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AN ANALYTICAL SOLUTION OF BOHR'S COLLECTIVE HAMILTONIAN

M.K. EL-ADAWI, H.A. ISMAIL, S.A. SHALABY and E.M. SAYED

Department of Physics
Faculty of Education
Ain - Shams University
Heliopolis, Cairo, EGYPT

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ABSTRACT. The collective states of ^{126}Xe, ^{128}Xe, ^{130}Xe, and ^{130}Ba, $52 < Z, N \leq 116$ are studied using Bohr's Collective Hamiltonian and a simple analytic form for the potential. The energy levels are calculated for this model and the results are compared with previously published experimental and calculated values.

KEY WORDS AND PHRASES.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES.

1. INTRODUCTION.

The lowest excited states in even-even nuclei are usually treated using Bohr's collective model [1] as a quadrupole motion of the nuclear surface. Bohr's Hamiltonian depends on the intrinsic collective variables β and ∂ and also on Euler's angles specifying the orientation of the intrinsic system. Depending on the shape of the potential energy of deformation, this Hamiltonian can describe both the vibrational limit and the rotational limit. In both cases, the solution of the collective Schrödinger equation is simple. However, there are many cases which cannot be fitted using the above treatment. Therefore, many attempts [2,3,4,5,6,7] have been made in order to relax some of the requirements characterizing the rotational and vibrational descriptions. Usually, the problem is approximated by a simple model in which the potential is replaced by a simple analytic function where the Schrödinger equation can be easily solved.

The present work represents one such trial in which some even-even nuclei in the region $52 < Z, N \leq 116$, susceptible to gamma deformation, are studied. These nuclei are: ^{126}Xe, ^{128}Xe, ^{130}Xe and ^{130}Ba. The collective states of these nuclei are studied using Bohr's collective Hamiltonian.

2. THEORETICAL CONSIDERATIONS.

Different microscopic computations [5,8,9,10,11] which have been performed for the potentials of nuclei belonging to the investigated region indicated a weak dependence of their potentials on ∂.

According to these considerations, we assumed the ∂ weakly dependent potential energy to be in the form:

$$V(\beta,\partial) = V_0(\beta) - \xi\beta^3 \cos 3\partial \tag{2.1}$$

The second term on the right hand side of (2.1) can be treated as a perturbation, where ξ is an adjustable parameter. The potential $V_0(\beta)$ is given in the form:

$$V_0(\beta) = \frac{1}{8} C(\beta^2 + \beta_0{}^4/\beta^2 - 2\,\beta_0{}^2) - D \tag{2.2}$$

Where the equilibrium deformation β_0, stiffness C, and depth D are defined as follows:

$$\left.\frac{dV_0}{d\beta}\right|_{\beta=\beta_0} = 0 \tag{2.3}$$

$$C = \left.\frac{d^2V_0}{d\,\beta^2}\right|_{\beta=\beta_0} > 0 \tag{2.4}$$

$$D = -\,V_0(\beta_0) \tag{2.5}$$

The Schrödinger equation can be written as

$$H\Psi_{n,\lambda I,M}(\beta,\partial,\theta_i) = E\Psi_{n,\lambda,I,M}(\beta,\partial,\theta_i) \tag{2.6}$$

where the collective Hamiltonian H can be written in the form

$$H = H_0 - \xi\,\beta^3 \cos 3\partial \tag{2.7}$$

H_0 is the unperturbed Hamiltonian given by Wilets and Jean [5]. The unperturbed Schrödinger equation can be written in the form

$$H_0\Psi^0_{n,\lambda,I,M}(\beta,\partial,\theta_i) = E_0\Psi^0_{n,\lambda,I,M}(\beta,\partial,\theta_i) \tag{2.8}$$

If the potential depends only on β, the Hamiltonian H_0 is separable and the solution $\Psi^0_{n,\lambda,I,M}(\beta,\partial,\theta_i)$ can be assumed as

$$\Psi^0_{n,\lambda,I,M}(\beta,\partial,\theta_i) = f_{n,\lambda}(\beta)\;\Phi_{\lambda IM}(\partial,\theta_i) \tag{2.9}$$

The radial part of equation (2.8) can be written in the form [5]

$$E[\beta^2 f(\beta)] = \frac{h^2}{2\beta}\left(-\frac{\partial^2}{\partial\beta^2} + \frac{(\lambda+1)(\lambda+2)}{\beta^2}\right) + V_0(\beta)\quad[\beta^2 f(\beta)] \tag{2.10}$$

Equation (2.10) is solved in the present work using the expression for $V_0(\beta)$ of (2.2). This can be shown as follows.

