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EQUIVALENCE CLASSES OF FUNCTIONS ON FINITE SETS

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ABSTRACT. By using Pólya's theorem of enumeration and de Bruijn's generalization of Pólya's theorem, we obtain the numbers of various weak equivalence classes of functions in R^D relative to permutation groups G and H where R^D is the set of all functions from a finite set D to a finite set R, G acts on D and H acts on R. We present an algorithm for obtaining the equivalence classes of functions counted in de Bruijn's theorem, i.e., to determine which functions belong to the same equivalence class. We also use our algorithm to construct the family of non-isomorphic f_m-graphs relative to a given group.

KEY WORDS AND PHRASES. Enumerations, equivalence classes of functions on finite sets, algorithm, f_m-graphs.

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1. INTRODUCTION.

Motivated by Carlitz's work in [1] on the invariantive properties over a finite field K, Cavior ([2],[3]) and Mullen ([4],[5],[6],[7]) studied several families of equivalence relations of functions from K into K. These equivalence relations can be described in more general forms as follows: Let $D = \{1,2,\ldots,m\}$, $R = \{1,2,\ldots n\}$, R^D be the set of all functions from D into R, G be a permutation group acting on D and H be a permutation group acting on R.

(I) Let $f,g \in R^D$. f is said to be weakly equivalent to g relative to G and H, if and only if there exist a $\sigma \in G$ and a $\tau \in H$ such that $\tau^{-1}f\sigma = g$, i.e., $\tau^{-1}f(\sigma d) = g(d)$ for every $d \in D$. There are three subfamilies:

(a) When H is the identity group, f is said to be right equivalent to g relative to G, i.e., $f\sigma = g$.

(b) When G is the identity group, f is said to be left equivalent to g relative to H, i.e., $\tau^{-1}f = g$.

(c) When $G = H$ and $\sigma^{-1}f\sigma = g$, f is said to be similar to g relative to G.

(II) Let $f,g \in R^D$. f is said to be strongly equivalent to g relative to G and H, if and only if there exist a $\sigma \in G$ and $\tau \in H$ such that $f\sigma = g$ and $\tau f = g$.

Clearly, all of these relations above are equivalence relations. One of Cavior's and Mullen's main results was to obtain the number of equivalence classes of functions over K relative to symmetric groups, and to cyclic groups. Here by using Pólya's theorem of enumeration and de Bruijn's generalization of Pólya's theorem, we shall point out that the numbers of various weak equivalence classes of functions in R^D relative to G and H can be obtained. We shall present an algorithm for obtaining the equivalence classes of functions counted in de Bruijn's theorem, i.e., to determine which functions belong to the same equivalence class. Our method is to associate each function with its incidence matrix. Various weak equivalence relations correspond to products of matrices, and from the entries of the incidence matrices, equivalent functions can be obtained. Our algorithm does not use the cycle indices of the permutation groups. We use our algorithm to construct the family of non-isomorphic f_m-graphs relative to a given group. The numbers of strong equivalence classes do not appear to be obtainable from Pólya's and de Bruijn's theorems. Cavior, in [2], obtained the number of strong equivalence classes relative to the symmetric groups. We apply our algorithm to strongly equivalent functions. With the help of Pólya's and de Bruijn's theorems, our algorithm enables us to determine the numbers of strong equivalence classes relative to some subgroups of the symmetric groups.

2. THEOREMS OF PÓLYA AND DE BRUIJN.

Let G be a permutation group acting on a set $D = \{1,2,\ldots,m\}$. Since every permutation can be uniquely written as a product of disjoint cycles, the cycle index of G is defined as the following polynomial in $Q[x_1,x_2,\ldots,x_m]$ where Q is the field of rational numbers and $x_i x_j = x_j x_i$ for $i,j = 1,2,\ldots,m$:

$$P_G(x_1,x_2,\ldots,x_m) = \frac{1}{|G|} \sum_{\sigma\in G} x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m}$$

where $|G|$ is the order of G and b_i is the number of cycles of length i in the disjoint cycle decomposition of σ for $i = 1,2,\ldots,m$.

THEOREM 1. (Pólya [8],[9],[10]). Let R^D be the set of all functions from a finite set D into a finite set R, G be a permutation group acting on D, w be a function from R into R' where R' is a commutative ring with an identity containing the rational numbers Q, and a relation $\sim$ be defined on R^D such that $f \sim g$ if and only if there exists a $\sigma \in G$ with $f(\sigma d) = g(d)$ for every $d \in D$. (This relation is an equivalence relation. Consequently, R^D is partitioned into disjoint equivalence classes $\{F\}$, where each F is called a pattern.) Then the total patterns, denoted by $\sum_F W(F)$, is

$$\sum_F W(F) = P_G\Big(\sum_{r\in R} w(r), \sum_{r\in R} (w(r))^2,\ldots, \sum_{r\in R} (w(r))^k,\ldots\Big) \tag{1}$$

where P_G is the cycle index relative to G. If $w(r) = 1$ for every $r \in R$, then the number of total patterns is

$$\Big|\sum_F W(F)\Big| = P_G(|R|,|R|,\ldots,|R|,\ldots) \tag{2}$$

where $|R|$ is the cardinality of R.

