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Research Article

Stability and Global Sensitivity Analysis for an Agree-Disagree Model: Partial Rank Correlation Coefficient and Latin Hypercube Sampling Methods

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In this paper, we present a new mathematical model that describes agree-disagree opinions during polls. We first present the model and its different compartments. Then, we use the next-generation matrix method to compute thresholds of equilibrium stability. We perform the stability analysis of equilibria to determine under which conditions these equilibrium points are stable or unstable. We show that the existence and stability of these equilibria are controlled by the calculated thresholds. Finally, we also perform several computational and statistical experiments to validate the theoretical results obtained in this work. To study the influence of various parameters on these thresholds and to identify the most influential parameters, a global sensitivity analysis is carried out based on the partial rank correlation coefficient method and the Latin hypercube sampling.

1. Introduction

Participation in political life requires citizens' attention, time, knowledge, money, and motivation. Citizens will participate if they receive benefits commensurate with the cost of participation. Perhaps, the debate on voting is the subject of the most studied political behavior. Individual economic, social, and psychological costs and voting benefits are well known [1]. Candidates and the media often use polls in election campaigns to determine which candidates are in the lead and who are likely to emerge victoriously. Opinion polls are surveys of intent for a sample of voters, while future results allow participants to negotiate and discuss opinions based on a particular outcome. Opinion polls revealed the expected voting quotas. Public opinion polls are widely used to identify people's political positions, vote, and other behaviors by asking questions about their opinions, activities, and personal characteristics. Answers to these questions are then counted, statistically analyzed, and interpreted. Polls' result also provides parties with the opportunity to refine their campaign strategies. In the long run, parties change their positions in public opinion [2].

Polls then play a key role in contemporary political campaigns. Party performance updates receive considerable media attention and often serve as a basis for political commentary in the weeks leading up to the election day. We know that knowing the positions of the electorate can affect voter behavior [3, 4]. However, it is unclear how voting shapes party leaders' strategies during the election campaign. Public opinion is often criticized for influencing voter perceptions and focusing on popularity rather than politics. By strategically communicating the poll results, parties can shape voters' vision of competition and encourage popular mobilization [5]. This explains why, at the beginning of the 20th century, more than thirty democratic countries banned the publication of opinion polls near elections [6, 7].

To fully understand the role of polls before the elections, it is necessary to take into account the offer of campaigns. This is particularly important in contexts of increased polarization, where holidays play an increasing role in determining the type of information that reaches voters [8]. At any time, party leaders may choose to speak of candidates from their party or other candidates for election, focusing on various political or gender issues. Voter signals in opinion polls can influence party leaders' decisions to balance these different elements of campaign speech [9].

In the electoral process, where there are only two political parties, the chances of creating strong political polarization are increasing. For example, U. S. citizens face a polarizing scenario during a presidential election, where they must vote for a two-party decision: Republicans and Democrats. Previous studies have shown that U. S. elections are attracting the Internet, where both Twitter and blogs are showing high political polarization [10, 11].

Other political scenarios requiring a bilateral decision are second-round electoral processes. While citizens of the first round can vote from a wide range of different political parties, in the last round, they can vote only for the two final candidates. Voters may not be fully identified with either side, but they still have to take a side. Previous articles have shown that this second round increases political polarization in the country [12]. In [13], the authors analyzed the 2017 presidential election in Chile and measured the resulting political polarization. Of a minority of qualified users, they could estimate the opinion of the majority.

In this contribution, we examine how voters' preferences and expectations compete with each other when dealing with voter support information to candidates. So, we start by developing a mathematical model to describe the evolution of opinions to predict the probability of results, and then we calculate and analyze the equilibrium points of the model by deriving the important stability thresholds for each equilibrium state.

Through the development of more efficient models, reliable statistical and mathematical methods are needed to improve the accuracy of modeling. One of the most recently popular ways is sensitivity analysis (SA) methods. Sensitivity analysis is used for a variety of reasons, such as developing decisions or recommendations, communicating, understanding or quantifying the system, and developing models. In the development of the model, it can be used to verify the validity or accuracy of the model, to simplify, calibrate, weak process, or lost data, and even to identify the important parameter for other studies [14].

The paper is organized as follows: Section 2 introduces our new model, giving some details about interactions between different compartments and parameters of the model. In Section 3, we derive basic reproduction numbers. Section 4 provides the results of the stability analysis of equilibria. In Section 5, sensitivity analysis is performed to identify the most important parameter in the proposed model, and Section 6 concludes the paper.

2. Presentation of the Model

There are many scenarios involving a binary decision. The poll model that we present here describes the opinions of agreement (or approval) and disagreement (or disapproval), regarding a candidate or idea, in polls preceding the elections. Note that the model can also describe situations in which there are more than two candidate parties because we can always reduce the situation to two decisions. For example, if there are four parts A, B, C, and D, we wish to study the political position of party A, and we can then examine the two subsets {A} and {B, C, D} of the investigation. Votes for A are considered agree, and votes for B, C, or D are considered disagree. More simply, we consider the opinion poll of voters on the performance of the party studied. As an example, we can cite the "approve or disapprove" poll done by Thomson Reuters on how Donald Trump fulfills his role as the president [15].

Without loss of generality, we devise a mathematical model describing the evolution of agree and disagree opinions during polls (surveys), and the types of surveys we consider here are surveys that can be answered in agreement, disagreement, or otherwise. Thus, the population targeted by the poll is regrouped into three groups: agree, disagree, and ignorant individuals.

The model herewith has been formulated using three compartments. Each of them has been described as follows:

(1) Ignorant (I): people who do not know about the poll or those who abstain from voting for personal reasons

(2) Agree (A): people in agreement with the idea being studied

(3) Disagree (D): people in disagreement with the idea being studied

All contacts are modeled by the standard incidence rate. For the modeling processes, a set of assumptions has been used. These are as follows:

(1) The targeted population is well mixed, that is, the ignorant individuals are homogeneously spread throughout the population

(2) Recruitment and mortality are negligible under the temporal scale consideration; therefore, no individual is recruited and no individual dies during the poll

(3) Individuals have the right to communicate with each other and can thus convince one another

(4) People who are unsure of their opinion are ignorant

(5) People who abstain from voting are ignorant

Everyone has their reasons for agreement or disagreement. An ignorant person can be persuaded by someone who agrees at a rate β_1 or by someone who does not agree with the opinion at a rate β_2. A person agreeing with the opinion may be persuaded by someone who does not agree at a rate α_2, or a person who disagrees with the opinion may be persuaded by someone who agrees at a rate α_1. People can abstain from voting or lose interest without any direct contact with individuals from the opposite opinion group, then agree people become ignorants at the rate γ_1, and disagree people become ignorants at the rate γ_2. A flowchart describing different interactions between the compartments of the model is presented in Figure 1.

All these assumptions and considerations are written as the following system of ordinary differential equations:

$$I' = -\beta_1 \frac{AI}{N} - \beta_2 \frac{DI}{N} + \gamma_1 A + \gamma_2 D, \tag{1}$$

$$A' = \beta_1 \frac{AI}{N} + \alpha_1 \frac{AD}{N} - \alpha_2 \frac{AD}{N} - \gamma_1 A, \tag{2}$$

$$D' = \beta_2 \frac{DI}{N} + \alpha_2 \frac{AD}{N} - \alpha_1 \frac{AD}{N} - \gamma_2 D, \tag{3}$$

where $I(0) \geq 0$, $A(0) \geq 0$, $D(0) \geq 0$, and $N = I + A + D$. Note that $N' = I' + A' + D' = 0$; thus, the population size N is considered as a constant in time. We can easily prove that, for nonnegative initial conditions, the solutions of systems (1)–(3) are nonnegative. To do this, recall that by [16] the system of equation

$$x' = f(x_1, x_2, \ldots, x_k), \tag{4}$$

with

$$x(0) = x_0 \geq 0, \tag{5}$$

is a positive system if and only if $\forall i = 1, 2, \ldots, k$.

$$x_i' = f_i(x_1 \geq 0, x_2 \geq 0, \ldots, x_i = 0, \ldots, x_k \geq 0) \geq 0. \tag{6}$$

Thus, for models (1)–(3), it is easy to verify that

$$\begin{gathered} I = 0 \Longrightarrow I' \geq 0, \\ A = 0 \Longrightarrow A' \geq 0, \\ D = 0 \Longrightarrow D' \geq 0. \end{gathered} \tag{7}$$

Therefore, all the solutions of systems (1)–(3) are nonnegative.

It is also clear that the solutions of models (1)–(3) are bounded based on the fact that $N = I + A + D$ is constant, and then $S \leq N$, $A \leq N$, and $D \leq N$. Therefore, we will focus to study models (1)–(3) in the closed positively invariant feasible set given by

$$\Omega = \left(\frac{(I, A, D) \in \mathbb{R}_+^3}{I + A + D = N} \right). \tag{8}$$

A summary of parameter description is given in Table 1.

