 \Delta y(u).
\end{aligned} \tag{33}$$

Then from Theorem 4 and (14), we can estimate the above as follows:

$$\begin{aligned}
&\|\mathcal{G}(y(u), z)\|_2 \\
&\quad \le \left(1 + \|y(u)\|_{C([0,T];H_0^1)}^2\right)\|\Delta z\|_2 \\
&\qquad + 2\|y(u)\|_{C([0,T];H_0^1)}\|\nabla z\|_2\|y(u)\|_{C([0,T];D(\Delta))} \\
&\quad \le (\text{with (14) and (8)}) \\
&\quad \le \left(1 + \|y(u)\|_{S(0,T)}^2\right)\|\Delta z\|_2 + C\|y(u)\|_{S(0,T)}^2\|\Delta z\|_2 \\
&\quad \le C\left(1 + \|y(u)\|_{S(0,T)}^2\right)\|\Delta z\|_2 \\
&\quad \le C\left(1 + \|(y_0, y_1, f, u)\|_{\mathcal{P}}^2\right)\|\Delta z\|_2.
\end{aligned} \tag{34}$$

Hence, by (34) we know that

$$\mathcal{G}(y(u), \cdot) \in \mathcal{L}\left(D(\Delta), L^2\right). \tag{35}$$

To estimate the solution z of (32), we take the scalar product of (32) with $-\Delta z' - \Delta z$ in L^2 :

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla z'\right\|_2^2 + \frac{\mu}{2}\frac{d}{dt}\|\Delta z\|_2^2 + \mu\left\|\Delta z'\right\|_2^2 \\
&\quad = \left(z'', \Delta z\right)_2 - \left(\mathcal{G}(y(u), z), \Delta z' + \Delta z\right)_2 \\
&\qquad - \left(uz + wy(u), \Delta z' + \Delta z\right)_2.
\end{aligned} \tag{36}$$

Integrating (36) over $[0,t]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\nabla z'(t)\right\|_2^2 + \frac{\mu}{2}\|\Delta z(t)\|_2^2 + \mu\int_0^t \left\|\Delta z'\right\|_2^2 ds \\
&\quad = -\left(\nabla z'(t), \nabla z(t)\right)_2 + \int_0^t \left\|\nabla z'\right\|_2^2 ds \\
&\qquad - \int_0^t \left(\mathcal{G}(y(u), z), \Delta z' + \Delta z\right)_2 ds \\
&\qquad - \int_0^t \left(uz + wy(u), \Delta z' + \Delta z\right)_2 ds.
\end{aligned} \tag{37}$$

The right hand side of (37) can be estimated as follows:

$$\begin{aligned}
&\left|\left(\nabla z'(t), \nabla z(t)\right)_2\right| = \left|\left(\nabla z'(t), \int_0^t \nabla z' ds\right)_2\right| \\
&\quad \le \left\|\nabla z'(t)\right\|_2 \left\|\int_0^t \nabla z' ds\right\|_2 \\
&\quad \le \sqrt{T}\left\|\nabla z'(t)\right\|_2 \left\|\nabla z'\right\|_{L^2(0,t;L^2)} \\
&\quad \le (\text{with the Young inequality}) \\
&\quad \le \epsilon\left\|\nabla z'(t)\right\|_2^2 + \frac{T}{\epsilon}\int_0^t \left\|\nabla z'\right\|_2^2 ds;
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\left|\int_0^t \left(\mathcal{G}(y(u), z), \Delta z' + \Delta z\right)_2 ds\right| \\
&\quad \le \int_0^t \|\mathcal{G}(y(u), z)\|_2 \left(\left\|\Delta z'\right\|_2 + \|\Delta z\|_2\right) ds \\
&\quad \le (\text{with (35)}) \le C\int_0^t \left(\|\Delta z\|_2 \left\|\Delta z'\right\|_2 + \|\Delta z\|_2^2\right) ds \\
&\quad \le (\text{with the Young inequality}) \\
&\quad \le \epsilon\int_0^t \left\|\Delta z'(t)\right\|_2^2 ds + C\int_0^t \|\Delta z\|_2^2 ds;
\end{aligned} \tag{39}$$

$$\begin{aligned}
&\left|\int_0^t \left(uz, \Delta z' + \Delta z\right)_2 ds\right| \\
&\quad \le \int_0^t \|uz\|_2 \left(\left\|\Delta z'\right\|_2 + \|\Delta z\|_2\right) ds \\
&\quad \le \|u\|_{\mathcal{F}}\int_0^t \|z\|_2 \left(\left\|\Delta z'\right\|_2 + \|\Delta z\|_2\right) ds \le (\text{with (8)}) \\
&\quad \le C\int_0^t \left(\|\Delta z\|_2 \left\|\Delta z'\right\|_2 + \|\Delta z\|_2^2\right) ds \\
&\quad \le (\text{with the Young inequality}) \\
&\quad \le \epsilon\int_0^t \left\|\Delta z'(t)\right\|_2^2 ds + C\int_0^t \|\Delta z\|_2^2 ds;
\end{aligned} \tag{40}$$

$$\begin{aligned}
&\left|\int_0^t \left(wy(u), \Delta z' + \Delta z\right)_2 ds\right| \\
&\quad \le \int_0^t \|wy(u)\|_2 \left(\left\|\Delta z'\right\|_2 + \|\Delta z\|_2\right) ds \\
&\quad \le (\text{with the Young inequality}) \\
&\quad \le \epsilon \int_0^t \left\|\Delta z'(t)\right\|_2^2 ds + C\int_0^t \|\Delta z\|_2^2 ds \\
&\qquad + C\int_0^t \|wy(u)\|_2^2 ds.
\end{aligned} \tag{41}$$

Considering (38)-(41) and taking $\epsilon = (1/6)\min\{1/2, \mu/2\}$, we can obtain the following from (37):

$$\begin{aligned}
&\left\|\nabla z'(t)\right\|_2^2 + \|\Delta z(t)\|_2^2 + \int_0^t \left\|\Delta z'\right\|_2^2 ds \\
&\quad \le C\int_0^t \left(\left\|\nabla z'\right\|_2^2 + \|\Delta z\|_2^2\right) ds + C\|wy(u)\|_{L^2(Q)}^2.
\end{aligned} \tag{42}$$

Applying Lemma 3 to (42), we obtain

$$\begin{aligned}
&\left\|\nabla z'(t)\right\|_2^2 + \|\Delta z(t)\|_2^2 + \int_0^t \left\|\Delta z'\right\|_2^2 ds \\
&\quad \le C\|wy(u)\|_{L^2(Q)}^2.
\end{aligned} \tag{43}$$

In view of (32), (43) implies that

$$\left\|z''\right\|_{L^2(Q)} \le C\|wy(u)\|_{L^2(Q)}. \tag{44}$$

Therefore, from (43) and (44), we can know that $z \in S(0,T)$, and the solution $z(= z(w))$ of (32) satisfies

$$\begin{aligned}
\|z(w)\|_{S(0,T)} &\le C\|wy(u)\|_{L^2(Q)} \le C\|w\|_{\mathcal{F}} \|y(u)\|_{L^2(Q)} \\
&\le C\|y(u)\|_{S(0,T)} \|w\|_{\mathcal{F}} \\
&\le C\|(y_0, y_1, f, u)\|_{\mathcal{P}} \|w\|_{\mathcal{F}}.
\end{aligned} \tag{45}$$

Hence, from (45), the mapping $w \in \mathcal{F} \longmapsto z(w) \in S(0,T)$ is linear and bounded. From this, we can infer that there exists $T \in \mathcal{L}(\mathcal{F}, S(0,T))$ such that $Tw = z(w)$ for each $w \in \mathcal{F}$.