THEOREM 2. (de Bruijn [11],[12]). Let R^D be the set of all functions from a finite set D into a finite set R, G be a permutation group action on D, H be a permutation group acting on R, and a relation $\sim$ be defined on R^D such that $f \sim g$ if and only if there exist a $\sigma \varepsilon G$ and a $\tau \varepsilon H$ with $f(\sigma d) = \tau g(d)$ for every $d \varepsilon D$. (This relation is an equivalence relation. Consequently, R^D is partitioned into disjoint equivalence classes $\{F\}$, where each F is called a pattern.) Then the number of total patterns is

$$[P_G(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \ldots) P_H(e^{z_1+z_2+\ldots}, e^{2(z_2+z_4+\ldots)}, e^{3(z_3+z_6+\ldots)}, \ldots)] \tag{3}$$

evaluated at $z_1 = z_2 = \ldots = 0$.

If H is the identity group acting on R, then (3) is (2) in Pólya's theorem.

THEOREM 3. Let D^D be the set of all functions from a finite set D whose cardinality is m into itself, G be a permutation group acting on D, and a relation $\sim$ be defined on D^D such that for every f and $g \varepsilon D^D$, $f \sim g$ if and only if there exists a $\sigma \varepsilon G$ with $\sigma^{-1} f(\sigma d) = g(d)$ for every $d \varepsilon D$. (This is an equivalence relation. Consequently, D^D is partitioned into disjoint equivalence classes $\{F\}$, where each F is called a pattern.) Then the number of total patterns is

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} \prod_{i=1}^{m} \Big(\sum_{j|i} j\, c_j \Big)^{c_i} \tag{4}$$

where c_i is the number of cycles of lenght i in the disjoint cycle decomposition of σ for $i = 1,2,\ldots,m$.

The number of equivalence classes in D^D is $\frac{1}{|G|} \sum_{\sigma \varepsilon G}$ (number of functions f such that $f\sigma = \sigma f$) and the number of $f \varepsilon D^D$ such that

$$f\sigma = \sigma f \quad \text{is} \quad \prod_{i=1}^{m} \Big(\sum_{j|i} j\, c_j \Big)^{c_i}. \quad \text{For details, see [7].}$$

3. AN ALGORITHM.

Let G and H be permutation groups acting on $D = \{1,2,\ldots,m\}$ and $R = \{1,2,\ldots,n\}$ respectively. For convenience, we shall call the weak equivalence relation in R^D relative to G and H the G-H-relation, i.e., f and g in R^D are said to be G-H-related if and only if there exists a $\sigma \varepsilon G$ and a $\tau \varepsilon H$ such that

$\tau^{-1}f(\sigma d) = g(d)$ for every $d \in D$. Clearly, it is an equivalence relation, and R^D is partitioned into disjoint classes each of which is called a G-H-class. Let G^* be the $m \times m$ permutation group corresponding to G, i.e., $G^* \cong G$, H^* be the $n \times n$ permutation group corresponding to H, i.e., $H^* \cong H$, and I be the set of all $m \times n$ (0,1)-matrices $A = (a_{ij})$ where each row of A consists of exactly one 1 and all other entires are zero. Two matrices A and B are said to be G^*-H^*-related if and only if there exist a $P \in G^*$ and a $Q \in H^*$ such that $PAQ^{-1} = B$. Clearly, this relation is an equivalence relation called a G^*-H^*-relation, and I is partitioned into disjoint equivalence classes each of which is called a G^*-H^*-class.

Similar to Lemma 1 in [13], we have

THEROEM 4. Let $\eta: R^D \to I$ be defined by $\eta(f) = A$ where $A = (a_{ij})$ with $a_{i,f(i)} = 1$ for $i = 1,2,\ldots,m$, and all other entries 0. Then

(1) η is a bijective map, and

(2) η preserves the G-H-relation in R^D and the G^*-H^*-relation in I.

PROOF. (1) Clearly, η is well defined. Let $\eta(f) = A = (a_{ij})$ with $a_{i,f(i)} = 1$ for $i = 1,2,\ldots,m$ and all other entries 0, and $\eta(g) = B = (b_{ij})$ with $b_{i,g(i)} = 1$ for $i = 1,2,\ldots,m$ and all other entries 0. If $A = B$, then $a_{i,f(i)} = b_{i,g(i)}$ for $i = 1,2,\ldots,m$, i.e., $f = g$. Hence, η is injective. Since $|R^D| = |I| = n^m$, η is bijective.

(2) Let f and g belong to the same G-H-class, i.e., there exist a $\sigma \in G$ and a $\tau \in H$ such that $\tau^{-1}f\sigma = g$, $\eta(f) = A = (a_{ij})$ and $\eta(g) = B = (b_{ij})$ with $a_{i,f(i)} = 1$ and $b_{i,g(i)} = 1$ for $i = 1,2,\ldots,m$ and all other entries 0. Then $b_{i,\tau^{-1}f(\sigma i)} = 1$ for $i = 1,2,\ldots,m$ and all other entries 0.