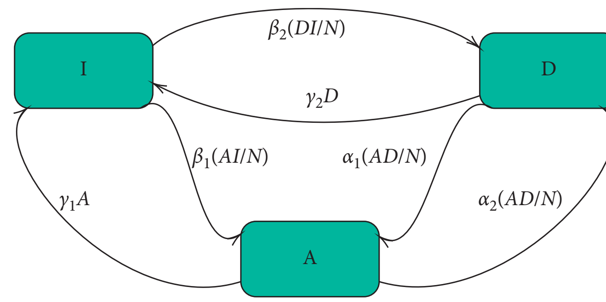


Figure 1: Flowchart for models (1)–(3).

Table 1: Parameter descriptions.

Parameter	Description
β_1	Ignorant to agree transmission rate
β_2	Ignorant to disagree transmission rate
α_1	Disagree to agree transmission rate
α_2	Agree to disagree transmission rate
γ_1	Interest loss factor of agree individuals
γ_2	Interest loss factor of disagree individuals

3. Thresholds: Basic Reproductive Numbers

In epidemiology, the basic reproductive number R_0 (or epidemic threshold) is defined as the average number of secondary cases of an infection produced by a "typical" infected individual during his/her entire life as infectious when introduced in a population of susceptibles [17–21]. The threshold R_0 is mathematically characterized in terms of infection transmission as a "demographic process," but offspring production is not seen as giving birth in a demographic sense, but it causes new infections through transmission [22–24]. Thus, the infection process can be considered as a "consecutive generation of infected individuals." The following growing generations indicate a growing population (i.e., an epidemic), and the growth factor for each generation indicates the potential for growth. So, the mathematical characterization of R_0 is this growth factor [22]. Generally, if $R_0 > 1$, an epidemic occurs, whereas if $R_0 < 1$, there will probably be no epidemic.

Following this definition, we will define our thresholds as follows: R_{D0} is the average number of new disagreements produced by an individual disagreeing introduced in a population of ignorant people during the period in which he or she was in this opinion. And, R_{A0} is the average number of new agreements produced by an individual in agreement and that was introduced into a population of ignorant people during the period in which he or she was in that opinion.

For the analysis of epidemic models, the first step is to calculate the disease-free equilibrium (DFE). This equilibrium point is then used to calculate the basic reproductive number using the next-generation matrix method. The objective of this section is only the calculation of the thresholds and not the equilibrium states of the model. But, in this method, we have to determine the equilibrium states when $A = 0$ and when $D = 0$.

In this contribution, let R_{X0} be the threshold of growing of the opinion X (either $X = A$ "agree" or $X = D$ "disagree"). Then, R_{D0} is the threshold associated with the disagree-free equilibrium, while R_{A0} is the one associated with the agree-free equilibrium.

We calculate the equilibria for the aforementioned model, and based on the next-generation matrix approach, we derive associated thresholds.

The points of equilibrium of systems (1)–(3) are the solutions of

$$I' = A' = D' = 0, \tag{9}$$

for the disagree-free equilibrium when there is no negative opinion, i.e., if we put $D = 0$. This gives

$$I^* = \frac{\gamma_1}{\beta_1} N,$$
$$A^* = \frac{N(\beta_1 - \gamma_1)}{\beta_1}, \tag{10}$$

where I^* and A^* represent the numbers of ignorant and agree individuals, respectively, in the absence of disagree people.

Therefore, for the system governed by (1)–(3), the disagree-free equilibrium is

$$e_1 = \left(\frac{\gamma_1}{\beta_1} N, \frac{N(\beta_1 - \gamma_1)}{\beta_1}, 0\right). \tag{11}$$

Following the second generation approach [22], we compute the threshold R_{D0} associated to the disagree-free equilibrium, which is

$$R_{D0} = \frac{\alpha_2\beta_1 - \alpha_2\gamma_1 + \beta_2\gamma_1}{\alpha_1\beta_1 - \alpha_1\gamma_1 + \beta_1\gamma_2} \tag{12}$$

for the agree-free equilibrium when there is no positive opinion, i.e., if we put $A = 0$. This gives

$$I^* = \frac{\gamma_2}{\beta_2} N,$$
$$D^* = \frac{N(\beta_2 - \gamma_2)}{\beta_2}, \tag{13}$$

where I^* and D^* represent the numbers of ignorant and disagree individuals, respectively, in the absence of agree people.

Therefore, for the system governed by (1)–(3), the agree-free equilibrium is

$$e_2 = \left(\frac{\gamma_2}{\beta_2} N, 0, \frac{N(\beta_2 - \gamma_2)}{\beta_2}\right). \tag{14}$$

By also following the second generation approach [22], we compute the threshold R_{A0} associated to the agree-free equilibrium, which is

$$R_{A0} = \frac{\alpha_1\beta_2 - \alpha_1\gamma_2 + \beta_1\gamma_2}{\alpha_2\beta_2 - \alpha_2\gamma_2 + \beta_2\gamma_1}. \tag{15}$$

Throughout this paper, we consider the following assumption: the model parameters verify

$$\beta_1 - \gamma_1 > -\frac{\beta_1\gamma_2}{\alpha_1},$$
$$\beta_1 - \gamma_1 > -\frac{\beta_2\gamma_1}{\alpha_2},$$
$$\beta_2 - \gamma_2 > -\frac{\beta_2\gamma_1}{\alpha_2},$$
$$\beta_2 - \gamma_2 > -\frac{\beta_1\gamma_2}{\alpha_1}. \tag{16}$$

Therefore, from the above assumption, we get

$$R_{D0} = \frac{\alpha_2\beta_1 - \alpha_2\gamma_1 + \beta_2\gamma_1}{\alpha_1\beta_1 - \alpha_1\gamma_1 + \beta_1\gamma_2} > 0,$$
$$R_{A0} = \frac{\alpha_1\beta_2 - \alpha_1\gamma_2 + \beta_1\gamma_2}{\alpha_2\beta_2 - \alpha_2\gamma_2 + \beta_2\gamma_1} > 0. \tag{17}$$

4. Stability Analysis

In order to analyze in terms of the proportions of ignorant, agree, and disagree individuals, let $i = (I/N)$, $a = (A/N)$, and $d = (D/N)$ denote the fraction of the classes I, A, and D in the population, respectively. After some calculations and replacing I by i, A by a, and D by d, equations (1)–(3) can be written as

$$i' = -\beta_1 ai - \beta_2 di + \gamma_1 a + \gamma_2 d, \tag{18}$$

$$a' = \beta_1 ai + \alpha_1 ad - \alpha_2 ad - \gamma_1 a, \tag{19}$$

$$d' = \beta_2 di + \alpha_2 ad - \alpha_1 ad - \gamma_2 d. \tag{20}$$

From the fact $N = I + A + D$, we have $i + a + d = 1$. Then, model systems (18)–(20) will be reduced to the following two differential equations:

$$a' = \beta_1 a(1 - a - d) + \alpha_1 ad - \alpha_2 ad - \gamma_1 a,$$
$$d' = \beta_2 d(1 - a - d) + \alpha_2 ad - \alpha_1 ad - \gamma_2 d, \tag{21}$$

which can be reduced to

$$a' = \beta_1 a(1 - a - d) + \alpha a\, \gamma_1 a,$$
$$d' = \beta_2 d(1 - a - d) - \alpha ad - \gamma_2 d, \tag{22}$$

where $\alpha = \alpha_1 - \alpha_2$.

4.1. Steady States.

4.1. Steady States. The steady states of system (22) are obtained by solving the system of equations

$$0 = \beta_1 a(1 - a - d) + \alpha ad - \gamma_1 a,$$
$$0 = \beta_2 d(1 - a - d) - \alpha ad - \gamma_2 d. \tag{23}$$

This system has four equilibrium points, and the trivial equilibrium $E_0 = (0, 0)$ is an equilibrium that exists always without any condition. It means there is no survey and there is no need to opinions.

One disagree-free equilibrium $E_1 = (1 - (\gamma_1/\beta_1), 0)$ exists if the condition $\beta_1 > \gamma_1$ holds, and the agree-free equilibrium $E_2 = (0, 1 - (\gamma_2/\beta_2))$ exists if $\beta_2 > \gamma_2$ holds.

The fourth and the positive equilibrium $E^* = (a^*, d^*)$, where

$$a^* = \frac{\alpha\beta_2 - \alpha\gamma_2 + \beta_1\gamma_2 - \beta_2\gamma_1}{\alpha(\alpha - \beta_1 + \beta_2)},$$
$$d^* = -\frac{\alpha\beta_1 - \alpha\gamma_1 + \beta_1\gamma_2 - \beta_2\gamma_1}{\alpha(\alpha - \beta_1 + \beta_2)}, \tag{24}$$

exists if one of the following conditions holds: "$R_{D0} > 1$ and $R_{A0} > 1$ and $\alpha(\alpha - \beta_1 + \beta_2) > 0$" or "$R_{D0} < 1$

and $R_{A0} < 1$ and $\alpha(\alpha - \beta_1 + \beta_2) < 0$." In fact, by a simple calculation, we get

$$\begin{aligned} &\alpha\beta_2 - \alpha\gamma_2 + \beta_1\gamma_2 - \beta_2\gamma_1 > 0 \Longleftrightarrow R_{A0} > 1, \\ &\alpha\beta_1 - \alpha\gamma_1 + \beta_1\gamma_2 - \beta_2\gamma_1 < 0 \Longleftrightarrow R_{D0} > 1. \end{aligned} \tag{25}$$

Thus, from (25), we deduce that $a^* > 0$ and $d^* > 0$. Throughout the article and without loss of generality, we just consider the first condition as a sufficient condition for the existence of E_*.