(ii) We set the difference $\delta = y(u+w) - y(u) - z$. Then, from (30) and (32), we can have the following:

$$\begin{aligned}
\delta'' - \mu\Delta\delta' &= (u+w)\,y(u+w) - uy(u) - uz \\
&\quad - wy(u) + \left(1 + \|\nabla y(u+w)\|_2^2\right)\Delta y(u+w) \\
&\quad - \left(1 + \|\nabla y(u)\|_2^2\right)\Delta y(u) - \left(1 + \|\nabla y(u)\|_2^2\right)\Delta z \\
&\quad - 2\left(\nabla y(u), \nabla z\right)_2 \Delta y(u) = u\delta \\
&\quad + w\left(y(u+w) - y(u)\right) + \left(1 + \|\nabla y(u)\|_2^2\right)\Delta\delta \\
&\quad + \left(\|\nabla y(u+w)\|_2^2 - \|\nabla y(u)\|_2^2\right)\Delta y(u+w) \\
&\quad - 2\left(\nabla y(u), \nabla z\right)_2 \Delta y(u) = (u+w)\,\delta + wz \\
&\quad + \left(1 + \|\nabla y(u)\|_2^2\right)\Delta\delta \\
&\quad + \left(\nabla y(u+w) - \nabla y(u), \nabla y(u+w) + \nabla y(u)\right)_2 \\
&\quad \cdot \Delta y(u+w) - 2\left(\nabla y(u), \nabla z\right)_2 \Delta y(u) = (u+w)\,\delta \\
&\quad + wz + \left(1 + \|\nabla y(u)\|_2^2\right)\Delta\delta \\
&\quad + \left(\nabla\delta, \nabla y(u+w) + \nabla y(u)\right)_2 \Delta y(u+w) \\
&\quad + \left(\nabla z, \nabla y(u+w) + \nabla y(u)\right)_2 \Delta y(u+w) \\
&\quad - 2\left(\nabla y(u), \nabla z\right)_2 \Delta y(u) = (u+w)\,\delta + wz \\
&\quad + \left(1 + \|\nabla y(u)\|_2^2\right)\Delta\delta \\
&\quad + \left(\nabla\delta, \nabla y(u+w) + \nabla y(u)\right)_2 \Delta y(u+w) \\
&\quad + \left(\nabla z, \nabla y(u+w) - \nabla y(u)\right)_2 \Delta y(u+w) \\
&\quad + 2\left(\nabla z, \nabla y(u)\right)_2 \left(\Delta y(u+w) - \Delta y(u)\right) \quad \text{in } Q.
\end{aligned} \tag{46}$$

Thus, we know from (46) that δ satisfies

$$\begin{aligned}
&\delta'' - \left(1 + \|\nabla y(u)\|_2^2\right)\Delta\delta \\
&\quad - \left(\nabla\delta, \nabla y(u+w) + \nabla y(u)\right)_2 \Delta y(u+w) \\
&\quad - \mu\Delta\delta' = (u+w)\,\delta + wz + I_1 + I_2 \quad \text{in } Q, \\
&\delta = 0 \quad \text{on } \Sigma, \\
&\delta(0,x) = 0, \\
&\delta'(0,x) = 0 \\
&\qquad\qquad \text{in } \Omega,
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
I_1 &= \left(\nabla z, \nabla y(u+w) - \nabla y(u)\right)_2 \Delta y(u+w), \\
I_2 &= 2\left(\nabla z, \nabla y(u)\right)_2 \left(\Delta y(u+w) - \Delta y(u)\right).
\end{aligned} \tag{48}$$

If we let

$$\begin{aligned}
&\mathcal{H}\left(y(u+w), y(u), z\right) \\
&\quad := \left(1 + \|\nabla y(u)\|_2^2\right)\Delta\delta \\
&\qquad + \left(\nabla\delta, \nabla y(u+w) + \nabla y(u)\right)_2 \Delta y(u+w),
\end{aligned} \tag{49}$$

then by similar arguments used for (34), we have

$$\mathcal{H}\left(y(u+w), y(u), \cdot\right) \in \mathcal{L}\left(D(\Delta), L^2\right). \tag{50}$$

Thanks to (50), if we follow similar arguments as in (i), then we can arrive at

$$\|\delta\|_{S(0,T)} \le C\|wz + I_1 + I_2\|_{L^2(Q)}. \tag{51}$$

From (14), Theorem 4, and (45), we can deduce the following:

$$\begin{aligned} \|wz\|_{L^2(Q)} &\le \|w\|_{\mathcal{F}} \|z\|_{L^2(Q)} \le C\|w\|_{\mathcal{F}} \|z\|_{S(0,T)} \\ &\le C\|w\|_{\mathcal{F}}^2; \end{aligned} \tag{52}$$

$$\begin{aligned} \|I_1\|_{L^2(Q)} &\le \|z\|_{C([0,T];H_0^1)} \|y(u+w)-y(u)\|_{C([0,T];H_0^1)} \\ &\times \|\Delta y(u+w)\|_{L^2(Q)} \le C\|z\|_{S(0,T)} \\ &\cdot \|y(u+w)-y(u)\|_{S(0,T)} \|y(u+w)\|_{S(0,T)} \\ &\le C\|w\|_{\mathcal{F}} \|u+w-u\|_{\mathcal{F}} \|(y_0,y_1,f,u+w)\|_{\mathcal{P}} \\ &\le C\|w\|_{\mathcal{F}}^2; \end{aligned} \tag{53}$$

$$\begin{aligned} \|I_2\|_{L^2(Q)} &\le 2\|z\|_{C([0,T];H_0^1)} \|y(u)\|_{C([0,T];H_0^1)} \\ &\times \|\Delta y(u+w)-\Delta y(u)\|_{L^2(Q)} \le C\|z\|_{S(0,T)} \\ &\cdot \|y(u)\|_{S(0,T)} \|y(u+w)-y(u)\|_{S(0,T)} \le C\|w\|_{\mathcal{F}} \\ &\cdot \|(y_0,y_1,f,u)\|_{\mathcal{P}} \|u+w-u\|_{\mathcal{F}} \le C\|w\|_{\mathcal{F}}^2. \end{aligned} \tag{54}$$

Hence, from (51) to (54), we can obtain

$$\begin{aligned} \|\delta\|_{S(0,T)} &\le C\|wz+I_1+I_2\|_{L^2(Q)} \\ &\le C\left(\|wz\|_{L^2(Q)} + \|I_1\|_{L^2(Q)} + \|I_2\|_{L^2(Q)}\right) \\ &\le C\|w\|_{\mathcal{F}}^2, \end{aligned} \tag{55}$$

which immediately implies that $\|\delta\|_{S(0,T)} = o(\|w\|_{\mathcal{F}})$ as $\|w\|_{\mathcal{F}} \longrightarrow 0$.

This completes the proof. □

The following result plays an important role in proving the existence of optimal controls in the next section.

Proposition 8. *Given $w \in \mathcal{F}$, the Fréchet derivative $Dy(u)w$ is locally Lipschitz continuous on $\mathcal{F}$ with $L^2(Q)$ topology. Indeed, it is satisfied that*

$$\begin{aligned} &\|Dy(u_1)w - Dy(u_2)w\|_{S(0,T)} \\ &\quad \le C\|u_1-u_2\|_{L^2(Q)} \|w\|_{L^2(Q)}, \end{aligned} \tag{56}$$

where $C > 0$ is a constant depending on the data.

Proof. Let $z_i = Dy(u_i)w$, $(i = 1,2)$ be the solutions of (32) corresponding to u_i, $(i = 1,2)$, and we set $\phi = z_1 - z_2$. Then, by similar calculations as in (46), we can deduce that ϕ satisfies

$$\begin{aligned} &\phi'' - \left(1+\|\nabla y(u_1)\|_2^2\right)\Delta\phi - 2(\nabla\phi, \nabla y(u_1))_2 \Delta y(u_1) \\ &\quad - \mu\Delta\phi' = u_1\phi + \sum_{i=1}^4 I_i \quad \text{in } Q, \\ &\phi = 0 \quad \text{on } \Sigma, \\ &\phi(0,x) = 0, \\ &\phi'(0,x) = 0 \\ &\qquad \text{in } \Omega, \end{aligned} \tag{57}$$

where

$$\begin{aligned} I_1 &= 2(\nabla z_2, \nabla y(u_1) - \nabla y(u_2))_2 \Delta y(u_1), \\ I_2 &= 2(\nabla z_2, \nabla y(u_2))_2 (\Delta y(u_1) - \Delta y(u_2)), \\ I_3 &= (\nabla y(u_1) - \nabla y(u_2), \nabla y(u_1) + \nabla y(u_2))_2 \Delta z_2, \\ I_4 &= (u_1-u_2) z_2 + w(y(u_1) - y(u_2)). \end{aligned} \tag{58}$$