Let $P = (p_{ij})$ and $Q = (q_{ij})$ be the permutation matrices corresponding to σ and τ respectively. By using the properties of permutation matrices, we have

$$(PAQ^{-1})_{ij} = \sum_t \sum_s p_{is}a_{st}q_{jt} = p_{iu}a_{uv}q_{jv} = a_{uv} = a_{\sigma i,\tau j} \tag{5}$$

for all $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$ with $\sigma i = u$ and $\tau j = v$. But since all

$a_{\sigma i,\tau j} = 0$ except $a_{\sigma i,f(\sigma i)} = 1$ for $i = 1,2,\ldots,m$, $\tau j = f(\sigma i)$, i.e., $j = \tau^{-1}f(\sigma i)$ and $(PAQ^{-1})_{i,\tau^{-1}f(\sigma i)} = 1$ for $i = 1,2,\ldots,m$ and all other entries 0. Hence, $PAQ^{-1} = B$, and A and B belong to the same G*-H*-class.

Conversely, if A and B belong to the same G*-H*-class, then there exist a $P \varepsilon G^*$ and a $Q \varepsilon H^*$ such that $PAQ^{-1} = B$. Since η is bijective, η^{-1} exists, say $\eta^{-1}A = f$ and $\eta^{-1}B = g$. Since by (5) $(PAQ^{-1})_{ij} = a_{\sigma i,\tau j}$, all $(PAQ^{-1})_{ij}$ are 0, except $a_{\sigma i,f(\sigma i)} = 1$ for $i = 1,2,\ldots,m$, i.e., $\tau j = f(\sigma i)$ or $j = \tau^{-1}f(\sigma i)$. Also, all the entries of $B = (b_{ij})$ are 0, except $b_{i,g(i)} = 1$ for $i = 1,2,\ldots,m$. Since $(PAQ^{-1})_{ij} = (B)_{ij}$ for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$, $\tau^{-1}f(\sigma i) = g(i)$ for $i = 1,2,\ldots,m$, i.e., $\tau^{-1}f\sigma = g$, and f and g belong to the same G-H-class.

From our Theorem 4, we know that for each $f \varepsilon R^D$ there exists a unique $\eta(f) = A = (a_{ij})$ in I with $a_{i,f(i)} = 1$ for $i = 1,2,\ldots,m$ and all other entries 0 (A is called the incidence matrix of f). Now for every $\sigma \varepsilon G$ and every $\tau \varepsilon H$, $a_{\sigma^{-1}i,\tau^{-1}f(i)} = 1$ for $i = 1,2,\ldots,m$ and all other entries 0 determine a matrix B in I. Let $P = (p_{ij})$ and $Q = (Q_{ij})$ be the permutation matrices corresponding to σ and τ respectively. Then $PAQ^{-1} = B$ and A and B are G*-H*-related. From Theorem 4, for each $B \varepsilon I$ there exists a unique $\eta^{-1}(B) = g$ in R^D and f and g are G-H-related. Consequently, we have the following algorithm for obtaining all equivalence classes, i.e., for determining which functions are in the same equivalence class:

Step 1. Select any $f \varepsilon R^D$ and write

$$a_{1,f(1)} = a_{2,f(2)} = \ldots = a_{m,f(m)} = 1.$$

Step 2. For every $\sigma \varepsilon G$ and every $\tau \varepsilon H$, compute

$$a_{\sigma^{-1}1,\tau^{-1}f(1)} = a_{\sigma^{-1}2,\tau^{-1}f(2)} = \ldots = a_{\sigma^{-1}m,\tau^{-1}f(m)} = 1.$$

Each computation determines a function in R^D, and the equivalence class containing f consists of these distinct functions.

Step 3. Select a function in R^D which is not a member of the equivalence class obtained in Step 2. Repeat Steps 1 and 2. Continue the process until every function in R^D appears in an equivalence class.

EXAMPLE 1. Let $G = \{\sigma_1 = (1), \sigma_2 = (23)\}$ act on $D = \{1,2,3\}$ and $H = \{\tau_1 = (1), \tau_2 = (12)\}$ act on $R = \{1,2\}$. Then the cycle indices of G and H are, respectively,

$$P_G(x_1,x_2,x_3) = \frac{1}{2}(x_1^3 + x_1x_2), \quad \text{and}$$

$$P_H(y_1,y_2) = \frac{1}{2}(y_1^2 + y_2).$$

Therefore, by using (3), the number of weak equivalence classes in R^D is 3. If the 8 functions in R^D are given as

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
1	1	1	1	2	1	2	2	2
2	1	1	2	1	2	1	2	2
3	1	2	1	1	2	2	1	2

then the algorithm can be used to determine the 3 classes so that

$$R^D = \{f_1,f_8\} \cup \{f_2, f_3, f_6, f_7\} \cup \{f_4,f_5\}.$$

4. APPLICATIONS.

We consider the weak equivalence classes in R^D relative to the groups G and H. Let $f \in R^D$ and $C(f) = \{(\sigma,\tau) \in G \times H;\ \tau^{-1}f\sigma = f\}$. Clearly, $C(f)$ is a subgroup of the product group $G \times H$.

THEOREM 5. The cardinality of the weak equivalence class $\bar{f}$ containing f in R^D relative to the groups G and H is equal to the index of $C(f)$ in the product group $G \times H$. Hence, $|\bar{f}|$ divides $|G|\cdot|H|$.

