Internat. J. Math. & Math. Sci.
Vol. 5 No. 4 (1982) 779-784

ON SEPARABLE ABELIAN EXTENSIONS OF RINGS

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(Received February 9, 1982)

ABSTRACT. Let R be a ring with 1, G $(= \langle \rho_1 \rangle \times \ldots \times \langle \rho_m \rangle)$ a finite abelian automorphism group of R of order n where $\langle \rho_i \rangle$ is cyclic of order n_i for some integers n, n_i, and m, and C the center of R whose automorphism group induced by G is isomorphic with G. Then an abelian extension $R[x_1,\ldots,x_m]$ is defined as a generalization of cyclic extensions of rings, and $R[x_1,\ldots,x_m]$ is an Azumaya algebra over K $(= C^G = \{c \text{ in } C \,/\, (c)\rho_i = c \text{ for each } \rho_i \text{ in } G\})$ such that $R[x_1,\ldots,x_m] \cong R^G \otimes_K C[x_1,\ldots,x_m]$ if and only if C is Galois over K with Galois group G (the Kanzaki hypothesis).

KEY WORDS AND PHRASES. Abelian ring extensions, separable algebras, Azumaya algebras, Galois extensions.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 16A16, 13A20, 13B05.

1. INTRODUCTION.

Cyclic extensions of rings have been intensively investigated by Nagahara and Kishimoto [1], Parimula and Sridharan [2], the present author [3,4,5], and others. In [3], a separable cyclic extension R[x] with respect to a cyclic automorphism group $\langle \rho \rangle$ of R of order n for some integer n over a noncommutative ring R was studied. It was shown ([3], Theorem 3.3) that if R is Galois over $R^{\langle \rho \rangle}$ $(= \{r \text{ in } R \,/\, (r)\rho = r\})$ with Galois group $\langle \rho \rangle$ and if $R^{\langle \rho \rangle}$ is contained in the center **C** of R, then R[x] is an Azumaya algebra over $R^{\langle \rho \rangle}$, where x^n (= b for some b in R) and n are units in $R^{\langle \rho \rangle}$. Let G be an abelian automorphism group of R of order n such that G = $\langle \rho_1 \rangle \times \ldots \times \langle \rho_m \rangle$

where $\langle \rho_i \rangle$ is a cyclic subgroup of order n_i for some integers n, m, and n_i. Noting that $(C)\rho_i = C$ for each ρ_i, we shall study an abelian extension $R[x_1,\ldots,x_m]$ with respect to G, where $rx_i = x_i(r\rho_i)$ for each r in R, $x_i^{n_i} = b_i$ which is a unit in C^G, $x_ix_j = x_jx_i$ for all i and j, and the set $\{x_1^{k_1}\ldots x_m^{k_m} / 0 \leq k_i < n_i\}$ is a basis over R. A ring R is called to satisfy the Kanzaki hypothesis ([6], P. 110) if R is Azumaya over C with a finite automorphism group G and C is Galois over K ($= C^G$) with Galois group induced by and isomorphic with G. DeMeyer [7] has shown that $R \cong R^G \otimes_K C$ under the Kanzaki hypothesis for R. The present paper will generalize the Parimula-Sridharan theorem from cyclic extensions ([2], Proposition 1.1, [3], Theorem 3.3) to abelian extensions $R[x_1,\ldots,x_m]$ with respect to an abelian automorphism group G ($= \langle \rho_1 \rangle \times \ldots \times \langle \rho_m \rangle$) of R. Let G restricted to C be isomorphic with G. Then we shall show that C is Galois over K ($= C^G$) if and only if $R[x_1,\ldots,x_m]$ is an Azumaya algebra over K such that $R[x_1,\ldots,x_m] \cong R^G \otimes_K C[x_1,\ldots,x_m]$ where R^G is an Azumaya K-algebra. Thus, a structure of $R[x_1,\ldots,x_m]$ is obtained. Moreover, a structure of $C[x_1,\ldots,x_m]$ is also obtained when each direct summand of G is a G-subgroup (see definition below).

2. PRELIMINARIES.

Throughout, let R be a ring with 1, C the center of R, G ($= \langle \rho_1 \rangle \times \ldots \times \langle \rho_m \rangle$) an abelian automorphism group of R of order n where ρ_i is cyclic of order n_i for some integers n, n_i, and m. Then $R[x_1,\ldots,x_m]$ is the abelian extension of R with respect to G as defined in Section 1. We denote C^G by K, and assume that the automorphism group of C is isomorphic with G. The Azumaya algebra R is called to satisfy the Kanzaki hypothesis ([6], P. 110) if C is Galois over K with Galois group induced by and isomorphic with G. For separable extensions, Azumaya algebras, and Galóis extensions, see [3], [4], and [5].

3. ABELIAN EXTENSIONS.

Keeping the notations of Sections 1 and 2, we shall show the Parimula-Sridharan theorem ([2], Proposition 1.1, [3], Theorem 3.3) and two structural theorems for abelian extensions $R[x_1,\ldots,x_m]$. We begin with a proposition on separable abelian extensions.