4.2. Stability of Steady States. The Jacobian matrix of system (22) is

$$J = \begin{pmatrix} J_{11} & a\alpha - a\beta_1 \\ -\alpha d - \beta_2 d & J_{22} \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned} J_{11} &= \alpha d - a\beta_1 - \gamma_1 - \beta_1 (a + d - 1), \\ J_{22} &= -\gamma_2 - a\alpha - \beta_2 d - \beta_2 (a + d - 1). \end{aligned} \tag{27}$$

Proposition 1. *The equilibrium $E_0 = (0, 0)$ is unstable if $\beta_1 > \gamma_1$ or $\beta_2 > \gamma_2$. Otherwise, it is stable.*

Proof. The Jacobian matrix at this equilibrium is

$$J(E_0) = \begin{pmatrix} \beta_1 - \gamma_1 & 0 \\ 0 & \beta_2 - \gamma_2 \end{pmatrix}. \tag{28}$$

It is clear that if $\beta_1 > \gamma_1$ or $\beta_2 > \gamma_2$, we get one positive eigenvalue of $J(E_0)$, and then E_0 is unstable. Else, we have all eigenvalues of $J(E_0)$ having the negative real part, which completes the proof. □

Remark 1. Note that the conditions in the previous proposition imply the existence of E_1 or E_2. Therefore, E_0 is unstable whenever there exists E_1 or E_2.

Proposition 2. *The equilibrium $E_1 = (1 - (\gamma_1/\beta_1), 0)$ is unstable if $R_{D0} > 1$. Otherwise, it is stable.*

Proof. The Jacobian matrix at this equilibrium is

$$J(E_1) = \begin{pmatrix} \gamma_1 - \beta_1 & \dfrac{(\alpha - \beta_1)(\beta_1 - \gamma_1)}{\beta_1} \\ 0 & \alpha\left(\dfrac{\gamma_1}{\beta_1} - 1\right) - \gamma_2 + \dfrac{\beta_2\gamma_1}{\beta_1} \end{pmatrix}. \tag{29}$$

The eigenvalues of $J(E_1)$ are

$$\begin{aligned} \lambda_1 &= \gamma_1 - \beta_1, \\ \lambda_2 &= \alpha\left(\frac{\gamma_1}{\beta_1} - 1\right) - \gamma_2 + \frac{\beta_2\gamma_1}{\beta_1} \\ &= -\frac{\alpha\beta_1 - \alpha\gamma_1 + \beta_1\gamma_2 - \beta_2\gamma_1}{\beta_1}. \end{aligned} \tag{30}$$

It is clear from the existence condition of this equilibrium that $\gamma_1 - \beta_1 < 0$. Thus, the stability of the point of equilibrium E_1 is based on the eigenvalue λ_2 of the matrix $J(E_1)$. By a simple calculation, we have

$$\alpha\beta_1 - \alpha\gamma_1 + \beta_1\gamma_2 - \beta_2\gamma_1 < 0 \Longleftrightarrow R_{D0} > 1. \tag{31}$$

This implies that if $R_{D0} > 1$, E_1 is unstable; else, E_1 is stable. □

Proposition 3. *The equilibrium $E_2 = (0, 1 - (\gamma_2/\beta_2))$ is unstable if $R_{A0} > 1$. Otherwise, it is stable.*

Proof. The Jacobian matrix at this equilibrium is

$$J(E_2) = \begin{pmatrix} \dfrac{\beta_1\gamma_2}{\beta_2} - \alpha\left(\dfrac{\gamma_2}{\beta_2} - 1\right) - \gamma_1 & 0 \\ -\dfrac{(\alpha + \beta_2)(\beta_2 - \gamma_2)}{\beta_2} & \gamma_2 - \beta_2 \end{pmatrix}. \tag{32}$$

The eigenvalues of $J(E_2)$ are

$$\begin{aligned} \lambda_1 &= \frac{\beta_1\gamma_2}{\beta_2} - \alpha\left(\frac{\gamma_2}{\beta_2} - 1\right) - \gamma_1 \\ &= \frac{\alpha\beta_2 - \alpha\gamma_2 + \beta_1\gamma_2 - \beta_2\gamma_1}{\beta_2}, \\ \lambda_2 &= \gamma_2 - \beta_2. \end{aligned} \tag{33}$$

It is clear from the existence condition of this equilibrium that $\gamma_2 - \beta_2 < 0$. Thus, the stability of the point of equilibrium E_2 is based on the eigenvalue λ_1 of the matrix $J(E_2)$. By a simple calculation, we have

$$\alpha\beta_2 - \alpha\gamma_2 + \beta_1\gamma_2 - \beta_2\gamma_1 > 0 \Longleftrightarrow R_{A0} > 1. \tag{34}$$

This implies that if $R_{A0} > 1$, E_2 is unstable; else, E_2 is stable. □

Proposition 4. *The equilibrium $E^* = (a^*, d^*)$, where*

$$\begin{aligned} a^* &= \frac{\alpha\beta_2 - \alpha\gamma_2 + \beta_1\gamma_2 - \beta_2\gamma_1}{\alpha(\alpha - \beta_1 + \beta_2)}, \\ d^* &= -\frac{\alpha\beta_1 - \alpha\gamma_1 + \beta_1\gamma_2 - \beta_2\gamma_1}{\alpha(\alpha - \beta_1 + \beta_2)}, \end{aligned} \tag{35}$$

is stable if "$\beta_1 < \gamma_1$ and $\gamma_2 < \beta_2$ and $\alpha > 0$" or "$\beta_1 > \gamma_1$ and $\gamma_2 > \beta_2$ and $\alpha < 0$."

Proof. The Jacobian matrix at this equilibrium is

$$J(E^*) = \begin{pmatrix} J_{11} & a^*\alpha - a^*\beta_1 \\ -\alpha d^* - \beta_2 d^* & J_{22} \end{pmatrix}, \tag{36}$$

where

$$\begin{aligned} J_{11} &= \alpha d^* - a^*\beta_1 - \gamma_1 - \beta_1 (a^* + d^* - 1), \\ J_{22} &= -\gamma_2 - a^*\alpha - \beta_2 d^* - \beta_2 (a + d - 1). \end{aligned} \tag{37}$$

Using fact (23), we have

$$J_{11} = -\beta_1 a^*,$$
$$J_{22} = -\beta_2 d^*. \tag{38}$$

Then,

$$J(E^*) = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix},$$
$$J_1 = -\frac{\beta_1(\alpha\beta_2 - \alpha\gamma_2 + \beta_1\gamma_2 - \beta_2\gamma_1)}{\alpha(\alpha - \beta_1 + \beta_2)},$$
$$J_2 = \frac{(\alpha - \beta_1)(\alpha\beta_2 - \alpha\gamma_2 + \beta_1\gamma_2 - \beta_2\gamma_1)}{\alpha(\alpha - \beta_1 + \beta_2)}, \tag{39}$$
$$J_3 = \frac{(\alpha + \beta_2)(\alpha\beta_1 - \alpha\gamma_1 + \beta_1\gamma_2 - \beta_2\gamma_1)}{\alpha(\alpha - \beta_1 + \beta_2)},$$
$$J_4 = \frac{\beta_2(\alpha\beta_1 - \alpha\gamma_1 + \beta_1\gamma_2 - \beta_2\gamma_1)}{\alpha(\alpha - \beta_1 + \beta_2)},$$

where the characteristic polynomial of is

$$\lambda^2 + c_1\lambda + c_2, \tag{40}$$

where

$$\begin{aligned} c_1 &= -\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\alpha}, \\ c_2 &= \frac{-1}{\alpha(\alpha - \beta_1 + \beta_2)} \\ &\quad \times \left(\alpha^2\beta_1\beta_2 - \alpha^2\beta_1\gamma_2 - \alpha^2\beta_2\gamma_1 + \alpha^2\gamma_1\gamma_2\right. \\ &\quad + \alpha\beta_1^2\gamma_2 - \alpha\beta_1\beta_2\gamma_1 + \alpha\beta_1\beta_2\gamma_2 - \alpha\beta_1\gamma_1\gamma_2 \\ &\quad - \alpha\beta_1\gamma_2^2 - \alpha\beta_2^2\gamma_1 + \alpha\beta_2\gamma_1^2 + \alpha\beta_2\gamma_1\gamma_2 \\ &\quad \left.+ \beta_1^2\gamma_2^2 - 2\beta_1\beta_2\gamma_1\gamma_2 + \beta_2^2\gamma_1^2\right), \\ &= \frac{-1}{\alpha(\alpha - \beta_1 + \beta_2)} \\ &\quad \times (\alpha\beta_1 - \alpha\gamma_1 + \beta_1\gamma_2 - \beta_2\gamma_1)(\alpha\beta_2 - \alpha\gamma_2 + \beta_1\gamma_2 - \beta_2\gamma_1). \end{aligned} \tag{41}$$