By similar arguments as in the proof of (i) of Theorem 7, ϕ in (57) can be estimated as follows:

$$\|\phi\|_{S(0,T)} \le C\left\|\sum_{i=1}^4 I_i\right\|_{L^2(Q)}. \tag{59}$$

From Theorem 4, the embedding $S(0,T) \hookrightarrow C^0(\overline{Q})$, and the first inequality of (45), we can deduce

$$\begin{aligned} \|z_2\|_{S(0,T)} &\le C\|wy(u_2)\|_{L^2(Q)} \\ &\le C\|y(u_2)\|_{C^0(\overline{Q})} \|w\|_{L^2(Q)} \\ &\le (\text{with } (10) \text{ and } (14)) \\ &\le C\|y(u_2)\|_{S(0,T)} \|w\|_{L^2(Q)} \\ &\le C\|(y_0,y_1,f,u_2)\|_{\mathcal{P}} \|w\|_{L^2(Q)} \\ &\le C\|w\|_{L^2(Q)}. \end{aligned} \tag{60}$$

We can estimate I_i $(i = 1,\ldots,4)$ of (57) as follows:

$$\begin{aligned} \|I_1\|_{L^2(Q)} &\le 2\|z_2\|_{C([0,T];H_0^1)} \|y(u_1)-y(u_2)\|_{C([0,T];H_0^1)} \\ &\cdot \|\Delta y(u_1)\|_{L^2(Q)} \le (\text{with } (14)) \le C\|z_2\|_{S(0,T)} \\ &\cdot \|y(u_1)-y(u_2)\|_{S(0,T)} \|y(u_1)\|_{S(0,T)} \\ &\le (\text{with Corollary 5, Theorem 4 and } (60)) \\ &\le C\|w\|_{L^2(Q)} \|u_1-u_2\|_{L^2(Q)} \|(y_0,y_1,f,u_1)\|_{\mathcal{P}} \\ &\le C\|u_1-u_2\|_{L^2(Q)} \|w\|_{L^2(Q)}; \end{aligned} \tag{61}$$

$$\begin{aligned} \|I_2\|_{L^2(Q)} &\le 2\|z_2\|_{C([0,T];H_0^1)} \|y(u_2)\|_{C([0,T];H_0^1)} \|\Delta y(u_1) \\ &- \Delta y(u_2)\|_{L^2(Q)} \\ &\le (\text{with an arguments similar to } (61)) \\ &\le C\|z_2\|_{S(0,T)} \|y(u_2)\|_{S(0,T)} \|y(u_1)-y(u_2)\|_{S(0,T)} \\ &\le C\|w\|_{L^2(Q)} \|(y_0,y_1,f,u_2)\|_{\mathcal{P}} \|u_1-u_2\|_{L^2(Q)} \\ &\le C\|u_1-u_2\|_{L^2(Q)} \|w\|_{L^2(Q)}; \end{aligned} \tag{62}$$

$$\begin{aligned}
&\|I_3\|_{L^2(Q)} \le \|y(u_1)-y(u_2)\|_{C([0,T];H_0^1)} \|y(u_1)\\
&\quad + y(u_2)\|_{C([0,T];H_0^1)} \times \|\Delta z_2\|_{L^2(Q)}\\
&\quad \le (\text{with an arguments similar to } (61))\\
&\quad \le C\|y(u_1)-y(u_2)\|_{S(0,T)} \|y(u_1)+y(u_2)\|_{S(0,T)}\\
&\quad \cdot \|z_2\|_{S(0,T)} \le C\|u_1-u_2\|_{L^2(Q)} (\|(y_0,y_1,f,u_1)\|_{\mathscr{P}}\\
&\quad + \|(y_0,y_1,f,u_2)\|_{\mathscr{P}}) \|w\|_{L^2(Q)} \le C\|u_1-u_2\|_{L^2(Q)}\\
&\quad \cdot \|w\|_{L^2(Q)};
\end{aligned} \tag{63}$$

$$\begin{aligned}
&\|I_4\|_{L^2(Q)} \le \|(u_1-u_2) z_2\|_{L^2(Q)}\\
&\quad + \|w(y(u_1)-y(u_2))\|_{L^2(Q)} \le \|z_2\|_{C^0(\overline{Q})} \|u_1\\
&\quad - u_2\|_{L^2(Q)} + \|w\|_{L^2(Q)} \|(y(u_1)-y(u_2))\|_{C^0(\overline{Q})}\\
&\quad \le \left(\text{with } (10) \text{ and } S(0,T) \hookrightarrow C^0(\overline{Q})\right)\\
&\quad \le C\left(\|z_2\|_{S(0,T)} \|u_1-u_2\|_{L^2(Q)}\right.\\
&\quad \left.+ \|w\|_{L^2(Q)} \|(y(u_1)-y(u_2))\|_{S(0,T)}\right)\\
&\quad \le (\text{with Corollary 5 and } (60)) \le C\|u_1\\
&\quad - u_2\|_{L^2(Q)} \|w\|_{L^2(Q)}.
\end{aligned} \tag{64}$$

From (61) to (64), we can obtain the following from (59):

$$\|\phi\|_{S(0,T)} \le C\|u_1-u_2\|_{L^2(Q)} \|w\|_{L^2(Q)}. \tag{65}$$

This completes the proof. □

4. Quadratic Cost Minimax Control Problems

In this section, we study the quadratic cost minimax optimal control problems for a damped Kirchhoff-type equation. Let the following be the set of the admissible controls:

$$\mathscr{U}_{ad} = \{u \in \mathscr{F} \mid a \le u \le b \text{ a.e. in } Q\}. \tag{66}$$

Let the following be the set of the admissible disturbance or noises:

$$\mathscr{V}_{ad} = \{v \in \mathscr{F} \mid c \le v \le d \text{ a.e. in } Q\}. \tag{67}$$

To perform our variational analysis, $L^2(Q)$ norms of $\mathscr{U}_{ad}$ and $\mathscr{V}_{ad}$ are preferable, even though $\mathscr{U}_{ad}$ and $\mathscr{V}_{ad}$ are subsets of $\mathscr{F}$. For simplicity, let $\mathscr{F}_{ad}$ be a product space defined by $\mathscr{F}_{ad} = \mathscr{U}_{ad} \times \mathscr{V}_{ad}$.

Using Theorem 4, we can uniquely define the solution mapping $\mathscr{F}_{ad} \longrightarrow S(0,T)$, which maps the term $q = (u,v) \in \mathscr{F}_{ad}$ to the solution $y(q) \in S(0,T)$, which satisfies the following equation:

$$\begin{aligned}
&y''(q) - \left(1+\|\nabla y(q)\|_2^2\right)\Delta y(q) - \mu\Delta y'(q)\\
&\quad = (u+v)\, y(q) + f \quad \text{in } Q,\\
&y(q) = 0 \quad \text{on } \Sigma,\\
&y(q;0,x) = y_0(x),\\
&y'(q;0,x) = y_1(x)\\
&\qquad\qquad \text{in } \Omega,
\end{aligned} \tag{68}$$

The solution $y(q)$ of (68) is the state of the control system (68). From Theorem 7, we can deduce that the map $q = (u,v) \longrightarrow y(q)$ of $\mathscr{F}_{ad}$ to $S(0,T)$ is Fréchet differentiable at $q = q^* = (u^*, v^*)$, and the Fréchet derivative of $y(q)$ at $q = q^*$ in the direction $w = (h,l) \in \mathscr{F}^2$, say $z = Dy(q^*)w$ is a unique solution of the following problem:

$$\begin{aligned}
&z'' - \left(1+\|\nabla y(q^*)\|_2^2\right)\Delta z\\
&\quad - 2(\nabla y(q^*), \nabla z)_2\, \Delta y(q^*) - \mu\Delta z' = (u^*+v^*)\, z\\
&\quad + (h+l)\, y(q^*) \quad \text{in } Q,\\
&z = 0 \quad \text{on } \Sigma,\\
&z(0,x) = 0,\\
&z'(0,x) = 0\\
&\qquad\qquad \text{in } \Omega.
\end{aligned} \tag{69}$$

The quadratic cost function associated with the control system (68) is

$$\begin{aligned}
J(u,v) = {}& \frac{1}{2}\|\mathscr{C} y(q) - Y_d\|_M^2 + \frac{\alpha}{2}\|u\|_{L^2(Q)}^2\\
&- \frac{\beta}{2}\|v\|_{L^2(Q)}^2,
\end{aligned} \tag{70}$$

where M is a Hilbert space of observation variables, the operator $\mathscr{C} \in \mathscr{L}(S(0,T), M)$ is an observer, $Y_d \in M$ is a desired value, and the positive constants α and β are the relative weights of the second and the third terms on the RHS of (70).