PROPOSITION 3.1. Let G $(= \langle \rho_1 \rangle \times \cdots \times \langle \rho_m \rangle)$ be an abelian automorphism group of R of order n. If n and $x_i^{n_i}$ $(= b_i)$ are units in C^G for each i, then $R[x_1,\dots,x_m]$ is a separable extension of R.

PROOF. Since n_i divides n, n_i is a unit in C^G. Hence the cyclic extension $R[x_1]$ with respect to $\langle \rho_1 \rangle$ is a separable extension over R ([3], Lemma 3.1). Now $\langle \rho_2 \rangle$ is extended to an automorphism group of $R[x_1]$ by $(x_1)\rho_2 = x_1$, so $(R[x_1])[x_2]$ is a separable extension over $R[x_1]$ by a similar reason. Thus $R[x_1,x_2]$ $(= (R[x_1])[x_2])$ is a separable extension over R by the transitivity of separable extensions. By repeating the above argument $(m-2)$ times, $R[x_1,\dots,x_m]$ is a separable extension over R.

We now show the Parimula-Sridharan theorem for $R[x_1,\dots,x_m]$.

THEOREM 3.2. By keeping the notations of Proposition 3.1, if R satisfies the Kanzaki hypothesis, then $R[x_1,\dots,x_m]$ is an Azumaya K-algebra.

PROOF. By Proposition 3.1, $R[x_1,\dots,x_m]$ is a separable extension over R. By the Kanzaki hypothesis for R, R is separable over C and C is Galois over K, so $R[x_1,\dots,x_m]$ is a separable extension over K by the transitivity of separable extensions. So, it suffices to show that the center of $R[x_1,\dots,x_m]$ is K. It is easy to see that K is contained in the center. Since $\{x_1^{k_1}\cdots x_m^{k_m} \,/\, 0 \le k_i < n_i\}$ is a basis of $R[x_1,\dots,x_m]$ over R, we can take f in the center of $R[x_1,\dots,x_m]$ such that $f = a_o + x_1^{k_1}\cdots x_m^{k_m}\cdot a$ where a_o and a are in R, and $0 \le k_i < n_i$. Then, $rf = fr$ for each r in R. This implies that $ra_o = a_o r$ and $ar = (r)\rho_1^{k_1}\cdots\rho_m^{k_m}\cdot a$. Hence a_o is in C, and the second equation implies that $a(r-(r)\rho_1^{k_1}\cdots\rho_m^{k_m}) = 0$ for each r in C. Thus a is in the annihilator ideal I of $\{r-(r)\rho_1^{k_1}\cdots\rho_m^{k_m} \,/\, r \text{ in } C\}$ of R. Since R is Azumaya over C, $I = I_o R$ where I_o is the annihilator ideal of $\{r-(r)\rho_1^{k_1}\cdots\rho_m^{k_m} \,/\, r \text{ in } C\}$ of C. $I_o = \{0\}$ ([7], Proposition 1.2) because C is Galois over K with Galois group induced by and isomorphic with G. Thus $I = \{0\}$, and so $a = 0$. Therefore, $f = a_o$ in C. Also, $x_i f = f x_i$ for each i, so $a_o = (a_o)\rho_i$ for each i. Thus a_o is in K. This completes the proof.

Next is a structural theorem for $R[x_1,\dots,x_m]$ under the Kanzaki hypothesis.

THEOREM 3.3. If R satisfies the Kanzaki hypothesis, then $R[x_1,\dots,x_m] \cong R^G \otimes_K C[x_1,\dots,x_m]$ as Azumaya K-algebras.

PROOF. By Proposition 3.1, $C[x_1,\dots,x_m]$ is an Azumaya algebra over K. Then, similar to the arguments used in the proof of Theorem 3.2, we shall show that the commutant of $C[x_1,\dots,x_m]$ in $R[x_1,\dots,x_m]$ is R^G. Clearly, R^G is contained in the commutant. Now, let $f = a_0 + x_1^{k_1}\cdots x_m^{k_m}\cdot a$ be an element in the commutant for some a_0 and a in R and $0 \leq k_i < n_i$. Then cf = fc for each c in C. This implies that a = 0. Also, $x_i f = f x_i$ for each i, so a_0 is in R^G. Thus f (= a_0) is in R^G. Noting that $C[x_1,\dots,x_m]$ and $R[x_1,\dots,x_m]$ are Azumaya algebras over K, we have that $R[x_1,\dots,x_m] \cong R^G \otimes_K C[x_1,\dots,x_m]$ by the well known commutant theorem for Azumaya algebras ([7], Theorem 4.3, P. 57).

COROLLARY 3.4. If R satisfies the Kanzaki hypothesis, then R^G is an Azumaya algebra over K.

PROOF. This is a consequence of Theorem 3.3 and the commutant theorem for Azumaya algebras.

We are going to show a converse of Theorem 3.3.

THEOREM 3.5. If $R[x_1,\dots,x_m]$ is an Azumaya algebra over K such that $R[x_1,\dots,x_m] \cong R^G \otimes_K C[x_1,\dots,x_m]$ where R^G is an Azumaya K-algebra, then C is Galois over K with Galois group induced and isomorphic with G.