By the conditions "$\beta_1 < \gamma_1$ and $\gamma_2 < \beta_2$ and $\alpha > 0$" or "$\beta_1 > \gamma_1$ and $\gamma_2 > \beta_2$ and $\alpha < 0$," we have $-(\beta_1\gamma_2 - \beta_2\gamma_1/\alpha) > 0$, and by existence conditions and

$$\alpha\beta_2 - \alpha\gamma_2 + \beta_1\gamma_2 - \beta_2\gamma_1 > 0 \Longleftrightarrow R_{A0} > 1,$$
$$\alpha\beta_1 - \alpha\gamma_1 + \beta_1\gamma_2 - \beta_2\gamma_1 < 0 \Longleftrightarrow R_{D0} > 1, \tag{42}$$

we have $c_1 > 0$ and $c_2 > 0$. Using the Routh–Hurwitz stability criterion, we conclude that the equilibrium point E^* is locally asymptotically stable. □

Remark 2

(1) Note that the conditions $R_{D0} > 1$ and $R_{A0} > 1$ lead to the instability of E_1 and E_2 and help in the existence of E^*.

(2) Sufficient conditions in the above proposition could be reduced to $\alpha\ (\beta_1\gamma_2 - \beta_1\gamma_2) < 0$.

A summary of sufficient existence and stability conditions of the steady states of model (11) is given in Table 2.

4.3. Examples. It can be seen from Table 3 that the equilibrium E_0 exists and it is stable in the four cases: example 1: $R_{D0} > 1$ and $R_{A0} > 1$, example 2: $R_{A0} < 1$ and $R_{D0} < 1$, example 3: $R_{D0} < 1$ and $R_{A0} > 1$, and example 4: $R_{D0} > 1$ and $R_{A0} < 1$.

Example 5 shows the existence and the stability of the steady state E_1 in the case of $R_{A0} < 1$ and $R_{D0} < 1$ and in the case of $R_{A0} > 1$ and $R_{D0} < 1$ in example 6. Examples 7 and 8 show the existence and the stability of the equilibrium E_2 in the cases of "$R_{A0} < 1$ and $R_{D0} < 1$" and "$R_{A0} < 1$ and $R_{D0} > 1$," respectively.

Examples 9 and 10 show the possibility of the existence and stability of the two steady states E_1 and E_2 at the same time, that is, "$\beta_1 > \gamma_1$ and $\beta_2 > \gamma_2$ and $R_{A0} < 1$ and $R_{D0} < 1$." These examples give an insight about the stability of each equilibrium for given parameters' values. After some numerical calculations, we noted that, in this situation, the parameters γ_1 and γ_2 may switch from E_1 to E_2 and the inverse, while if $\gamma_1 > \gamma_2$, then E_1 will be more attractive, and if $\gamma_2 > \gamma_1$, then E_2 will be more attractive and that with the same initial conditions. We simulate the model with different initial conditions to illustrate the impact of initial conditions on the stability of E_1 and E_2 in this situation. Figure 2 depicts the stability of E_1 and E_2 at the same time, where we consider the same set of parameters from example 9 in Table 3. By changing the initial conditions, we can see that E_1 is stable when $I(0) = 100$, $A(0) = 100$, and $D(0) = 80$, while it can be seen that E_2 is also stable with this set of parameters but after choosing $I(0) = 100$, $A(0) = 80$, and $D(0) = 100$.

Example 11 shows the existence and the stability of the equilibrium E_*, where "$R_{D0} > 1$ and $R_{A0} > 1$ and $\alpha(\alpha - \beta_1 + \beta_2) > 0$" and "$\beta_1 < \gamma_1$ and $\gamma_2 < \beta_2$ and $\alpha > 0$," while example 12 shows the existence and the stability of E_*, where "$R_{D0} > 1$ and $R_{A0} > 1$ and $\alpha(\alpha - \beta_1 + \beta_2) > 0$" and "$\beta_1 > \gamma_1$ and $\gamma_2 > \beta_2$ and $\alpha < 0$."

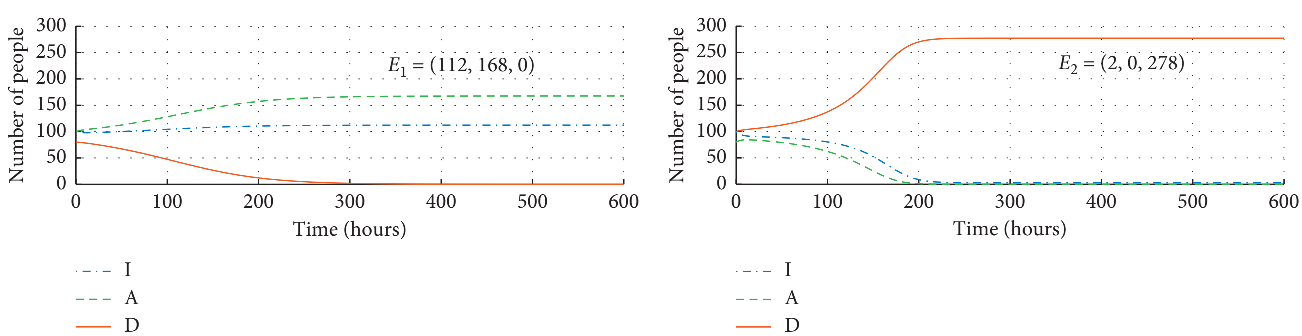
Figure 3 depicts examples of the existence and stability of the equilibrium state E_0 for different parameters' values and threshold R_{D0} and R_{A0} values simulated with initial conditions and parameters' values from Table 3. Figure 4 depicts examples of the existence and stability of the equilibrium state E_1 for different parameters' values and threshold R_{D0} and R_{A0} values simulated with initial conditions and parameters' values from Table 3. It can be seen from Figure 4(b) that the function D decreases towards zero, but it will take a long time. In the Figure 4(c), we have considered 1,200 hours to show that the function D will go to zero but very slowly. Figure 5 depicts examples of the existence and stability of the equilibrium state E_2 for different parameters' values and threshold R_{D0} and R_{A0} values simulated with initial conditions and parameters' values from Table 3. We can see from Figure 5(b) that the function A will also take a very long time to go to zero. In Figure 5(c), we have considered 1,200 hours to show that function A will tend to zero but very slowly. Figure 6 depicts examples of the existence and stability of the equilibrium state E_* for different

Table 2: Summary of sufficient existence and stability conditions of equilibria.

Equilibrium	Existence conditions	Stability conditions
E_0	—	$\beta_1 < \gamma_1$ and $\beta_2 < \gamma_2$
$E_1 = (1 - (\gamma_1/\beta_1), 0)$	$\beta_1 > \gamma_1$	$R_{D0} < 1$
$E_2 = (0, 1 - (\gamma_2/\beta_2))$	$\beta_2 > \gamma_2$	$R_{A0} < 1$
$E_* = (a^*, d^*)$	$R_{D0} > 1$ and $R_{A0} > 1$ and $\alpha(\alpha - \beta_1 + \beta_2) > 0$	$\beta_1 < \gamma_1$ and $\gamma_2 < \beta_2$ and $\alpha > 0$ or $\beta_1 > \gamma_1$ and $\gamma_2 > \beta_2$ and $\alpha < 0$

Table 3: Steady states and parameters' values used in example simulations, where $I(0) = A(0) = D(0) = 100$.

		β_1	β_2	α_1	α_2	γ_1	γ_2	R_{D0}	R_{A0}
1	$E_0 = (300, 0, 0)$	0.0010	0.1010	0.1010	0.3010	0.5010	0.3010	1.9901	2.0730
2	$E_0 = (300, 0, 0)$	0.0010	0.2010	0.2010	0.3010	0.1010	0.5010	0.5000	0.8543
3	$E_0 = (300, 0, 0)$	0.0010	0.0010	0.2010	0.1010	0.1010	0.1010	0.5000	2.0001
4	$E_0 = (300, 0, 0)$	0.0010	0.0010	0.1010	0.2010	0.1010	0.1010	2.0001	0.5000
5	$E_1 = (2.9703, 297.0297, 0)$	0.1010	0.0010	0.2010	0.2010	0.0010	0.0010	0.6634	0.4925
6	$E_1 = (2.9703, 297.0297, 0)$	0.1010	0.0010	0.0010	0.0010	0.0010	0.0010	0.5025	101
7	$E_2 = (2.9703, 0, 297.0297)$	0.0010	0.1010	0.2010	0.2010	0.1010	0.0010	0.4925	0.6634
8	$E_2 = (2.9703, 0, 297.0297)$	0.0010	0.1010	0.0010	0.0010	0.0010	0.0010	101	0.5025
9	$E_1^* = (120.3593, 179.6407, 0)$	0.5010	0.1010	0.1010	0.0010	0.2010	0.0010	0.6688	0.5196
10	$E_2^* = (120.3593, 0, 179.6407)$	0.1010	0.5010	0.0010	0.1010	0.0010	0.2010	0.5196	0.6688
11	$E_* = (133.3333, 10.5556, 156.1111)$	0.1010	0.7010	0.3010	0.0010	0.2010	0.3010	467.7774	1.0672
12	$E_* = (214.2857, 80.4643, 5.2500)$	0.3010	0.4010	0.0010	0.8010	0.2010	0.5010	1.0649	300.8004



(a) (b)

Figure 2: Illustration of the bistability of E_1 and E_2 at the same time using the set of parameters of example 9 from Table 3. Stability of E_1 (a) when $I(0) = 100$, $A(0) = 100$, and $D(0) = 80$. Stability of E_2 (b) when $I(0) = 100$, $A(0) = 80$, and $D(0) = 100$.

parameters' values and threshold R_{D0} and R_{A0} values simulated with initial conditions and parameters' values from examples 11 and 12 of Table 3.