To pursue our objective, we assume that the observer $\mathscr{C}(\in \mathscr{L}(S(0,T), M))$ in (70) is a compact operator. As mentioned in the introduction, the minimax optimal control problem can be summarized as follows:

(i) Find an admissible control $u^* \in \mathscr{U}_{ad}$ and a noise (or disturbance) $v^* \in \mathscr{V}_{ad}$ such that (u^*, v^*) is a saddle point of the functional $J(u,v)$ of (70). That is,

$$J(u^*, v) \le J(u^*, v^*) \le J(u, v^*), \quad \forall (u,v) \in \mathscr{F}_{ad}. \tag{71}$$

(ii) Characterize (u^*, v^*) (optimality condition).

Such a pair (u^*, v^*) in (71) is called an optimal pair (or an optimal strategy pair) for the problem (70).

4.1. Existence of Optimal Pairs. To study the existence of optimal pairs, we present the following results.

Proposition 9. *The solution mapping from $\mathcal{F}_{ad}$ to $S(0,T)$ is continuous from the weakly-star topology of $\mathcal{F}_{ad}$ to the weak topology of $S(0,T)$.*

In proving the Proposition 9, we need the following compactness lemma.

Lemma 10. *Let X, Y and Z be Banach spaces such that the embeddings $X \hookrightarrow Y \hookrightarrow Z$ are continuous and the imbedding $X \hookrightarrow Y$ is compact. Then a bounded set of $W^{1,\infty}(0,T;X,Z) = \{g \mid g \in L^\infty(0,T;X),\ g' \in L^\infty(0,T;Z)\}$ is relatively compact in $C([0,T];Y)$.*

Proof. See Simon [26]. □

Proof of Proposition 9. Let $q = (u,v) \in \mathcal{F}_{ad}$ and let $q_n = (u_n, v_n) \in \mathcal{F}_{ad}$ be a sequence such that

$$q_n \rightharpoonup q \quad \text{weakly-star in } \mathcal{F}_{ad} \text{ as } n \longrightarrow \infty. \tag{72}$$

For simplicity, we let each state $y_n = y(q_n)$ be a solution of

$$\begin{gathered} y_n'' - \left(1 + \|\nabla y_n\|_2^2\right)\Delta y_n - \mu\Delta y_n' = (u_n + v_n)\, y_n + f \quad \text{in } Q, \\ y_n = 0 \quad \text{on } \Sigma, \\ y_n(0,x) = y_0(x), \\ y_n'(0,x) = y_1(x) \quad \text{in } \Omega. \end{gathered} \tag{73}$$

We conduct the scalar product of (73) with $-\Delta y_n' - \Delta y_n$ in L^2 :

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\nabla y_n'\right\|_2^2 + \frac{\mu+1}{2}\frac{d}{dt}\|\Delta y_n\|_2^2 + \mu\left\|\Delta y_n'\right\|_2^2 \\ &\quad + \left(1 + \|\nabla y_n\|_2^2\right)\|\Delta y_n\|_2^2 + \|\nabla y_n\|_2^2 \frac{1}{2}\frac{d}{dt}\|\Delta y_n\|_2^2 \\ &= \left(y_n'', \Delta y_n\right)_2 - \left((u_n + v_n)\, y_n + f, \Delta y_n' + \Delta y_n\right)_2, \end{aligned} \tag{74}$$

which immediately implies

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\nabla y_n'\right\|_2^2 + \frac{\mu+1}{2}\frac{d}{dt}\|\Delta y_n\|_2^2 + \mu\left\|\Delta y_n'\right\|_2^2 \\ &\quad + \|\nabla y_n\|_2^2 \frac{1}{2}\frac{d}{dt}\|\Delta y_n\|_2^2 \\ &\le \left(y_n'', \Delta y_n\right)_2 - \left((u_n + v_n)\, y_n + f, \Delta y_n' + \Delta y_n\right)_2. \end{aligned} \tag{75}$$

The integration of (75) over $[0,t]$ implies

$$\begin{aligned} &\frac{1}{2}\left\|\nabla y_n'(t)\right\|_2^2 + \frac{\mu+1}{2}\|\Delta y_n(t)\|_2^2 + \mu\int_0^t \left\|\Delta y_n'\right\|_2^2 ds \\ &\quad + \frac{1}{2}\|\nabla y_n(t)\|_2^2 \|\Delta y_n(t)\|_2^2 \\ &\le \mathcal{I}(y_0, y_1) - \left(\nabla y_n'(t), \nabla y_n(t)\right)_2 + \int_0^t \left\|\nabla y_n'\right\|_2^2 ds \\ &\quad + \int_0^t \left(\nabla y_n, \nabla y_n'\right)_2 \|\Delta y_n\|_2^2 ds \\ &\quad - \int_0^t \left((u_n + v_n)\, y_n + f, \Delta y_n' + \Delta y_n\right)_2 ds, \end{aligned} \tag{76}$$

where

$$\begin{aligned} \mathcal{I}(y_0, y_1) &= \frac{1}{2}\|\nabla y_1\|_2^2 + \frac{\mu+1}{2}\|\Delta y_0\|_2^2 \\ &\quad + \frac{1}{2}\|\nabla y_0\|_2^2 \|\Delta y_0\|_2^2 + (\nabla y_1, \nabla y_0)_2. \end{aligned} \tag{77}$$

By conducting similar calculations to the proof of (i) of Theorem 7, we can obtain the following from (76):

$$\begin{aligned} &\left\|\nabla y_n'(t)\right\|_2^2 + \|\Delta y_n(t)\|_2^2 + \int_0^t \left\|\Delta y_n'\right\|_2^2 ds \\ &\le C\Bigg(\left|\mathcal{I}(y_0, y_1)\right| + \|f\|_{L^2(Q)}^2 \\ &\quad + \int_0^t \left(\left\|\nabla y_n'\right\|_2^2 + \|\Delta y_n\|_2^2\right) ds \\ &\quad + \int_0^t \left|\left(\nabla y_n, \nabla y_n'\right)_2\right| \|\Delta y_n\|_2^2 ds\Bigg). \end{aligned} \tag{78}$$

Since we know from Theorem 4 that $y_n \in S(0,T)$, we can note that

$$\begin{aligned} \left|\left(\nabla y_n(\cdot), \nabla y_n'(\cdot)\right)_2\right| &\le \|y_n\|_{C([0,T];H_0^1)} \left\|y_n'\right\|_{C([0,T];H_0^1)} \\ &\le (\text{with } (14)) \le C\|y_n\|_{S(0,T)}^2 \\ &\le C\left\|(y_0, y_1, f, u_n + v_n)\right\|_{\mathcal{P}}^2. \end{aligned} \tag{79}$$

From (78) and (79), we can infer

$$\begin{aligned} &\left\|\nabla y_n'(t)\right\|_2^2 + \|\Delta y_n(t)\|_2^2 + \int_0^t \left\|\Delta y_n'\right\|_2^2 ds \\ &\le C\left(1 + \int_0^t \left(\left\|\nabla y_n'\right\|_2^2 + \|\Delta y_n\|_2^2\right) ds\right). \end{aligned} \tag{80}$$

Applying Lemma 3 to (80), we have

$$\left\|\nabla y_n'(t)\right\|_2^2 + \|\Delta y_n(t)\|_2^2 + \int_0^t \left\|\Delta y_n'\right\|_2^2 ds \le C. \tag{81}$$

Theorem 4 and (81) imply that y_n remains in a bounded set of $S(0,T) \cap W^{1,\infty}(0,T;D(\Delta),H_0^1)$. Therefore, by using Rellich's extraction theorem, we can find a subsequence of $\{y_n\}$ also called $\{y_n\}$, and find $y \in S(0,T) \cap W^{1,\infty}(0,T; D(\Delta), H_0^1)$ such that

$$y_n \rightharpoonup y \quad \text{weakly in } S(0,T) \text{ as } n \longrightarrow \infty, \tag{82}$$

$$y_n \rightharpoonup y \quad \text{weakly-star in } L^\infty(0,T; D(\Delta)) \text{ as } n \longrightarrow \infty, \tag{83}$$

$$y_n' \rightharpoonup y' \quad \text{weakly-star in } L^\infty\left(0,T; H_0^1\right) \text{ as } n \longrightarrow \infty. \tag{84}$$

