Internat. J. Math. & Math. Sci.
Vol. 6 No. 3 (1983) 605-608

RESEARCH NOTES

A CHARACTERIZATION OF THE DESARGUESIAN PLANES OF ORDER q^2 BY SL(2,q)

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(Received January 25, 1982)

ABSTRACT. The main result is that if the translation complement of a translation plane of order q^2 contains a group isomorphic to SL(2,q) and if the subgroups of order q are elations (shears), then the plane is Desarguesian. This generalizes earlier work of Walker, who assumed that the kernel of the plane contained GF(q).

KEY WORDS AND PHRASES. *Translation planes, translation complement, elations.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 51A40, 20B25.

THEOREM. Let π be a translation plane of order q^2, where $q = p^r$ and p is a prime. Let $G \cong SL(2,q)$ be a subgroup of the translation complement of π whose elements of order p are elations. Then π is a Desarguesian plane.

This theorem is a special case required in the classification of all translation planes π of order q^2 which admit a collineation group $G \cong SL(2,q)$ [1, 2]. That classification is a generalization of the work of Walker and Schaeffer [3, 4], who assume, in addition, that the kernel of π contains GF(q).

To begin the proof, let W be a vector space of dimension 2r over GF(p). Since

π is a $4r$-dimensional vector space over $GF(p)$, we may represent π as $W \oplus W$ so that the points of π are vectors (x,y), where $x,y \in W$. The components of π (i.e., the lines containing $(0,0)$) have the form $\{(0,y): y \in W\}$ and $\{(x,xA): x \in W\}$ for various $GF(p)$—linear transformations $A: W \to W$. We will denote the components by their defining equations $x = 0$ and $y = xA$, respectively. Next, note that each Sylow p-subgroup Q of G is abelian and hence all the elements ($\neq 1$) of Q have the same elation axis. Let S denote the set of all components of π and let N be the subset of elation axes; thus $|S| = q^2+1$ and $|N| = q+1$.

LEMMA 1. (Hering [5], Ostrom [6]). We may coordinatize π as above such that

$$G = \left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix} : A,B,C,D \in K;\ AD - BC = I\right\}$$

where K is a field of $2r \times 2r$ matrices over $GF(p)$ and $K \cong GF(q)$. Further, the elation axes (that is, the elements of N) have the form $y = xA$ $(A \in K)$ and $x = 0$.

LEMMA 2. There is an element $g \in G$ such that the following conditions are satisfied: (i) $|g| \,\big|\, q+1$; (ii) $|g| \nmid p^t - 1$ for $t < 2r$; and (iii) g fixes a component of π which is not in the set N.

PROOF. The integer s is a <u>p-primitive</u> <u>prime</u> <u>divisor</u> of q^2-1 if s is a prime, $s|q^2-1$, and $s \nmid p^t-1$ for $0 < t < 2r$ (hence $s|q+1$). q^2-1 has a p-primitive prime divisor s unless $q = 8$ or $q = p$ and $p+1 = 2^a$ [7]. In the first case, let $|g| = s$ so that g satisfies conditions (i) and (ii). Then g also satisfies condition (iii) because $|g|$ is a prime and g permutes the $q(q-1)$ components in $S\backslash N$. If $q = 8$, choose g such that $|g| = 9$. Since $|S\backslash N| = 56 \not\equiv 0 \pmod 3$, g must fix one of the elements of $S\backslash N$. Finally, if $q = p$ and $p+1 = 2^a$, choose h of order 8 in G and let $g = h^2$. Then g^2 has order 2 in $G = SL(2,K)$, so $g^2 = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}$ fixes every component of π. Hence, h has orbits of lengths 1, 2, and 4 in S, and since $4 \nmid p(p-1)$ then h has an orbit of length 1 or 2 on $S\backslash N$. Therefore $g = h^2$ fixes an element of $S\backslash N$.

LEMMA 3. Choose $g \in G$ so that g satisfies the conditions of Lemma 2, and let $L(g)$ be the ring of matrices generated by g over $GF(p)$. Then $L(g)$ is a field $\cong GF(q^2)$ and $L(g)$ contains the subfield

$$\widetilde{K} = \left\{\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in K\right\}.$$

PROOF. $g \in G \subset GL(2,K)$ by Lemma 1. As a 2×2 matrix over K, g has a minimum polynomial $f(x)$ over K of degree ≤ 2. Since $|g| \nmid q(q-1)$, then the degree of f is 2 and f is irreducible over K. Therefore, g and K generate a field $U \cong GF(q^2)$ which contains $L(g)$ as a subfield. Since $|g| \nmid p^t - 1$ (for $t < 2r$), then $L(g) = U$ and $L(g) \supset \tilde{K}$.

LEMMA 4. Let g of Lemma 2 fix the component $y = xT$ of $S \backslash N$. Then $K[T]$ is a field isomorphic to $GF(q^2)$.

PROOF. $L(g)$ and hence $\tilde{K} = \left\{\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in K\right\}$ fix the component $y = xT$, and thus K centralizes T. T and the elements of K are $2r \times 2r$ matrices which act on a vector space $V = V(2r,p)$ of dimension 2r over GF(p). K makes V into a 2-dimensional vector space and T acts as a K-linear transformation of V. Hence, the minimum polynomial $f(x)$ of T over K has degree ≤ 2. If T has an eigenvalue A in K, then the distinct components $y = xT$ and $y = xA$ of π must intersect, which is impossible. Therefore, T is irreducible over K and $K[T] \cong GF(q^2)$.

We can now complete the proof of the Theorem. Let π^* denote the Desarguesian affine plane of order q^2 coordinatized by the field $L = K[T]$; i.e., the points of π^* are $\{(x,y): x,y \in L\}$ and the components of π^* are $\{y = xC: C \in L\} \cup \{x = 0\}$. Clearly, $GL(2,L)$ acts as a collineation group of π^*. We superimpose π^* on π by identifying the points of π^* and π. Since $K \subset L$ and $T \in L$, the components $y = xA$ of N and $y = xT$ are components both of π^* and π. Since $G = SL(2,K) \subset GL(2,L)$, then G acts both as a collineation group of π^* and of π. Finally, recall that $SL(2,K)$ acts transitively on the $q(q-1)$ components of π^* outside of N (for example, the stabilizer subgroup in $SL(2,K)$ of a component of π^* outside N has order $q+1$). Therefore, the images of $y = xT$ under G constitute $q(q-1)$ components both of π^* and of π; so $\pi^* = \pi$ as required.

The research of all three authors was partially supported by the National Science Foundation.

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