5. Analysis of Thresholds

5.1. No Abstention: No Loss of Interest. In most of the situations, people abstain from voting for personal reasons. Here, we discuss the situation when there is no abstention and there is no loss of interest of voting, i.e., $\gamma_1 = \gamma_2 = 0$. Therefore, R_{D0} and R_{A0} become

$$R_{D0} = \frac{\alpha_2}{\alpha_1}, \quad R_{A0} = \frac{\alpha_1}{\alpha_2}, \tag{43}$$

which means that thresholds R_{D0} and R_{A0} depend only on α_1 and α_2, and the equilibria become

$$E_1 = (1, 0), \quad E_2 = (0, 1), \tag{44}$$

and from the existence conditions in Table 2, we can deduce that there is no E_*.

E_1 exists without any condition, and it is stable if $\alpha_2 < \alpha_1$, and E_2 exists without any condition, and it is stable if $\alpha_1 < \alpha_2$.

This result explains the effect of the polarization on the outcome of polls. When there is no abstention from voting and people keep their interest in voting, then the most influential parameters on the outcome of the polls are the polarization factors α_1 and α_2. For example, during a poll determining the political position of one candidate Y by voting with Approve or Disapprove or otherwise, if candidate Y can convince people by his political vision and/or by other ways, then he can change the course of events in his favor. Mathematically, he makes $\alpha_2 < \alpha_1$. But, if somehow he contributes to making $\alpha_1 < \alpha_2$, then things could spin out of control on the election day.

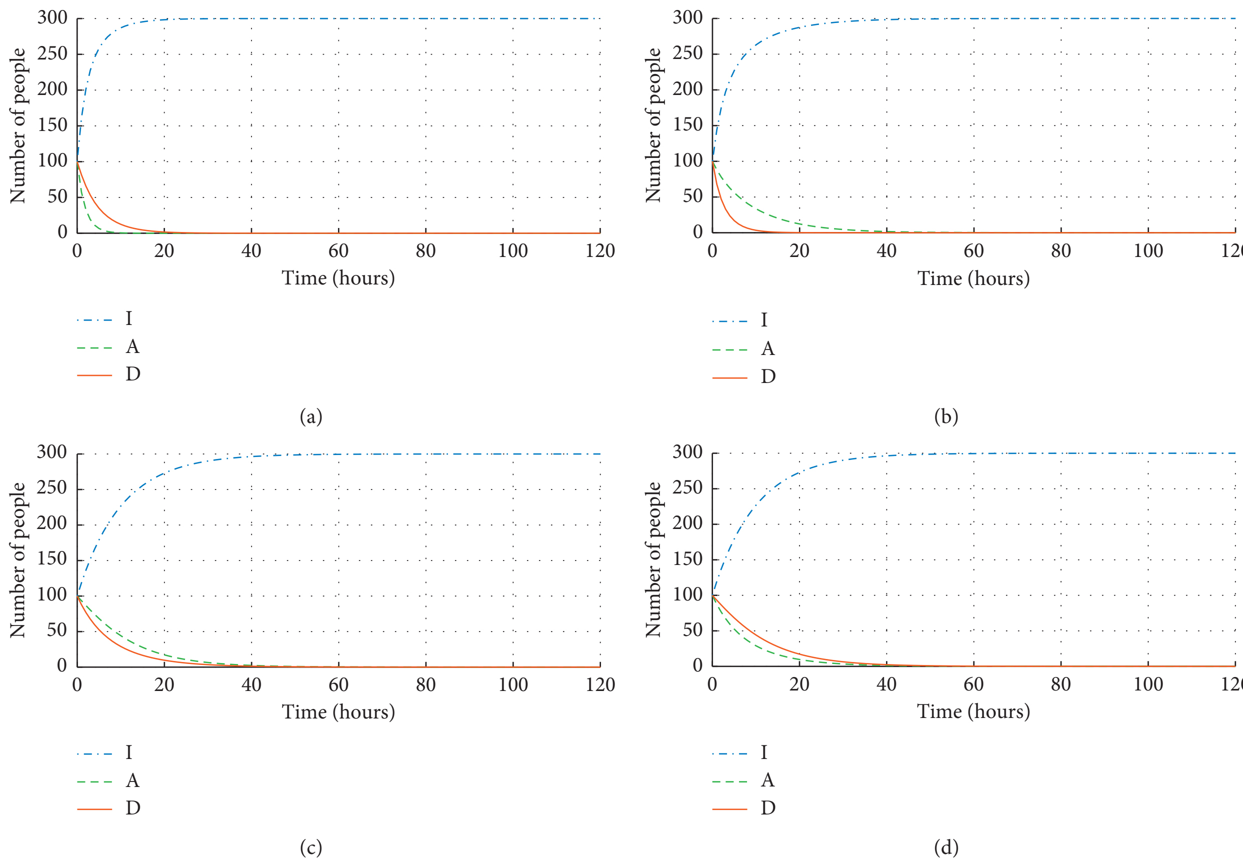


FIGURE 3: Examples of the steady state E_0 simulated with initial conditions $I(0) = 100$, $A(0) = 100$, and $D(0) = 100$ and parameters' values from Table 3. (a) Example 1. (b) Example 2. (c) Example 3. (d) Example 4.