The proof is not difficult and hence is omitted.

COROLLARY 5.1. The cardinality of the right equivalence class $\bar{f}$ containing f in R^D relative to the group G is equal to the index of the subgroup $C(f) = \{\sigma \in G;\ f\sigma = f\}$ in G. Hence, $|\bar{f}|$ divides $|G|$.

COROLLARY 5.2. The cardinality of the left equivalence class $\bar{f}$ containing f in R^D relative to the group H is equal to the index of the subgroup $C(f) = \{\tau \in H; \tau^{-1}f = f\}$ in H. Hence, $|\bar{f}|$ divides $|H|$.

COROLLARY 5.3. The cardinality of the similar class $\bar{f}$ containing f in R^D relative to the group G is equal to the index of the subgroup $C(f) = \{\sigma \in G; \sigma^{-1}f\sigma = f\}$ in G. Hence, $|\bar{f}|$ divides $|G|$.

A. The cycle indices for many families of groups are known, e.g., see p. 36 in [9]. In particular, the cycle index of the cyclic group C_q of order q on p points is

$$P_{C_q}(x_1,x_2,\ldots,x_q) = \frac{1}{q}\sum_{i|q} \phi(i)x_i^{\frac{q}{i}}$$

where $\phi(i)$ is the Euler's phi-function.

EXAMPLE 2. Let $G = <(123\ldots q)>$ be the cyclic group generated by $(123\ldots q)$ acting on $q+k$ points. Then

$$P_G(x_1,x_2,\ldots,x_q,x_{q+1},\ldots,x_{q+k}) = \frac{1}{q}\sum_{i|q} (\phi(i)x_i^{\frac{q}{i}}\, x_1^k). \tag{6}$$

Let G be the permutation group acting on $D = \{1,2,\ldots,q,q+1,\ldots,q+k = m\}$ and H be the identity group acting on $R = \{1,2,\ldots,n\}$. Then, by using (3), the number of the right equivalence classes in R^D relative to G is:

$$N = [P_G(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_m})(e^{n(z_1+z_2+\ldots+z_n+\ldots)})]_{z_1=z_2=\ldots=0}$$

$$= \frac{1}{q}\sum_{i|q} (\phi(i)n^{\frac{q}{i}}\, n^k). \tag{7}$$

In particular, if $G_1 = <(1234)>$ acts on $D = \{1,2,3,4,5\}$ and H is the identity group acting on $R = D$, then by (7) the number of right equivalence classes in D^D relative to G_1 is 825.

We show the following:

(a) There are 25 right equivalence classes each of cardinality 1. By using our algorithm, the function corresponding to

$$a_{1i} = a_{2i} = a_{3i} = a_{4i} = a_{5j} = 1$$

is not changed for any $\sigma \in G_1$, $i = 1,2,\ldots,5$ and $j = 1,2,\ldots,5$.

(b) There are 50 right equivalence classes each of cardinality 2. Applying $\sigma = (1234)$, $\sigma^2 = (13)(24)$, $\sigma^3 = (1432)$ and $\sigma^4 = (1)$ to

$$a_{1i} = a_{2j} = a_{3i} = a_{4j} = a_{5k} = 1$$

for $i \neq j$, $i,j = 1,2,\ldots,5$ and $k = 1,2,\ldots,5$, we have

$$a_{4i} = a_{1j} = a_{2i} = a_{3j} = a_{5k} = 1, \tag{8}$$

$$a_{3i} = a_{4j} = a_{1i} = a_{2j} = a_{5k} = 1, \tag{9}$$

$$a_{2i} = a_{3j} = a_{4i} = a_{1j} = a_{5k} = 1, \text{ and} \tag{10}$$

$$a_{1i} = a_{2j} = a_{3i} = a_{4j} = a_{5k} = 1. \tag{11}$$

Since (8) and (9) are the same as (10) and (11) respectively, there are $5 \cdot (\frac{5 \cdot 4}{2}) = 50$ right equivalence classes each of cardinality 2.

(c) Since by Corollary 5.1 there is no right equivalence class of cardinality 3, the number of right equivalence classes of cardinality 4 is 825-25-50 = 750. Our results in this example with $G_1 = \langle(1234)\rangle$ coincide with the results on p. 113 in [12].

EXAMPLE 3. Let $H = \langle(123\ldots q)\rangle$ be the cyclic group generated by $(123\ldots q)$ acting on $q+k$ points. Then the cycle index of H is the same as (6). Let G be the identity group acting on $D = \{1,2,\ldots,m\}$ and H be the group $\langle(12\ldots q)\rangle$ acting on $R = \{1,2,\ldots,q,q+1,\ldots,q+k = n\}$. Then, by using (3), the number N of the left equivalence classes in R^D relative to H is given by

$$N = \frac{1}{q}\,[n^m + \sum_{\substack{i|q \\ i>1}} \phi(i)k^m]. \tag{12}$$

In particular, let G be the identity group acting on $D = \{1,2,\ldots,5\}$ and H_1 be the group $\langle(1234)\rangle$ acting on $R = D$. Then the number of left equivalence classes in D^D relative to H_1 is 782 by using (12), or by using (3) and (7).