3. SOLUTION.

Putting
$$y = \beta^2 f(\beta) \tag{3.1}$$
and substituting for $V_0(\beta)$ from (2.2), we get
$$\frac{d^2y}{d\beta^2} + \frac{2B}{h^2}\left[A - \frac{h^2}{2B}\frac{(\lambda+1)(\lambda+2)}{\beta^2} - \frac{1}{8}(\beta^2 - \frac{1}{8}C\beta_0^4/\beta^2\right]y = 0 \tag{3.2}$$
where
$$A = E + D + 1/4\, C\beta_0^2 \tag{3.3}$$
The asymptotic solution of (3.2) is
$$y_\infty = \exp - Z^2/2 \tag{3.4}$$
where $Z = a\beta$ and $a^2 = \frac{1}{2h}\sqrt{CB}$.

The general solution of (3.2) can be written in the form
$$y = F(Z)\exp - Z^2/2 \tag{3.5}$$
where
$$F(Z) = Z^{(\frac{1}{2} + A_\lambda)}\, U(Z). \tag{3.6}$$
The function U(Z) satisfies the equation
$$U'' + \left[\frac{2\alpha}{Z} - 2Z\right]U' + [P - 2\alpha - 1]U = 0 \tag{3.7}$$
where
$$P = \frac{2BA}{h^2a^2}, \quad \alpha = (\frac{1}{2} + A), \text{ and} \tag{3.8}$$
$$A_\lambda = \frac{1}{2}\sqrt{(2\lambda+3)^2 + CB\,\beta_0^4/h^2}.$$
Using $X = Z^2$, (3.7) takes the form
$$XU'' + [\alpha + \frac{1}{2} - X]U' - \frac{1}{4}[2\alpha + 1 - P]U = 0 \tag{3.9}$$
Equation (3.9) has the power series solution
$$U = {}_1F_1(\frac{1}{4}(2\alpha + 1 - P), \alpha + \frac{1}{2}; X) \tag{3.10}$$
where ${}_1F_1(b,d;x)$ is the confluent hypergeometric function.

It is evident that F(b,d;x) is a polynomial iff b is a non-positive integer; therefore, the requirement that the solution (3.10) be bounded at ∞ is expressed by the condition
$$\frac{1}{4}(2\alpha + 1 - P) = -n \tag{3.11}$$
where n is a non-negative integer.

Substituting for P, , and A in (3.11), we get
$$E_{n_\beta,\lambda} = h\sqrt{\frac{C}{B}}\,[n_\beta + \frac{1}{2} + \frac{1}{2}A_\lambda] - \frac{1}{4}C\beta_0^4 - D \tag{3.12}$$

According to (3.12), the energy of the ground state E_{oo} is

$$E_{oo} = h\sqrt{\frac{c}{B}}\left[\frac{1}{2} + \frac{1}{2}A_o\right] - \frac{1}{4}C\beta_0{}^4 - D \qquad (3.13)$$

The energy difference ΔE is given by

$$\Delta E = E_{CaLC.} = h\sqrt{\frac{c}{B}}\left[n_\beta + \frac{1}{2}(A_\lambda - A_o)\right] \qquad (3.14)$$

This result was used to calculate the energy levels for ^{126}Xe, ^{128}Xe, ^{130}Xe, and ^{130}Ba in the region $52 < Z$, $N \leq 116$. The calculations were carried out at $n_\beta = 0$ and $\lambda = 1,2,3,4,5,6$. The mass parameter B was taken equal to 50 MeV^{-1} as given by Wilets [5] and the parameters β_0 and C were fitted to the experimental energy levels. For λ greater than 2, the present model was found to give a better fitting to the energy levels when a correction term was introduced with the limitation of $(2\lambda - 3)$ splittings of the levels. The comparison between the experimental and the calculated values without splittings for each of the nuclei enabled us to find the correction term in the form

$$\pm \frac{1}{4}\sqrt{\frac{c}{B}} \cdot n_\lambda \quad , \quad \text{where} \quad n_\lambda = 0,1,2,\ldots,(\lambda - 2) \qquad (3.15)$$