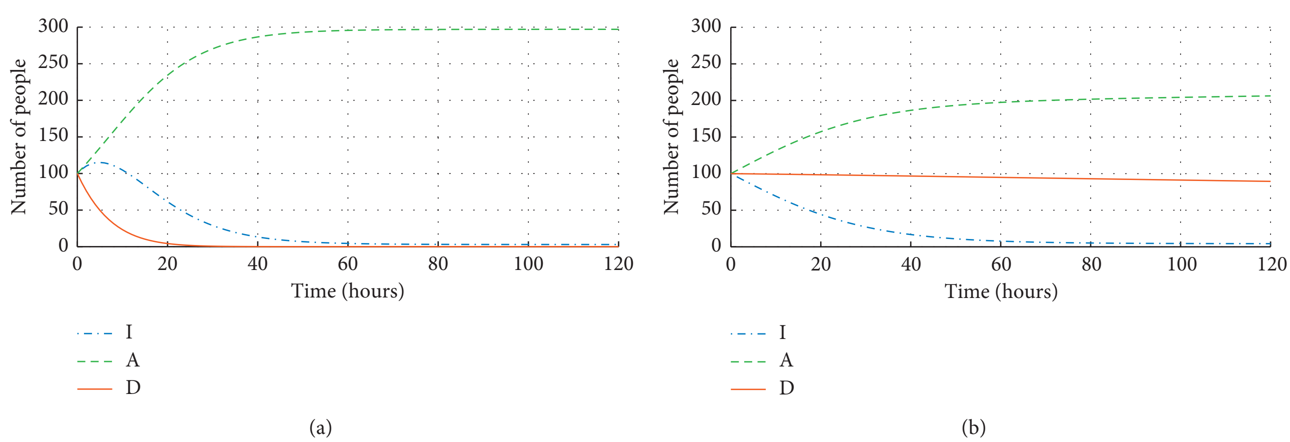
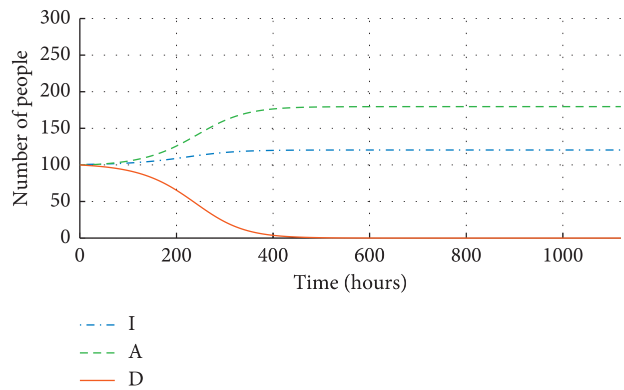
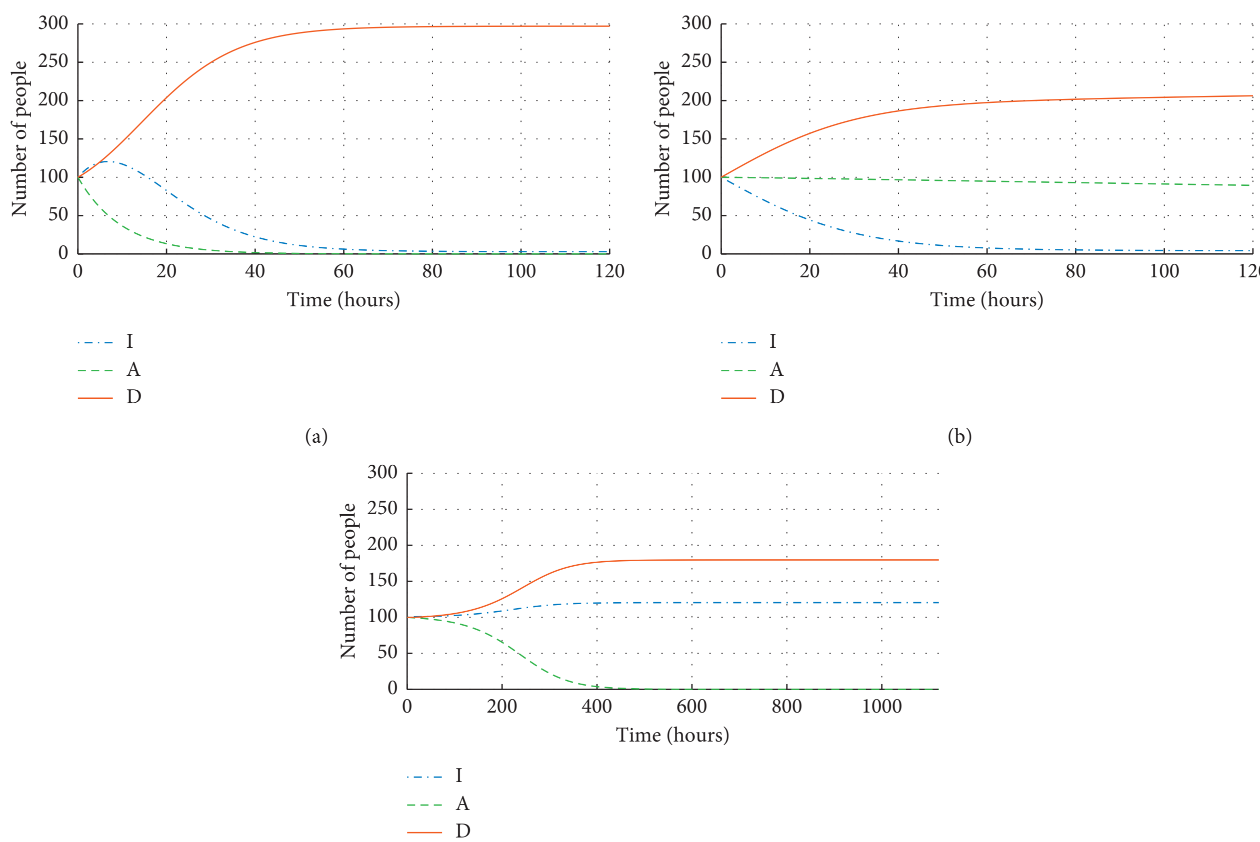


FIGURE 4: Continued.



(c)

Figure 4: Examples of the steady state E_1 simulated with initial conditions $I(0) = 100$, $A(0) = 100$, and $D(0) = 100$ and parameters' values from Table 3. (a) Example 5. (b) Example 6. (c) Example 9.



(a) (b) (c)

Figure 5: Examples of the steady state E_2 simulated with initial conditions $I(0) = 100$, $A(0) = 100$, and $D(0) = 100$ and parameters' values from Table 3. (a) Example 7. (b) Example 8. (c) Example 10.

5.2. One Chance. In some situations, voters are not allowed to modify their choice; then, they could take one side until the end of the survey. In this section, we discuss the situation when there is no persuasion and then there is no polarization, i.e., $\alpha_1 = \alpha_2 = 0$. Therefore, R_{D0} and R_{A0} become

$$\begin{aligned} R_{D0} &= \frac{\beta_2 \gamma_1}{\beta_1 \gamma_2}, \\ R_{A0} &= \frac{\beta_1 \gamma_2}{\beta_2 \gamma_1}, \end{aligned} \tag{45}$$

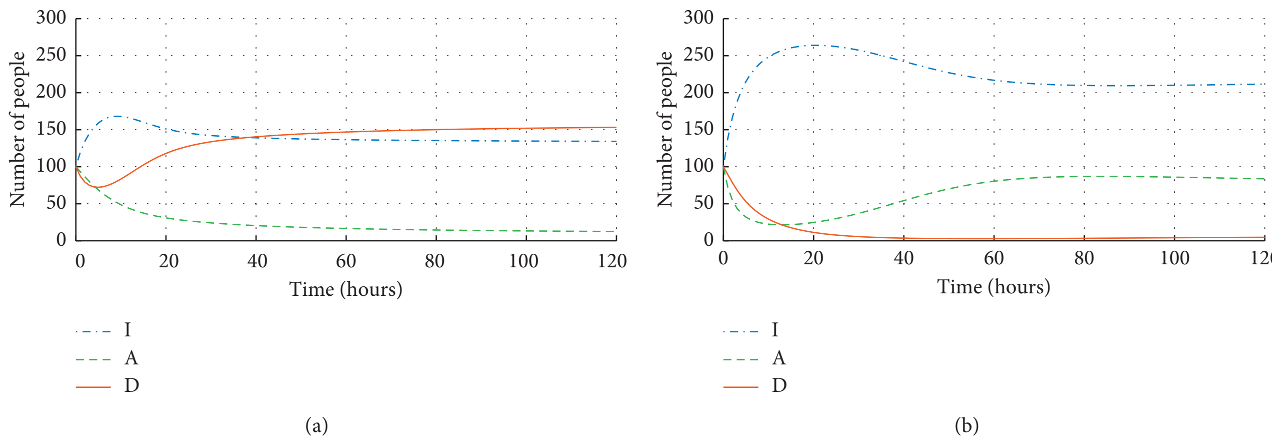


Figure 6: Examples of the steady state E_* simulated with initial conditions $I(0) = 100$, $A(0) = 100$, and $D(0) = 100$ and parameters' values from Table 3. (a) Example 11. (b) Example 12.

which means that thresholds R_{D0} and R_{A0} depend only on β_1, β_2, γ_1, and γ_2, and there is no change in the equilibria E_1 and E_2. From the conditions of the existence in Table 2, we can deduce that there is no E_*. A sufficient condition to make E_1 stable is "$\beta_2 < \beta_1$ and $\gamma_1 < \gamma_2$," and the sufficient condition to make E_2 stable is "$\beta_2 > \beta_1$ and $\gamma_1 > \gamma_2$." This result explains the efficiency of election campaigns. For instance, if there are two candidate parts X and Y and the poll is carried out to study the political position of Y, then votes for Y are considered as agree and votes for X are considered as disagree. If candidate Y presents a successful election campaign, it will attract more people alongside him and increase the number of people agreeing (i.e., $\beta_2 < \beta_1$), which may lead disagree people to lose interest or abstain from voting (i.e., $\gamma_1 < \gamma_2$).

5.3. Statistical Analysis. Here, we use probability distribution functions of the six parameters given in Table 4 sampled by using the Latin hypercube sampling, see Figure 7. We compute the probabilities of equilibria existence and stability conditions. It can be seen from Table 5 that the probability of E_1 to exist is about 0.5960, while its probability of stability is about 0.53. The probability of E_1 to exist and to be stable is 0.4040. The probability of the existence of E_2 is about 0.6250, while the probability of its stability is about 0.5370. The probability of E_1 to exist and to be stable is 0.43. The equilibrium E_* has a probability of existence about 0.0240 and a probability of stability about 0.2190, while E_* exists, and it is stable with a probability of 0.001.

6. Sensitivity Analysis

Global sensitivity analysis (GSA) approach helps to identify the effectiveness of model parameters or inputs and thus provides essential information about the model performance. Out of the many methods of carrying out sensitivity analysis is the partial rank correlation coefficient (PRCC) method that has been used in this paper. The PRCC is a method based on sampling. One of the most efficient methods used for sampling is the Latin hypercube sampling (LHS), which is a type of Monte Carlo sampling [25] as it densely stratifies the input parameters. As the name suggests, the PRCC measures the strength between the inputs and outputs of the model using correlation through the sampling done by the LHS method [26–28].

Table 4: Six input parameters' probability distribution functions.