We show the following: By using our algorithm, the function corresponding to

$$a_{15} = a_{25} = a_{35} = a_{45} = a_{55} = 1$$

is in an equivalence class by itself. Now consider

$$a_{1,j_1} = a_{2,j_2} = a_{3,j_3} = a_{4,j_4} = a_{5,j_5} = 1$$

where j_k are not all 5 for $k = 1,2,\ldots,5$.

Then with $\tau = (1234)$, we have

$$a_{1,\tau^{-1}j_1} = a_{2,\tau^{-1}j_2} = a_{3,\tau^{-1}j_3} = a_{4,\tau^{-1}j_4} = a_{5,\tau^{-1}j_5} = 1,$$

$$a_{1,(\tau^2)^{-1}j_1} = a_{2,(\tau^2)^{-1}j_2} = a_{3,(\tau^2)^{-1}j_3} = a_{4,(\tau^2)^{-1}j_4} = a_{5,(\tau^2)^{-1}j_5} = 1,$$

$$a_{1,(\tau^3)^{-1}j_1} = a_{2,(\tau^3)^{-1}j_2} = a_{3,(\tau^3)^{-1}j_3} = a_{4,(\tau^3)^{-1}j_4} = a_{5,(\tau^3)^{-1}j_5} = 1,$$

$$a_{1,j_1} = a_{2,j_2} = a_{3,j_3} = a_{4,j_4} = a_{5,j_5} = 1.$$

All of the functions corresponding to the above matrices are different and therefore, the cardinality of the equivalence class is 4. Hence, there is only one equivalence class of cardinality 1 and all the others are of cardinality 4, that is, there are $782 - 1 = 781$ equivalence classes of cardinality 4. Our results in this example thus coincide with the results on p. 353 in [4].

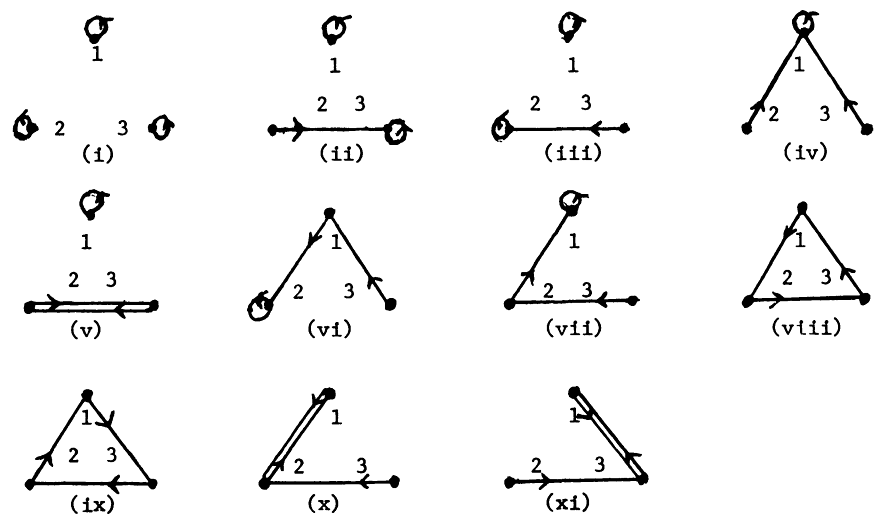
B. A labeled directed graph with m vertices is said to be an f_m-graph if the out-degree at every vertex is 1. Thus, the incidence matrix of an f_m-graph belongs to the set of $m \times m$ matrices I. Conversely, every matrix in I determines a unique f_m-graph. Let G be a permutation group acting on m points. Two f_m-graphs X_1 and X_2 are said to be G-isomorphic if and only if there exists a $\sigma \in G$ such that σ maps the vertices of X_1 onto the vertices of X_2, and σ preserves the directed edges, i.e., $[\sigma a, \sigma b]$ is a directed edge in X_2 if and only if $[a,b]$ is a directed edge in X_1. A G-isomorphism of X_1 onto itself is said to be an automorphism of X_1. Let A_1 and A_2 be the incidence matrices of X_1 and X_2 respectively. Then it is well known that X_1 and X_2 are G-isomorphic if and only if there exists a permutation matrix P corresponding to a $\sigma \in G$ such that $PA_1P^{-1} = A_2$.

Since the set of all functions from m points into itself is in one to one correspondence with the set of all f_m-graphs, we may use (4) to count the number of nonisomorphic classes of f_m-graphs relative to G, i.e., the number of nonisomorphic classes of f_m-graphs relative to G is

$$\frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^{m} \Big(\sum_{j|i} j\, c_j \Big)^{c_i} \tag{13}$$

where c_i is the number of cycles of lenght i in the disjoint cycle decomposition of σ for $i = 1,2,\dots,m$.