Since the embedding $D(\Delta) \hookrightarrow H_0^1$ is compact, we can apply Lemma 10 to (83) and (84) with $X = D(\Delta)$ and $Y = Z = H_0^1$ in Lemma 10 to verify that

$$y_n \quad \text{is pre-compact in } C\left([0,T]; H_0^1\right). \tag{85}$$

Hence, we can find a subsequence $\{y_{n_k}\} \subset \{y_n\}$ if necessary such that

$$y_{n_k}(t) \longrightarrow y(t) \quad \text{in } H_0^1 \text{ for } \forall t \in [0,T] \text{ as } k \longrightarrow \infty. \tag{86}$$

Therefore, (82) and (86) imply

$$\left\|\nabla y_{n_k}\right\|_2^2 \Delta y_{n_k} \rightharpoonup \|\nabla y\|_2^2 \Delta y \quad \text{weakly in } L^2(Q) \text{ as } k \longrightarrow \infty. \tag{87}$$

From (72) and (85), we can also extract a subsequence, if necessary, denoted again by $q_n \equiv (u_n, v_n)$ such that

$$(u_n + v_n) y_n \rightharpoonup (u+v) y \quad \text{weakly in } L^2(Q). \tag{88}$$

We replace y_n by y_{n_k}, if necessary, and take $k \longrightarrow \infty$ in (73). Then, by the standard argument in Dautray and Lions [23, pp.561-565], we conclude that the limit y is a solution of

$$\begin{gathered} y'' - \left(1 + \|\nabla y\|_2^2\right)\Delta y - \mu \Delta y' = (u+v) y + f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \\ y'(0,x) = y_1(x) \\ \text{in } \Omega. \end{gathered} \tag{89}$$

Moreover, from the uniqueness of solutions of (89), we conclude that $y = y(q)$ in $S(0,T)$, which implies that $y(q_n) \rightharpoonup y(q)$ weakly in $S(0,T)$.

This completes the proof. □

We now study the existence of optimal pairs.

Theorem 11. *Let the observer $\mathscr{C}$ in (70) be a compact operator. Then, for sufficiently large α and β in (70), there exists $(u^*, v^*) \in \mathscr{F}_{ad}$ such that (u^*, v^*) satisfies (71).*

Proof. Let $\mathscr{P}_v$ be the map $u \longrightarrow J(u,v)$ and let $\mathscr{Q}_u$ be the map $v \longrightarrow J(u,v)$. To obtain the existence of optimal pairs in the minimax control problem, we follow the steps given by [13]: We prove that $\mathscr{P}_v$ is convex and lower semicontinuous for all $v \in \mathscr{V}_{ad}$ and that $\mathscr{Q}_u$ is concave and upper semicontinuous for all $u \in \mathscr{U}_{ad}$. Then, we employ the minimax theorem in infinite dimensions (see Barbu and Precupanu [17]).

For sufficiently large α and β in (70), we first prove the convexity of $\mathscr{P}_v$ and the concavity of $\mathscr{Q}_u$. To prove the convexity of $\mathscr{P}_v$, which is a differentiable map, it is sufficient to show that

$$\left(D\mathscr{P}_v(u_1) - D\mathscr{P}_v(u_2)\right)(u_1 - u_2) \geq 0, \quad \forall u_1, u_2 \in \mathscr{U}_{ad}. \tag{90}$$

From Fréchet differentiability of the solution map $u \longrightarrow y(u,v)$, where v is fixed, (90) can be rewritten as

$$\begin{aligned} &\left(\mathscr{C} y(u_1,v) - Y_d, \mathscr{C} D_u y(u_1,v)(u_1-u_2)\right)_M \\ &\quad + \alpha \int_0^T (u_1, u_1 - u_2)_2 \, dt \\ &\quad - \left(\mathscr{C} y(u_2,v) - Y_d, \mathscr{C} D_u y(u_2,v)(u_1-u_2)\right)_M \\ &\quad - \alpha \int_0^T (u_2, u_1 - u_2)_2 \, dt \geq 0, \quad \forall u_1, u_2 \in \mathscr{U}_{ad}, \end{aligned} \tag{91}$$

where $D_u y(u_i,v)(u_1-u_2)$, $(i=1,2)$ are solutions of (69), in which $(u^*+v^*)z + (h+l)y(p)$ is replaced by $(u_i+v)z + (u_1 - u_2)y(u_i,v)$, $(i=1,2)$, respectively. We can easily deduce that (91) is equivalent again to

$$\begin{aligned} &\left(\mathscr{C}\left(y(u_1,v) - y(u_2,v)\right), \mathscr{C} D_u y(u_1,v)(u_1-u_2)\right)_M \\ &\quad + \left(\mathscr{C} y(u_2,v) - Y_d, \mathscr{C}\left(D_u y(u_1,v)(u_1-u_2)\right.\right. \\ &\quad \left.\left. - D_u y(u_2,v)(u_1-u_2)\right)\right)_M + \alpha \|u_1 - u_2\|_{L^2(Q)}^2 \geq 0, \\ &\qquad \forall u_1, u_2 \in \mathscr{U}_{ad}. \end{aligned} \tag{92}$$

From Corollary 5, Proposition 8, and (60), we can estimate the left hand side of (92) as follows:

$$\begin{aligned} &\left|\left(\mathscr{C}\left(y(u_1,v) - y(u_2,v)\right), \mathscr{C} D_u y(u_1,v)(u_1-u_2)\right)_M\right| \\ &\quad \leq \left\|\mathscr{C}\left(y(u_1,v) - y(u_2,v)\right)\right\|_M \left\|\mathscr{C} D_u y(u_1,v)(u_1 - u_2)\right\|_M \leq \|\mathscr{C}\|_{\mathscr{L}(S(0,T),M)}^2 \left\|y(u_1,v) - y(u_2,v)\right\|_{S(0,T)} \left\|D_u y(u_1,v)(u_1-u_2)\right\|_{S(0,T)} \\ &\quad \leq (\text{with Corollary 5 and } (60)) \leq C\|u_1 - u_2\|_{L^2(Q)}^2; \end{aligned} \tag{93}$$

$$\begin{aligned}
&\left|\left(\mathscr{C}y(u_2,v)-Y_d,\mathscr{C}(D_u y(u_1,v)(u_1-u_2)\right.\right.\\
&\quad\left.\left.-D_u y(u_2,v)(u_1-u_2))\right)_M\right|\le\|\mathscr{C}y(u_2,v)\\
&\quad-Y_d\|_M\|\mathscr{C}(D_u y(u_1,v)(u_1-u_2)-D_u y(u_2,v)\\
&\quad\cdot(u_1-u_2))\|_M\le\|\mathscr{C}\|_{\mathscr{L}(S(0,T),M)}\left(\|\mathscr{C}\|_{\mathscr{L}(S(0,T),M)}\right.\\
&\quad\cdot\|y(u_2,v)\|_{S(0,T)}+\|Y_d\|_M\left.\right)\times\|D_u y(u_1,v)(u_1\\
&\quad-u_2)-D_u y(u_2,v)(u_1-u_2)\|_{S(0,T)}\\
&\quad\le(\text{with Proposition 8})\le C\left(\|y(u_2,v)\|_{S(0,T)}\right.\\
&\quad\left.+\|Y_d\|_M\right)\|u_1-u_2\|^2_{L^2(Q)}\le C\left(\|(y_0,y_1,f,u_2\right.\\
&\quad\left.+v)\|_{\mathscr{P}}+\|Y_d\|_M\right)\|u_1-u_2\|^2_{L^2(Q)}.
\end{aligned}\tag{94}$$

Considering from (92) to (94), we can deduce that there exists a sufficiently large $\alpha_l(\mathscr{P},\mathscr{F}_{ad},Y_d,\mathscr{C})$ such that, for any $\alpha>\alpha_l(\mathscr{P},\mathscr{F}_{ad},Y_d,\mathscr{C})$, (92) holds true. Therefore, the map $\mathscr{P}_v$ is convex.

Similarly, we can also show that there exist a sufficiently large $\beta_l(\mathscr{P},\mathscr{F}_{ad},Y_d,\mathscr{C})$ such that the following inequality is satisfied for any $\beta>\beta_l(\mathscr{P},\mathscr{F}_{ad},Y_d,\mathscr{C})$:

$$\begin{aligned}
&(D\mathscr{Q}_u(v_1)-D\mathscr{Q}_u(v_2))(v_1-v_2)\le 0,\\
&\qquad\forall v_1,v_2\in\mathscr{V}_{ad}.
\end{aligned}\tag{95}$$

This also indicates the concavity of $\mathscr{Q}_u$.