Parameters	Pdf
β_1	Normal
β_2	Normal
α_1	Triangular
α_2	Triangular
γ_1	Triangular
γ_2	Triangular

Parameters β_1 and β_2 follow normal distribution with mean and standard deviation 0.5 and 0.01, respectively, while parameters α_1, α_2, γ_1, and γ_2 follow triangular distribution with minimum, maximum, and mode as 0.02, 0.8, and 0.51, respectively. A summary of probability distribution functions is given in Table 4. In Latin hypercube sampling approach, probability density function (given in Table 4) for each parameter is stratified into 100 equiprobable (1/100) serial intervals. Then, a single value is chosen randomly from each interval. This produces 100 sets of values of R_{D0} and R_{A0}, from 100 sets of different parameter values mixed randomly, calculated by using equations (4) and (5), respectively.

Sensitivity analysis has been done concerning the basic reproduction numbers R_{D0} and R_{A0}, and the main objective of this section is to identify which parameters are important in contributing variability to the outcome of basic reproduction numbers based on their estimation uncertainty.

To sort the model parameters according to the size of their effect on R_{D0} and R_{A0}, a partial rank correlation coefficient is calculated between the values of each of the six parameters and the values of R_{D0} and R_{A0} in order to identify and measure the statistical influence of any one of the six input parameters on thresholds R_{D0} and R_{A0}. The larger the partial rank correlation coefficient, the larger the influence is on the input parameter affecting the magnitude of R_{D0} and R_{A0}.

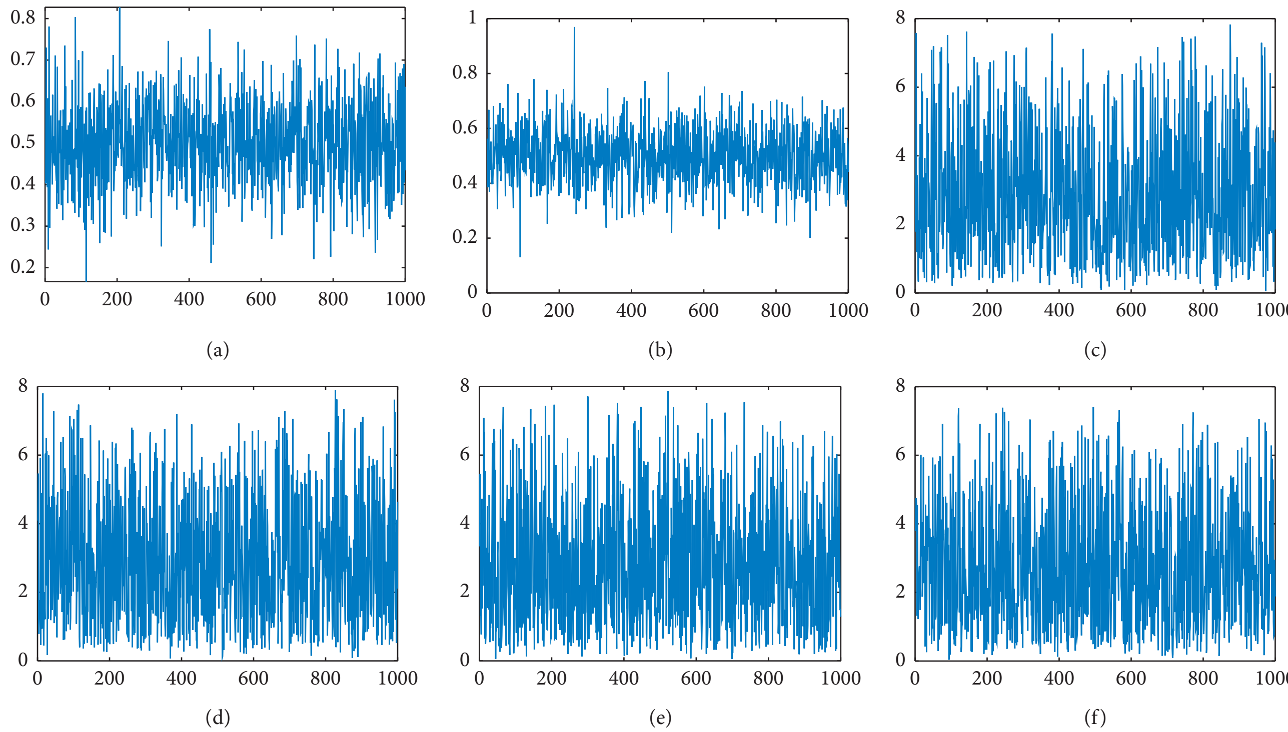


Figure 7: Probability distribution functions of the six sampled input parameters' values. These results have been obtained from Latin hypercube sampling using a sample size of 1000. (a) β_1. (b) β_2. (c) α_1. (d) α_2. (e) γ_1. (f) γ_2.

Table 5: Summary of probabilities of existence and stability conditions of equilibria.

Equilibrium	Probability of existence	Probability of stability	Probability of existence and stability
E_1	0.5960	0.5300	0.4040
E_2	0.6250	0.5370	0.4300
E_*	0.0240	0.2190	0.001

As shown in Table 6, the transmission rate, β_1, and parameters γ_1 and γ_2 are highly correlated with the threshold R_{D0} with corresponding values −0.1374 and 0.1613 and −0.2101, respectively. Moderate correlation exists between transmission rates α_1 and α_2 and R_{D0} with corresponding values as −0.0633 and 0.0669, respectively. Weak correlations have been observed between the transmission rate β_2 and R_{D0} with the corresponding value 0.0025.

As shown in the sensitivity index column of Table 6, the parameter γ_2 accounts for the maximum variability 0.8499 in the outcome of basic reproduction number R_{D0}. The parameter γ_1 is the next to account for the variability 0.7492 in the outcome of R_{D0}. Then, the transmission rate β_1 accounts for the variability 0.5596 in the outcome of R_{D0} followed by the transmission rate α_2 that accounts for the variability 0.5269 in the outcome of R_{D0}. Transmission parameters α_1 and β_2 account for the least variability 0.1065 and 0.0192 in the outcome of basic reproduction number R_{D0}, respectively. Hence, parameters γ_1 and γ_2 and the transmission rate β_1 are the most influential parameters in determining R_{D0}.

It can be seen from Table 7 that the parameter γ_1 is highly correlated with the threshold R_{A0} with the corresponding value

Table 6: PRCCs for R_{D0} and six input parameters and the corresponding sensitivity index.

Parameters	Sampling	PRCCs	p value	Sensitivity index
β_1	LHS	−0.1374	0.1728	0.5596
β_2	LHS	0.0025	0.9800	0.0192
α_1	LHS	−0.0633	0.5318	0.1065
α_2	LHS	0.0669	0.5085	0.5269
γ_1	LHS	0.1613	0.1090	0.7492
γ_2	LHS	−0.2101	0.0359	0.8499

Table 7: PRCCs for R_{A0} and six input parameters and the corresponding sensitivity index.

Parameters	Sampling	PRCCs	p value	Sensitivity index
β_1	LHS	0.2137	0.0328	0.5753
β_2	LHS	−0.1928	0.0546	0.2421
α_1	LHS	0.0857	0.3967	0.0159
α_2	LHS	−0.2076	0.0382	0.3872
γ_1	LHS	−0.3343	6.7425e−04	0.8004
γ_2	LHS	0.2523	0.0113	0.7122