PROOF. By the commutant theorem for Azumaya algebras, since $R[x_1,\dots,x_m]$ and R^G are Azumaya K-algebras, so is $C[x_1,\dots,x_m]$. Then, we claim that C is Galois over K with Galois group G. Suppose not. There is a non-identity g in G such that $\{c-(c)g \,/\, c \text{ in } C\}$ is not C ([7], Proposition 1.2). Let $g = \rho_1^{k_1}\cdots\rho_m^{k_m}$ for some k_i, $0 \leq k_i < n_i$. Since I generated by (c-(c)g) for c in C is a G-ideal of C (that is, (I)G = I), we have an Azumaya algebra $(C/I)[x_1,\dots,x_m]$ over $K/(K \cap I)$. On the other hand, one can show that $(x_1^{k_1}\cdots x_m^{k_m})$ is in the center of $(C/I)[x_1,\dots,x_m]$. This is a contradition. Thus C is Galois over K with Galois group G.

Let S be a ring Galois extension over a subring T with a finite Galois group G. A normal subgroup H of G is called a <u>G-subgroup</u> if S is Galois over S^H with Galois group H and S^H is Galois over T with Galois group G/H. Keeping the notations of Theorem 3.5, we give a structural theorem for $C[x_1,\ldots,x_m]$.

We denote the center of $C[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m]$ by $C_i^!$ for each i. Clearly, $C_i^! = C^{(G/\langle\rho_i\rangle)}$. Let each direct summand of G be a G-subgroup, we have:

THEOREM 3.6. If C is Galois over K with Galois group G, then the abelian extension $C[x_1,\ldots,x_m] \cong C_1^![x_1]\otimes_K\cdots\otimes_K C_m^![x_m]$ as Azumaya K-algebras.

PROOF. Extending ρ_i from C to $C[x_1,\ldots,x_m]$ by $(x_j)\rho_i = x_j$ for each i and j, we claim that $C[x_1,\ldots,x_m] \cong (C[x_1,\ldots,x_{m-1}])^{\rho_m}\otimes_K C_m^![x_m]$. In fact, since C is Galois over K, $C^{(G/\langle\rho_m\rangle)}$ is Galois over K with Galois group $\langle\rho_m\rangle$ (for $G/\langle\rho_m\rangle \cong \langle\rho_1\rangle\times\cdots\times\langle\rho_{m-1}\rangle$ is a G-subgroup of G by hypothesis). Now, the center of $C[x_1,\ldots,x_{m-1}]$ is $C^{(G/\langle\rho_m\rangle)}$, so $C[x_1,\ldots,x_{m-1}]$ satisfies the Kanzaki hypothesis; that is, $C[x_1,\ldots,x_{m-1}]$ has an automorphism group $\langle\rho_m\rangle$ such that its center $C^{(G/\langle\rho_m\rangle)}$ is Galois over $(C^{(G/\langle\rho_m\rangle)})^{\langle\rho_m\rangle}$ (= K) with Galois group induced by and isomorphic with $\langle\rho_m\rangle$. But $C[x_1,\ldots,x_m] \cong (C[x_1,\ldots,x_{m-1}])[x_m]$, so $C[x_1,\ldots,x_m] \cong (C[x_1,\ldots,x_{m-1}])^{\langle\rho_m\rangle}\otimes_K C_m^![x_m]$ by Theorem 3.3. Next, considering $(C[x_1,\ldots,x_{m-1}])^{\langle\rho_m\rangle}$, we have that $(C[x_1,\ldots,x_{m-1}])^{\langle\rho_m\rangle} \cong (C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-2}])[x_{m-1}]$ such that the center of $C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-2}] = C_{m-1}^!$ which is Galois over K with Galois group $\langle\rho_{m-1}\rangle$. Since $\langle\rho_{m-1}\rangle$ is an automorphism group of $C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-2}]$, $C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-2}]$ satisfies the Kanzaki hypothesis with a center which is Galois over K with Galois group $\langle\rho_{m-1}\rangle$. Hence $C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-1}] \cong C^{\langle\rho_m\rangle\times\langle\rho_{m-1}\rangle}[x_1,\ldots,x_{m-2}]\otimes_K C_{m-1}^![x_{m-1}]$. The above arguments can be repeated for (m-2) more times. Thus the proof is completed.

As immediate consequences of Theorem 3.5 and Theorem 3.6, we have the following:

COROLLARY 3.7. If R satisfies the Kanzaki hypothesis such that each direct summand of G is a G-subgroup, then $R[x_1,\ldots,x_m] \cong R^G\otimes_K C_1^![x_1]\otimes_K\cdots C_m^![x_m]$.

COROLLARY 3.8. If R satisfies the Kanzaki hypothesis such that the center C of R has no idempotents but 0 and 1, then $R[x_1,\ldots,x_m] \cong R^G\otimes_K C_1^![x_1]\otimes_K\cdots\otimes_K C_m^![x_m]$.

PROOF. Since C is Galois over K with no idempotents but 0 and 1, each direct summand of G is indeed a G-subgroup ([7], Theorem 1.1, P. 80, or [8]).

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