Next, we prove the existence of an optimal pair $(u^*,v^*)\in\mathscr{F}_{ad}$ by verifying that $\mathscr{P}_v$ is lower semicontinuous for all $v\in\mathscr{V}_{ad}$ and $\mathscr{Q}_u$ is upper semicontinuous for all $u\in\mathscr{U}_{ad}$. Let $\{u_n\}\subset\mathscr{U}_{ad}$ be a minimizing sequence of J. Thus

$$\liminf_{n\to\infty}J(u_n,v)=\min_{u\in\mathscr{U}_{ad}}J(u,v).\tag{96}$$

Since $\mathscr{U}_{ad}$ defined by (66) is a closed, bounded, and convex in $\mathscr{F}$, we can extract a subsequence $\{u_{n_k}\}\subset\{u_n\}$ such that

$$u_{n_k}\rightharpoonup u^*\quad\text{weakly-starin }\mathscr{U}_{ad}\text{ as }k\longrightarrow\infty.\tag{97}$$

Then, by Proposition 9, we have $\forall v\in\mathscr{V}_{ad}$,

$$\begin{aligned}
&y(u_{n_k},v)\rightharpoonup y(u^*,v)\\
&\qquad\text{weakly in }S(0,T)\text{ as }k\longrightarrow\infty.
\end{aligned}\tag{98}$$

Thus, by the assumption that $\mathscr{C}\in\mathscr{L}(S(0,T),M)$ is a compact operator, we can extract a subsequence of $\{u_{n_k}\}$, if necessary, denoted again by $\{u_{n_k}\}$, such that

$$\begin{aligned}
&\mathscr{C}y(u_{n_k},v)\longrightarrow\mathscr{C}y(u^*,v)\\
&\qquad\text{strongly in }M\text{ as }k\longrightarrow\infty,
\end{aligned}\tag{99}$$

$\forall v\in\mathscr{V}_{ad}$. From (97), it can be easily verified for the same subsequence $\{u_{n_k}\}$ in (97) that

$$u_{n_k}\rightharpoonup u^*\quad\text{weakly in }L^2(Q)\text{ as }k\longrightarrow\infty.\tag{100}$$

Due to the weakly lower semicontinuity in the $L^2(Q)$ norm topology, we can determine from (99) and (100) that the map $\mathscr{P}_v:u\longrightarrow J(u,v)$ is lower semicontinuous for all $v\in\mathscr{V}_{ad}$. By similar arguments, we can prove that $\mathscr{Q}_u$ is upper semicontinuous for all $u\in\mathscr{U}_{ad}$.

Hence, we know that

$$J_0(v)=\liminf_{n\to\infty}J(u_n,v)\ge J(u^*,v),\quad\forall v\in\mathscr{V}_{ad}.\tag{101}$$

But since $J_0(v)\le J(u^*,v)$, we have

$$J_0(v)=J(u^*,v)=\min_{u\in\mathscr{U}_{ad}}J(u,v),\quad\forall v\in\mathscr{V}_{ad}.\tag{102}$$

Similarly, we also know that there exists $v^*\in\mathscr{V}_{ad}$ such that

$$J_0(v^*)=\max_{v\in\mathscr{V}_{ad}}J_0(v).\tag{103}$$

From (102) and (103), we can conclude that $(u^*,v^*)\in\mathscr{F}_{ad}$ is an optimal pair for the cost (70).

This completes the proof. □

4.2. Necessary Conditions of Optimal Pairs. We now turn to the necessary optimality conditions that have to be satisfied by optimal pairs with the cost (70). For this purpose, we consider the following two types of observations C_i, $(i=1,2)$ of distributive and terminal values:

(1) we take $M_1=L^2(Q)\times L^2$ and $C_1\in\mathscr{L}(S(0,T),M_1)$ and observe $C_1y(q)=(y(q;\cdot),y(q;T))\in L^2(Q)\times L^2$;

(2) we take $M_2=L^2(Q)$ and $C_2\in\mathscr{L}(S(0,T),M_2)$ and observe $C_2y(q)=y'(q;\cdot)\in L^2(Q)$.

Remark 12. Clearly, the embedding $S(0,T)\hookrightarrow L^2(Q)$ is compact. From the embedding (14) we can utilize Lemma 10 in which $X=D(\Delta)$ and $Y=Z=L^2$ to obtain the embedding $S(0,T)\hookrightarrow C([0,T];L^2)$ is also compact. Consequently, the observer C_1 is a compact operator. Thus, C_1 satisfies the requirement for the existence of optimal pairs given in Theorem 11.

Remark 13. Since $y'(q)\in H^1(0,T;D(\Delta),L^2)\equiv\{g\mid g\in L^2(0,T;D(\Delta)),\ g'\in L^2(Q)\}$, and the embedding $D(\Delta)\hookrightarrow L^2$ is compact, we can employ the Aubin-Lions-Temam's compact embedding theorem (cf. Temam [27, p. 274]) to determine that the embedding $H^1(0,T;D(\Delta),L^2)\hookrightarrow L^2(Q)$ is compact. Consequently, the observer C_2 is a compact operator. Therefore, C_2 satisfies the requirement for the existence of optimal pairs given in Theorem 11.

4.2.1. Case of Distributive and Terminal Values Observations C_1. In this observation case, we consider the cost function associated with the control system (68):

$$\begin{aligned}
J(u,v)={}&\frac{1}{2}\|y(q)-Y_d\|^2_{L^2(Q)}+\frac{1}{2}\left\|y(q;T)-Y_d^T\right\|_2^2\\
&+\frac{\alpha}{2}\|u\|^2_{L^2(Q)}-\frac{\beta}{2}\|v\|^2_{L^2(Q)},
\end{aligned}\tag{104}$$

where $Y_d \in L^2(Q)$ and $Y_d^T \in L^2$ are desired values, and the positive constants α and β are the relative weight of the second and the third terms on the RHS of (104).

Now we formulate the following adjoint equation to describe the necessary optimality conditions for this observation:

$$\begin{aligned}
&p'' - \mathcal{G}(y(q^*), p) + \mu\Delta p' \\
&\quad = (u^* + v^*) p + y(q^*) - Y_d \quad \text{in } Q, \\
&p = 0 \quad \text{on } \Sigma, \\
&p(T, x) = 0, \\
&p'(T, x) = -y(q^*; T, x) + Y_d^T(x) \\
&\qquad\qquad\qquad\qquad \text{in } \Omega,
\end{aligned} \tag{105}$$

where $\mathcal{G}(\cdot,\cdot)$ is defined in (33). Using a similar estimation to (34), we can have

$$\mathcal{G}(y(q^*), \cdot) \in \mathcal{L}(H_0^1, H^{-1}). \tag{106}$$

Remark 14. By considering the observation conditions $y(q^*) - Y_d \in L^2(Q) \subset L^2(0,T;H^{-1})$ and $y(q^*;T) - Y_d^T \in L^2$ and (106), we can refer to the well-posedness result of Dautray and Lions [23, pp.558-570] to verify that (105), reversing the direction of time $t \longrightarrow T - t$, admits a unique weak solution $p \in W(0,T)$, which is given in Definition 2.

We now discuss the first-order optimality conditions for the minimax optimal control problem (71) for the quadratic cost function (104).

Theorem 15. *If α and β in the cost (104) are large enough, then an optimal control $u^* \in \mathcal{U}_{ad}$ and a disturbance $v^* \in \mathcal{V}_{ad}$, namely, an optimal pair $q^* = (u^*, v^*) \in \mathcal{F}_{ad}$ satisfying (71), can be given by*

$$\begin{aligned}
u^* &= \max\left\{a, \min\left\{-\frac{y(q^*)p}{\alpha}, b\right\}\right\}, \\
v^* &= \max\left\{c, \min\left\{\frac{y(q^*)p}{\beta}, d\right\}\right\},
\end{aligned} \tag{107}$$

where p is the weak solution of (105).

Proof. Let $q^* = (u^*, v^*) \in \mathcal{F}_{ad}$ be an optimal pair in (71) with the cost (104) and let $y(q^*)$ be the corresponding weak solution of (68).