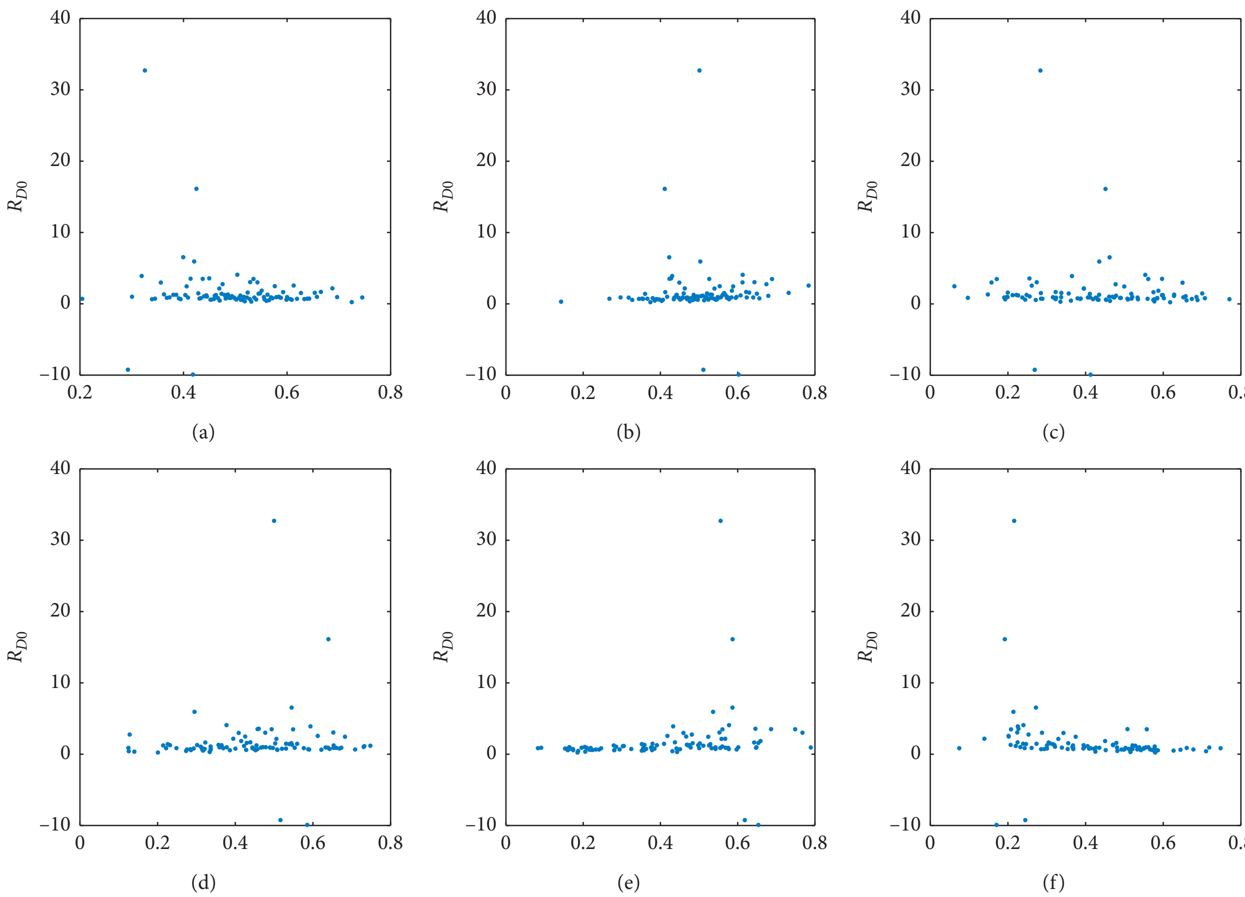


Figure 8: Scatter plots for the basic reproduction number R_{D0} and six sampled input parameters' values. These results have been obtained from Latin hypercube sampling using a sample size of 100. (a) β_1. (b) β_2. (c) α_1. (d) α_2. (e) γ_1. (f) γ_2.

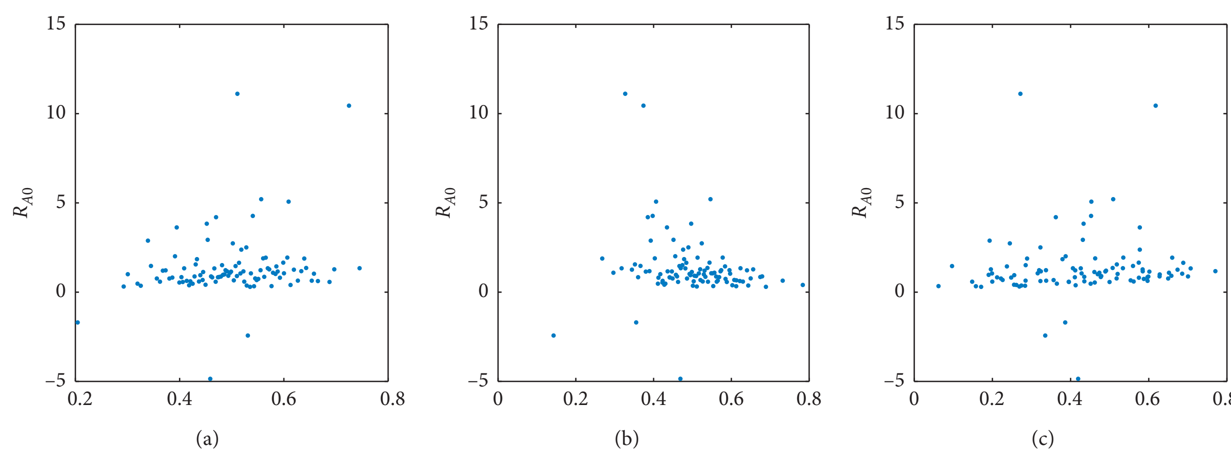


Figure 9: Continued.

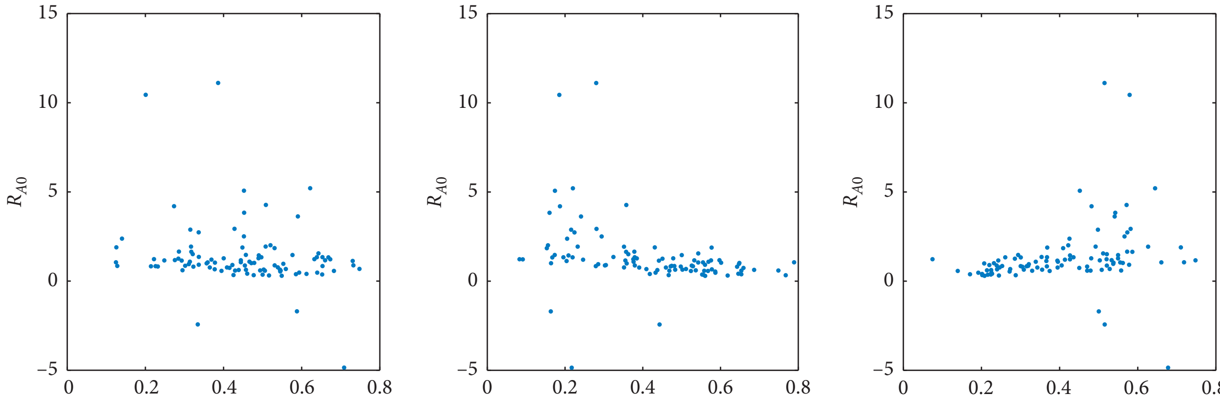


Figure 9: Scatter plots for the basic reproduction number R_{A0} and six sampled input parameters' values. These results have been obtained from Latin hypercube sampling using a sample size of 100. (a) β_1. (b) β_2. (c) α_1. (d) α_2. (e) γ_1. (f) γ_2.

−0.3343. Moderate correlation exists between transmission rates β_1, β_2, and α_2 and the parameter γ_2 and R_{A0} with corresponding values as 0.2137, −0.1928, −0.2076, and 0.2523, respectively. Weak correlations have been observed between the transmission rate α_1 and R_{A0} with the corresponding value 0.0857.

One can see in the sensitivity index column of Table 7 that the parameter γ_1 accounts for the maximum variability 0.8004 in the outcome of basic reproduction number R_{A0}. The parameter γ_2 is the next to account for the variability 0.7122 in the outcome of R_{A0}. The transmission rate β_1 accounts for the variability 0.5753 in the outcome of this threshold followed by the transmission rate α_2 that accounts for the variability 0.3872 in the outcome of R_{A0}. Transmission parameters β_2 and α_1 account for the least variability 0.2421 and 0.0159 in the outcome of basic reproduction number R_{A0}, respectively. Hence, parameters γ_1 and γ_2 and the transmission rate β_1 are also the most influential parameters in determining R_{A0}.

Scatter plots comparing the basic reproduction numbers R_{D0} and R_{A0} against each of the six parameters: $\beta_1, \beta_2, \alpha_1, \alpha_2, \gamma_1$, and γ_2 are shown in Figures 8 and 9, respectively, based on Latin hypercube sampling with a sample size of 100. These scatter plots clearly show the linear relationships between outcome of R_{D0} and R_{A0} and input parameters.

7. Conclusion

In this paper, an IAD model-type compartmental model has been considered to explore agree-disagree opinions during polls. The equations governing the system have been solved to compute equilibrium states, and the next-generation matrix method is used to derive basic reproduction numbers R_{A0} and R_{D0}.

The model exhibits four feasible points of equilibrium, namely, the trivial equilibrium, agree-free equilibrium, disagree-free equilibrium, and the positive equilibrium. Sufficient equilibrium conditions of existence are given, and stability analysis is performed to show under which conditions equilibrium states are stable or unstable. The stability of these points of equilibrium is controlled by the threshold number R_{A0} and R_{D0}. If the threshold, R_{D0}, is less than one, the disagree opinion dies out and the disagree-free equilibrium is stable. If R_{D0} is greater than one, the disagree opinion persists and the disagree-free equilibrium is unstable. If the threshold, R_{A0}, is less than one, the agree opinion dies out and the agree-free equilibrium is stable. If R_{A0} is greater than one, the agree opinion persists and the disagree-free equilibrium is unstable. We simulated some examples with different parameter values to show the existence and the stability of such equilibria.

The probabilities of the existence and probabilities of the stability of equilibria are computed based on the parameters' distribution function sampled with the Latin hypercube sampling method. To identify the most influential parameter in the proposed model, global sensitivity analysis is carried out based on the partial rank correlation coefficient method and Latin hypercube sampling. This statistical study shows that the most influential parameters in the determination of thresholds of equilibria stability are β_1, the polarization parameter of ignorant people by people agreeing, the parameter, γ_1, of the loss of interest of the people agreeing, and finally γ_2, the parameter of loss of interest of disagreeing people.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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