EXAMPLE 4. Let $G = \langle(123)\rangle$ act on $\{1,2,3\}$. Then the number of nonisomorphic classes of f_3-graphs relative to G is, by using (13), $\frac{1}{3}[(1\cdot 3)^3 + (3\cdot 1)^1 + (3\cdot 1)^1] = 11$. By using our algorithm, we have the following nonisomorphic f_3-graphs relative to G:



EXAMPLE 5. Replace $G = \langle(123)\rangle$ in Example 4 by the symmetric group S_3 on $\{1,2,3\}$. Then the number of nonisomorphic classes of f_3-graphs relative to S_3 is, by using (13), $\frac{1}{6}[(1\cdot 3)^3 + 3((1\cdot 1)^1(1\cdot 1+2\cdot 1)^1) + 2(3\cdot 1)^1] = 7$, and the nonisomorphic

f_3-graphs (relative to S_3) are: (i), (ii), (iv), (v), (vi), (viii) and (x), i.e., (ii) is isomorphic to (iii) by (23), (vi) is isomorphic to (vii) by (12), (viii) is isomorphic to (ix) by (13), and (x) is isomorphic to (xi) by (23).

By applying (4), de Bruijn in [12] obtained the number N_m of similar equivalence classes relative to the symmetric group S_m as

$$N_m = \sum_{(k)} \prod_{i=1}^{m} \left(\sum_{j|i} j\, k_j\right)^{k_i} (k_i!\; i^{k_i})^{-1} \tag{14}$$

where the first summation is over all m-tuples $(k_1,k_2,\ldots,k_m)$ of non-negative integers k_i which satisfy $k_1 + 2k_2 + \ldots + mk_n = m$. The first few values of N_m are given by $N_1 = 1$, $N_2 = 3$, $N_3 = 7$, $N_4 = 19$, $N_5 = 47$, $N_6 = 130$. Formula (14) gives the answer to the problem posed by Cavior in [2, p. 128] concerning the number of similarity classes relative to the symmetric group.

C. On p. 129 in [2], Cavior obtained the number of strong equivalence classes in D^D relative to S_m and S_m where $D = \{1,2,\ldots,m\}$ and S_m is the symmetric group on D. Here, with the help from the theorems of Pólya and de Bruijn, we apply our algorithm to obtain the following theorems.

THEOREM 6. Let $D = \{1,2,\ldots,m\}$ where m is an odd integer, $R = \{1,2\}$, G be any permutation group acting on D and H be the group $\langle(12)\rangle$ acting on R. Then every strong equivalence class in R^D relative to G and H consists of only one function, i.e., the number of strong equivalence classes is 2^m.

PROOF. Let f be any function in R^D and $A_f = (a_{ij})$ be the incidence matrix f with

$$a_{1,i_1} = a_{2,i_2} = a_{3,i_3} = \ldots = a_{m,i_m} = 1$$

and all other entries 0 where i_k is either 1 or 2 for $k = 1,2,\ldots,m$. Then, by using our algorithm, the right equivalence class relative to G containing f consists of the functions corresponding to the set of matrices $\{A_{f\sigma};\ \sigma \in G$, and $a_{\sigma^{-1}1,i_1} = a_{\sigma^{-1}2,i_2} = a_{\sigma^{-1}3,i_3} = \ldots = a_{\sigma^{-1}m,i_m} = 1$ and all other entries 0$\}$. The left equivalence class relative to H containing f consists of the set of matrices

$\{A_{\tau f};\ \tau \in H$ and $a_{1,\tau^{-1}i_1} = a_{2,\tau^{-1}i_2} = a_{3,\tau^{-1}i_3} = \ldots = a_{m,\tau^{-1}i_m} = 1$ and all other entries $0\}$. Since m is odd, the set $\{\tau^{-1}i_1, \tau^{-1}i_2, \ldots, \tau^{-1}i_m\} \neq \{i_1, i_2, \ldots, i_m\}$ for $\tau = (12)$. Hence, the intersection of the left equivalence class relative to G containing f and the right equivalence class relative to H containing f is $\{f\}$. Consequently, the number of strong equivalence classes relative to G and H is $|R^D| = |R|^{|D|} = 2^m$.

In [8], Theorem 5.3 states: Let S_n be the set of all one-to-one functions from a finite set $D = \{1,2,\ldots,n\}$ onto itself, G be a permutation group acting on the domain D, H be a permutation group acting on the range D and an equivalence relation relative to G and H in S_n be defined as follows: $f \sim g$ if and only if there exist a $\sigma \in G$ and a $\tau \in H$ such that $\tau^{-1}f(\sigma d) = g(d)$ for every $d \in D$. Then the number of patterns (equivalence classes) is

$$[P_G(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \ldots)\ P_H(z_1, 2z_2, 3z_3, \ldots)] \tag{15}$$

evaluated at $z_1 = z_2 = z_3 = \ldots = 0$. (15) is also equal to

$$[P_H(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \ldots)\ P_G(z_1, 2z_2, 3z_3, \ldots)] \tag{16}$$

evaluated at $z_1 = z_2 = z_3 = \ldots = 0$.

Let p be a prime, G be the cyclic group C_p of order p generated by $(12\ \ldots\ p)$ acting on $D = \{1,2,\ldots,p\}$, and H be the identity group acting on D. By using (15) and Corollary 5.1 restated for one-to-one functions, it can be shown that there are $(p-1)!$ right equivalence classes in S_p relative to C_p each of order p. Similarly, the number of left equivalence classes in S_p relative to $H = C_p$ is $(p-1)!$ and the cardinality of each equivalence class is p. These results are in agreement with those concerning permutation polynomials over finite fields obtained by Mullen in [6].