From Theorem 7, we know that the map $q = (u,v) \longrightarrow y(q)$ is Fréchet differentiable at $q = q^* = (u^*, v^*)$ in the direction $w = (h,l) \in \mathcal{F}^2$, which satisfies $q^* + \epsilon w \in \mathcal{F}_{ad}$ for sufficiently small $\epsilon > 0$. Thus, the map $q = (u,v) \longrightarrow y(q)$ is also (strongly) Gâteaux differentiable at $q = q^*$ in the direction $w = (h,l) \in \mathcal{F}^2$. Thus, we have

$$\begin{aligned}
&\frac{y(q^* + \epsilon w) - y(q^*)}{\epsilon} \longrightarrow z\ (= z(w)) \\
&\qquad\qquad \text{strongly in } S(0,T) \text{ as } \epsilon \longrightarrow 0^+,
\end{aligned} \tag{108}$$

where $z = Dy(q^*)w$ is a unique solution of (69). Therefore we can obtain the Gâteaux derivative of the cost (104) at $q = q^*$ in the direction $w = (h,l)$ as follows:

$$\begin{aligned}
&DJ(u^*, v^*)(h,l) \\
&\quad = \lim_{\epsilon\longrightarrow 0^+} \frac{J(u^* + \epsilon h, v^* + \epsilon l) - J(u^*, v^*)}{\epsilon} = \lim_{\epsilon\longrightarrow 0^+} \frac{1}{2} \\
&\quad \cdot \int_0^T \Big(y(q^* + \epsilon w) + y(q^*) \\
&\quad - 2Y_d, \frac{y(q^* + \epsilon w) - y(q^*)}{\epsilon} \Big)_2 dt + \lim_{\epsilon\longrightarrow 0^+} \frac{1}{2} \\
&\quad \cdot \Big(y(q^* + \epsilon w; T) + y(q^*; T) - 2Y_d^T, \\
&\quad \frac{y(q^* + \epsilon w; T) - y(q^*; T)}{\epsilon} \Big)_2 + \lim_{\epsilon\longrightarrow 0^+} \Big[\frac{\alpha}{2} \\
&\quad \cdot \int_0^T \left(2(u^*, h)_2 + \epsilon \|h\|_2^2\right) dt - \frac{\beta}{2} \\
&\quad \cdot \int_0^T \left(2(v^*, l)_2 + \epsilon \|l\|_2^2\right) dt \Big] = \int_0^T (y(q^*) \\
&\quad - Y_d, z)_2\, dt + \left(y(q^*; T) - Y_d^T, z(T)\right)_2 \\
&\quad + \alpha \int_0^T (u^*, h)_2\, dt - \beta \int_0^T (v^*, l)_2\, dt,
\end{aligned} \tag{109}$$

where $z = Dy(q^*)w$ is a solution of (69).

Before we proceed to the calculations, we note that

$$\begin{aligned}
&\langle \mathcal{G}(y(q^*), \varphi), \phi \rangle_{-1,1} \\
&\quad = -\left(1 + \|\nabla y(q^*)\|_2^2\right)(\nabla\varphi, \nabla\phi)_2 \\
&\qquad - 2(\nabla y(q^*), \nabla\varphi)_2 (\nabla y(q^*), \nabla\phi)_2 \\
&\quad = \left\langle \left(1 + \|\nabla y(q^*)\|_2^2\right)\Delta\phi, \varphi \right\rangle_{-1,1} \\
&\qquad + 2(\nabla y(q^*), \nabla\phi)_2 (\Delta y(q^*), \varphi)_2 \\
&\quad = \langle \varphi, \mathcal{G}(y(q^*), \phi) \rangle_{1,-1}, \quad \forall \varphi, \phi \in H_0^1.
\end{aligned} \tag{110}$$

We multiply both sides of the weak form of (105) by z, which is a solution of (69), and integrate it over $[0,T]$. Then, we have

$$\begin{aligned}
&\int_0^T \langle p'', z \rangle_{-1,1}\, dt \\
&\quad - \int_0^T \langle (\mathcal{G}(y(q^*), p) + (u^* + v^*) p, z \rangle_{-1,1}\, dt \\
&\quad + \mu \int_0^T \langle \Delta p', z \rangle_{-1,1}\, dt \\
&= \int_0^T (y(q^*) - Y_d, z)_2\, dt.
\end{aligned} \tag{111}$$

By integration by parts and the terminal value of the weak solution p of (105), (111) can be rewritten as

$$\begin{aligned}
&\int_0^T \left(p, z''\right)_2 dt + \left(p'(T), z(T)\right)_2 \\
&\quad - \int_0^T \langle \mathcal{G}(y(q^*), p), z\rangle_{-1,1} dt \\
&\quad - \mu \int_0^T \left(p, \Delta z'\right)_2 dt - \int_0^T (p, (u^* + v^*) z)_2 dt \\
&\quad = \left(\text{by } (110) \text{ and } p'(T) = -y(q^*; T) + Y_d^T\right) \\
&\quad = \int_0^T \left(p, z''\right)_2 dt - \left(y(q^*; T) - Y_d^T, z(T)\right)_2 \\
&\quad - \int_0^T (p, \mathcal{G}(y(q^*), z))_2 dt - \mu \int_0^T \left(p, \Delta z'\right)_2 dt \\
&\quad - \int_0^T (p, (u^* + v^*) z)_2 dt \\
&\quad = \int_0^T (y(q^*) - Y_d, z)_2 dt.
\end{aligned} \tag{112}$$

Since z is the solution of (69), we can obtain the following from (112):

$$\begin{aligned}
&\int_0^T (y(q^*) - Y_d, z)_2 dt + \left(y(q^*; T) - Y_d^T, z(T)\right)_2 \\
&\quad = \int_0^T ((h + l) y(q^*), p)_2 dt.
\end{aligned} \tag{113}$$

Therefore, we can deduce that (109) and (113) imply

$$\begin{aligned}
DJ(u^*, v^*)(h, l) = &\int_0^T (\alpha u^* + y(q^*) p, h)_2 dt \\
&+ \int_0^T (-\beta v^* + y(q^*) p, l)_2 dt.
\end{aligned} \tag{114}$$

Since $q^* = (u^*, v^*) \in \mathcal{F}_{ad}$ is an optimal pair in (71), we know that

$$\begin{aligned}
&D_u J(u^*, v^*)(h) \geq 0, \\
&D_v J(u^*, v^*)(l) \leq 0, \\
&\qquad (h, l) \in \mathcal{F}^2.
\end{aligned} \tag{115}$$

Therefore, we can obtain the following from (114) and (115):

$$\begin{aligned}
&\int_0^T (\alpha u^* + y(q^*) p, h)_2 dt \geq 0, \\
&\int_0^T (-\beta v^* + y(q^*) p, l)_2 dt \leq 0,
\end{aligned} \tag{116}$$

where $(h, l) \in \mathcal{F}^2$. By considering the signs of the variations h and l in (116), which depend on u^* and v^*, respectively, we can deduce the following from (116) (possibly not unique):

$$\begin{aligned}
u^* &= \max\left\{a, \min\left\{-\frac{y(q^*) p}{\alpha}, b\right\}\right\}, \\
v^* &= \max\left\{c, \min\left\{\frac{y(q^*) p}{\beta}, d\right\}\right\}.
\end{aligned} \tag{117}$$

This completes the proof. □

4.2.2. Case of Velocity Observation C_2. In this observation case, we consider the cost function associated with the control system (68):

$$\begin{aligned}
J(u, v) = &\frac{1}{2}\left\|y'(q) - Y_d\right\|_{L^2(Q)}^2 + \frac{\alpha}{2}\|u\|_{L^2(Q)}^2 \\
&- \frac{\beta}{2}\|v\|_{L^2(Q)}^2,
\end{aligned} \tag{118}$$

where $Y_d \in L^2(Q)$ is a desired value and the positive constants α and β are the relative weight of the second and the third terms on the RHS of (118). Now we turn to the necessary optimality conditions that have to be satisfied by each solution of the minimax optimal control problem with the cost (118). For this purpose, as proposed in a previous study [8], we introduce the following adjoint equation corresponding to (68), in which $q = (u, v)$ is replaced by $q^* = (u^*, v^*)$:

$$\begin{aligned}
&p' + \int_t^T (\mathcal{G}(y(q^*), p) + (u^* + v^*) p) ds + \mu \Delta p \\
&\quad = y'(q^*) - Y_d \quad \text{in } Q, \\
&p = 0 \quad \text{on } \Sigma, \\
&p(T, x) = 0 \quad \text{in } \Omega,
\end{aligned} \tag{119}$$

where $\mathcal{G}(\cdot, \cdot)$ is defined in (33).