Substituting for A and A_o in (3.14) and using the correction term (3.15), we get the following expression for the energy levels with splitting:

$$E_{n_\beta,\lambda} = \sqrt{\frac{c}{B}}\left[n_\beta + \frac{1}{4}\left\{\sqrt{CB\,{}_0^4 + (2\lambda+3)^2} - \sqrt{CB\beta_0^4 + g}\right\}\right] \pm \frac{1}{4}n_\lambda\sqrt{\frac{c}{B}} \,,\; n_\lambda = 0,1,2,\ldots,(\lambda-2) \qquad (3.16)$$

The calculated energy levels without splitting (3.14) and with splitting (3.15) were compared with the experimental data from references (5,11,13,14,15). The energy levels calculated according to the (VMI) model [16] are also given.

The results are given in Tables (1-4).

	TABLE (1) ^{126}Xe C = 320, B = 50, β_0 = 0.32			TABLE (2) ^{128}Xe C = 160, B = 50, β_0 = 0.28		
λ	$E_{exp.}$ Reference 5,11,13,14,15	$E_{calc.}$ present work	$E_{calc.}$ Reference 16	$E_{exp.}$ Reference 5,11,13,14,15	$E_{calc.}$ present work	$E_{calc.}$ Reference 16
1	0.386	0.3723	0.3880	0.4410	0.4406	0.4480
				0.949		
2	0.870	0.9029	0.9599	1.018	1.0202	
				1.040		1.0410
	0.930	0.9342			1.2443	
3	1.650	1.5666	1.6484	1.7500	1.6915	1.7454
		2.1991			2.1387	
		1.0738			1.5284	
		1.7063			1.9758	
4	2.450	2.3387	2.4261	2.530	2.4230	2.5328
		2.9712			2.8702	
		3.6036			3.3174	
		......				
5		3.1976	3.2773		3.1947	3.3892
		......				
		......			3.5469	
6		4.1255	4.1910		3.9941	
		......			4.4413	4.3052

TABLE (3)

^{130}Xe

C = 150, B = 50, β_0 = 0.24

TABLE (4)

^{130}Ba

C = 550, B = 50, β_0 = 0.33

λ	$E_{exp.}$ Reference 5,11,13,14,15 (^{130}Xe)	$E_{calc.}$ present work (^{130}Xe)	$E_{calc.}$ Reference 16 (^{130}Xe)	Reference 5,11,13,14,15 (^{130}Ba)	$E_{calc.}$ present work (^{130}Ba)
1	0.538	0.5378	0.5342	0.3574	0.3581
				0.9019	
2	1.205	1.2014	1.1924		0.8803
	1.632			0.9080	
		1.5021			0.7221
	1.794				
3	1.944	1.9352	1.9557	1.5575	1.5513
	2.017				
	2.369	2.3682		2.2544	2.3804
		1.8435			0.6956
		2.2765		1.4770	1.5247
4	2.780	2.7095	2.8014	2.3468	2.3539
		3.1425		3.1677	3.1831
		3.5755			4.0122
		2.2098			0.7835
		2.6428		1.8447	1.6127
		3.0758			2.4418
5		3.5088	3.716	3.2313	3.2710
		3.9418		4.0964	4.1002
		4.3748			4.9293
		4.8079			5.7585
		......		1.8830	1.7989
		......		2.6456	2.6281
		3.8914		3.6599	3.4573
6		4.3244	4.6911		4.2864
		4.7574			......
		......			......
		......			......

4. CONCLUSIONS.

The results showed that:

1) Some characteristics of the nuclei in the region $52 < Z, N \leq 116$ can be described by a model in which the potential is expressed in the form of a simple analytic function, such as the one used in the present study.

2) The effect of the perturbation term can be replaced by a correction term. The correction term in the present study is related to the quantum number with a limitation of $(2\lambda - 3)$ splittings for $\lambda > 2$.

The method in the present study has an advantage over numerical solutions, since it gives a simple physical picture by classifying the states according to known quantum numbers. The degree of accuracy as shown from the tables is satisfactory for the ordinary scope of the studies.

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