THEOREM 7. Let $D = \{1,2,\ldots,p\}$ and $R = \{1,2,\ldots,q\}$ where p is a prime and q is an integer greater than 1, C_p be the cyclic group of order p generated by $(12\ldots p)$ acting on D, and C_q be the cyclic group of order q generated by $(12\ldots q)$ acting on R.

(a) If q is not a multiple of p, then every strong equivalence class in R^D relative to C_p and C_q consists of only one function, i.e., the number of strong equivalence classes in R^D relative to C_p and C_q is q^p.

(b) If $q = p$ (i.e., $D = R$ and $C_p = C_q$), then the number of strong equivalence classes in D^D relative to C_p and C_p is $p^p - (p-1)^2$.

PROOF. (a) We claim that the number of weak equivalence classes relative to C_p and C_q is $\frac{1}{p}(q^{p-1} + p - 1)$, and that there are $\frac{1}{p}(q^{p-1}-1)$ weak equivalence classes each having cardinality pq, and there is one weak equivalence class having cardinality q. Since $P_{C_p}(x_1,x_2,\ldots,x_p) = \frac{1}{p}(x_1^p + (p-1)x_p)$ and $P_{C_q}(x_1,x_2,\ldots,x_q) = \frac{1}{q}\sum_{i|q}\phi(i)x_i^{q/i}$, by (3), the number of weak equivalence classes $|W|$ is

$$|W| = \frac{1}{p}\left[\left(\frac{\partial^p}{\partial z_1^p} + (p-1)\frac{\partial}{\partial z_p}\right)\frac{1}{q}\left(e^{q(z_1+z_2+\ldots)} + T\right)\right]_{z_1=z_2=\ldots=0} \tag{17}$$

where the function T does not involve z_1 and z_p. The reason is that every non-identity permutation in C_q has no fixed points and p and q are relatively prime. Hence, (17) is equal to

$$|W| = \frac{1}{pq}(q^p + (p-1)q) = \frac{1}{p}(q^{p-1} + p - 1). \tag{18}$$

Applying our algorithm to the function $f_1 \in R^D$ corresponding to $a_{11} = a_{21} = a_{31} = \ldots = a_{p1} = 1$, we have the weak equivalence class $\overline{f_1}$ consisting of the q functions corresponding to

$$\begin{aligned}
&a_{11} = a_{21} = a_{31} = \ldots = a_{p1} = 1,\\
&a_{12} = a_{22} = a_{32} = \ldots = a_{p2} = 1,\\
&a_{13} = a_{23} = a_{33} = \ldots = a_{p3} = 1,\\
&\quad\cdot \qquad \cdot \qquad \cdot \qquad \cdot\\
&\quad\cdot \qquad \cdot \qquad \cdot \qquad \cdot\\
&\quad\cdot \qquad \cdot \qquad \cdot \qquad \cdot\\
&a_{1q} = a_{2q} = a_{3q} = \ldots = a_{pq} = 1.
\end{aligned} \tag{19}$$

Not counting the weak equivalence class above, we still have

$\frac{1}{p}(q^{p-1} + p - 1) - 1 = \frac{1}{p}(q^{p-1} - 1)$ weak equivalence classes. Since there are $|R^D| = q^p$ functions and since each weak equivalence class can have its cardinality at most pq, the cardinality of each of these $\frac{1}{p}(q^{p-1} - 1)$ weak equivalence classes is pq.

Now we show that every strong equivalence class in R^D relative to C_p and C_q consists of only one function. Clearly, applying our algorithm to each function in $\bar{f}_1$, we have that each function belongs to a strong equivalence class consisting of only itself. Let f and g be strongly equivalent functions in R^D and not in $\bar{f}_1$, i.e., there exist a $\sigma \in C_p$ and a $\tau \in C_q$ such that $f\sigma = g$ and $\tau f = g$. Assume $f \neq g$. Then none of σ and τ could be the identity, and we would have $e_2^{-1} f\sigma$ and $(\tau^{-1})^{-1} fe_1$ in the same weak equivalence class in R^D relative to C_p and C_q where e_1 and e_2 are the identities of C_p and C_q respectively. Since $f \notin \bar{f}_1$, the weak equivalence class containing f has cardinality pq, i.e., $e_2^{-1} f\sigma \neq (\tau^{-1})^{-1} fe_1$. But $e_2^{-1} f\sigma = g = (\tau^{-1})^{-1} fe_1$. That is a contradiction, and the cardinality of every strong equivalence class in R^D relative to C_p and C_q is 1, i.e., the number of the strong equivalence classes in R^D relative to C_p and C_q is $|R^D| = q^p$.