Remark 16. Usually, adjoint systems of second order problems are also second order (cf. Lions [9]) as long as they are meaningful. However, we have a barrier in this quasilinear (68). If we derive a formal second order adjoint system related to the velocity observation with the cost (118), then it is hard to explain the well-posedness. To overcome this difficulty, we follow the idea given in [8, 11], in which it is adopted that the first-order integrodifferential system as an appropriate adjoint system instead of the formal second order adjoint system.

Proposition 17. *Equation (119) admits a unique weak solution p satisfying*

$$p \in H^1\left(0, T; H_0^1, L^2\right) \cap C\left([0, T]; H_0^1\right), \tag{120}$$

where $H^1(0, T; H_0^1, L^2)$ is the solution space of (119) given by

$$\begin{aligned}
&H^1\left(0, T; H_0^1, L^2\right) \\
&\quad = \left\{\phi \mid \phi \in L^2\left(0, T; H_0^1\right),\ \phi' \in L^2(Q)\right\}.
\end{aligned} \tag{121}$$

Proof. Since

$$\int_{T-t}^{T} \left(\mathcal{G}\left(y\left(q^*\right), p\right) + \left(u^* + v^*\right) p\right)(s)\, ds = \int_0^t \left(\mathcal{G}\left(y\left(q^*\right), p\right) + \left(u^* + v^*\right) p\right)(T-\sigma)\, d\sigma, \tag{122}$$

the time reversed equation of (119) ($t \longrightarrow T-t$ in (119)) is given by

$$\begin{aligned} &-\psi' + \int_0^t \left(\mathcal{G}\left(y\left(q^*\right), \psi\right) + \left(u^* + v^*\right)\psi\right) d\sigma + \mu\Delta\psi \\ &\quad = -y'\left(q^*\right) - Y_d \quad \text{in } Q, \\ &\psi = 0 \quad \text{on } \Sigma, \\ &\psi(0,x) = 0 \quad \text{in } \Omega, \end{aligned} \tag{123}$$

where $\psi(\cdot) = p(T-\cdot)$. From (106) and $-y'(q^*) - Y_d \in L^2(Q)$, it is verified that all requirements of Dautray and Lions [23, pp.656-661] are satisfied with (123). Therefore, it readily follows that there exists a unique weak solution $\psi \in H^1(0,T;H_0^1,L^2) \cap C([0,T];H_0^1)$ of (123).

This completes the proof. □

We now discuss the first-order optimality conditions for the minimax optimal control problem (71).

Theorem 18. *If α and β in the cost (118) are large enough, then an optimal control $u^* \in \mathcal{U}_{ad}$ and a disturbance $v^* \in \mathcal{V}_{ad}$, namely, an optimal pair $q^* = (u^*, v^*) \in \mathcal{F}_{ad}$ satisfying (71), can be given by:*

$$\begin{aligned} u^* &= \max\left\{a, \min\left\{\frac{y\left(q^*\right)p}{\alpha}, b\right\}\right\}, \\ v^* &= \max\left\{c, \min\left\{-\frac{y\left(q^*\right)p}{\beta}, d\right\}\right\}, \end{aligned} \tag{124}$$

where p is the weak solution of (119).

Proof. Let $q^* = (u^*, v^*) \in \mathcal{F}_{ad}$ be an optimal pair in (71) with the cost (118) and $y(q^*)$ be the corresponding weak solution of (68).

By analogy with the proof of Theorem 15, the Gâteaux derivative of the cost (118) at $q^* = (u^*, v^*)$ in the direction $w = (h,l) \in \mathcal{F}^2$ that satisfies $q^* + \epsilon w \in \mathcal{F}_{ad}$ for sufficiently small $\epsilon > 0$ is given by

$$\begin{aligned} &DJ\left(u^*, v^*\right)(h,l) \\ &\quad = \lim_{\epsilon \longrightarrow 0^+} \frac{J\left(u^* + \epsilon h, v^* + \epsilon l\right) - J\left(u^*, v^*\right)}{\epsilon} \\ &\quad = \int_0^T \left(y'\left(q^*\right) - Y_d, z'\right)_2 dt + \alpha \int_0^T \left(u^*, h\right)_2 dt \\ &\qquad - \beta \int_0^T \left(v^*, l\right)_2 dt, \end{aligned} \tag{125}$$

where $z = Dy(q^*)w$ is a solution of (69). We multiply both sides of the weak form of (119) by z' and integrate it over $[0,T]$. Then, we have

$$\begin{aligned} &\int_0^T \left(p', z'\right)_2 dt \\ &\quad + \int_0^T \left\langle \int_t^T \left(\mathcal{G}\left(y\left(q^*\right), p\right) + \left(u^* + v^*\right) p\right) ds, \right. \\ &\quad \left. z' \right\rangle_{-1,1} dt - \mu \int_0^T \left(\nabla p, \nabla z'\right)_2 dt \\ &\quad = \int_0^T \left(y'\left(q^*\right) - Y_d, z'\right)_2 dt. \end{aligned} \tag{126}$$

By integration by parts and the terminal value of the weak solution p of (119), (126) can be rewritten as

$$\begin{aligned} &-\int_0^T \left(p, z''\right)_2 dt \\ &\quad + \int_0^T \left\langle \mathcal{G}\left(y\left(q^*\right), p\right) + \left(u^* + v^*\right) p, z \right\rangle_{-1,1} dt \\ &\quad + \mu \int_0^T \left(p, \Delta z'\right)_2 dt = (\text{By } (110)) \\ &\quad = -\int_0^T \left(p, z''\right)_2 dt \\ &\quad + \int_0^T \left(p, \mathcal{G}\left(y\left(q^*\right), z\right) + \left(u^* + v^*\right) z\right)_2 dt \\ &\quad + \mu \int_0^T \left(p, \Delta z'\right)_2 dt = \int_0^T \left(y'\left(q^*\right) - Y_d, z'\right)_2 dt. \end{aligned} \tag{127}$$

Since z is the solution of (69), we can obtain the following from (127):

$$\int_0^T \left(y'\left(q^*\right) - Y_d, z'\right)_2 dt = -\int_0^T \left((h+l)\, y\left(q^*\right), p\right)_2 dt. \tag{128}$$

Therefore, we can deduce that (125) and (128) imply

$$\begin{aligned} DJ\left(u^*, v^*\right)(h,l) &= \int_0^T \left(\alpha u^* - y\left(q^*\right)p, h\right)_2 dt \\ &\quad + \int_0^T \left(-\beta v^* - y\left(q^*\right)p, l\right)_2 dt. \end{aligned} \tag{129}$$

Since $q^* = (u^*, v^*) \in \mathcal{F}_{ad}$ is an optimal pair in (71), we know that

$$\begin{aligned} &D_u J\left(u^*, v^*\right)(h) \geq 0, \\ &D_v J\left(u^*, v^*\right)(l) \leq 0, \\ &\qquad (h,l) \in \mathcal{F}^2. \end{aligned} \tag{130}$$

Therefore, we can obtain the following from (129) and (130):

$$\begin{aligned}&\int_0^T (\alpha u^* - y(q^*)\,p, h)_2\, dt \geq 0,\\&\int_0^T (-\beta v^* - y(q^*)\,p, l)_2\, dt \leq 0,\end{aligned} \tag{131}$$

where $(h, l) \in \mathcal{F}^2$. By considering the signs of the variations h and l in (131), which depend on u^* and v^*, respectively, we can deduce from (131) that (possibly not unique)

$$\begin{aligned}u^* &= \max\left\{a, \min\left\{\frac{y(q^*)\,p}{\alpha}, b\right\}\right\},\\v^* &= \max\left\{c, \min\left\{-\frac{y(q^*)\,p}{\beta}, d\right\}\right\}.\end{aligned} \tag{132}$$

This completes the proof. □

5. Conclusion

The Fréchet differentiability from a bilinear control input into the solution space of a damped Kirchhoff-type equation is verified. As an application of this result, we proposed a minimax optimal control problem for the above state equation by using quadratic cost functions that depend on control and disturbance (or noise) variables. By utilizing the Fréchet differentiability of the solution map and the continuity of the solution map in a weak topology, we have proven existence of the optimal control of the worst disturbance, called the optimal pair under some hypothesis. And we derived necessary optimality conditions that any optimal pairs must satisfy in some observation cases.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

Authors' Contributions

The author read and approved the final manuscript.

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