(b) First, we consider the set S_p of all one-to-one functions in D^D. We claim that if f belongs to normalizer of C_p in the group S_p, then the cardinality of the strong equivalence class containing f relative to C_p and C_p is p. The cardinality of any right equivalence class in S_p relative to C_p is p. Let f and g be any two right equivalent functions relative to C_p. Then there exists a $\sigma \in C_p$ such that $f\sigma = g$. Since $f \in S_p$, f^{-1} exists. Let $\tau = f\sigma f^{-1}$. Since f is a normalizer of C_p and since $\sigma \in C_p$, $\tau \in C_p$. Then $\tau f = (f\sigma f^{-1})f = f\sigma = g$. Consequently, the cardinality of the strong equivalence class containing f relative to C_p and C_p is p. We note that if f is a normalizer of C_p in S_p, then $f\sigma$ is also a normalizer of C_p in S_p for every $\sigma \in C_p$.

We claim that if f belongs to S_p and f is not normalizer of C_p in S_p, then the cardinality of the strong equivalence class containing f relative to C_p and C_p is 1. Let f and g be strongly equivalent functions, i.e., there exist σ and τ in C_p such that $f\sigma = g$ and $\tau f = g$. Assume that $f \neq g$. Then neither

σ nor τ is the identity, $f\sigma = \tau f$ and $f\sigma f^{-1} = \tau$. Since σ is not the identity, $f\sigma^i f^{-1} = (f\sigma f^{-1})(f\sigma f^{-1})\dots(f\sigma f^{-1}) = \tau^i \in C_p$ for $i = 1,2,\dots,p$, i.e., f is a normalizer of C_p in S_p. That is a contradiction. Hence, $f = g$, and the cardinality of the strong equivalence class containing f relative to C_p and C_p is 1.

We claim that if $f \in D^D$ and $f \notin S_p$, then the cardinality of the strong equivalence class containing f relative to C_p and C_p is 1. First, we shall consider the number $|W|$ of weak equivalence classes in D^D relative to C_p and C_p: By using (3), we have

$$\begin{aligned} |W| &= [\tfrac{1}{p}(\frac{\partial^p}{\partial z_1^p} + (p-1)\frac{\partial}{\partial z_p})(\tfrac{1}{p}(e^{p(z_1+z_2+\dots)} + (p-1)e^{p(z_p+\dots)})]z_1=z_2=\dots=0 \\ &= p^{p-2} + p - 1. \end{aligned} \tag{20}$$

Let $|\bar{S}_p|$ be the number of weak equivalence classes in S_p relative to C_p and C_p. By (15), we have

$$\begin{aligned} |\bar{S}_p| &= [\tfrac{1}{p}(\frac{\partial^p}{\partial z_1^p} + (p-1)\frac{\partial}{\partial z_p})(\tfrac{1}{p}(z_1^p + (p-1)(pz_p))]z_1=z_2=\dots=0 \\ &= \tfrac{1}{p}((p-1)! + (p-1)^2). \end{aligned} \tag{21}$$

Since a one-to-one function can only be weakly equivalent to a one-to-one function and since a non-one-to-one function can only be weakly equivalent to a non-one-to-one function, the number $|\bar{N}|$ of weak equivalence non-one-to-one functions in D^D relative to C_p and C_p is

$$|\bar{N}| = |W| - |\bar{S}_p| = (p^{p-2} + p - 1) - \tfrac{1}{p}((p-1)! + (p-1)^2). \tag{22}$$

Applying our algorithm to the function $f_2 \in D^D$ corresponding to $a_{11} = a_{21} = a_{31} = \dots = a_{p1} = 1$, we have the weak equivalence class $\bar{f}_2$ consisting of p functions corresponding to

$$
\begin{array}{l}
a_{11} = a_{21} = a_{31} = \dots = a_{p1} = 1, \\
a_{12} = a_{22} = a_{32} = \dots = a_{p2} = 1, \\
\quad \vdots \qquad\qquad\quad \vdots \\
a_{1p} = a_{2p} = a_{3p} = \dots = a_{pp} = 1.
\end{array}
\tag{23}
$$

Not counting the weak equivalence class above, we still have $|\bar{N}| - 1$ weak equivalence classes of non-one-to-one functions in D^D. Since there are $|D^D| - |S_p| = p^p - p!$ non-one-to-one functions and since each weak equivalence class can have its cardinality at most p^2, the cardinality of each of these $|\bar{N}| - 1$ weak equivalence classes is p^2, because

$$p + p^2(|\bar{N}|-1) = p + p^2[(p^{p-2} + p - 1) - \frac{1}{p}((p-1)! + (p-1)^2] - p^2 = p^p - p!.$$

Similar to the proof in (a) with $q = p$, we may conclude that every strong equivalence class of non-one-to-one functions relative to C_p and C_p consists of only one function.

We know that $|D^D| = p^p$, $|S_p| = p!$ and the cardinality of the normalizer of C_p in S_p is $p(p-1)$ (see 2.3 on p. 12 in [14]). Since the cardinality of the strong equivalence class $\bar{f}$ containing the normalizer f of C_p in S_p is p and since every function in $\bar{f}$ is also a normalizer of C_p in S_p, there are $p-1$ strong equivalence classes each of which has cardinality p. Since every other strong equivalence class has cardinality 1, the number of strong equivalence classes in D^D relative to C_p and C_p is

$$(p^p - p!) + (p! - p(p-1)) + (p-1) = p^p - (p-1)^2.$$

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