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AN HYPERVALUATION OF A RING ONTO A TOTALLY ORDERED NON-CANCELLATIVE SEMIGROUP WITHOUT ZERO DIVISORS

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ABSTRACT. In this paper we answer to a question posed by Marc KRANSER: It it possible to have a totally ordered noncancellative semigroup without zero divisors,and a ring hypervaluated by this semigroup? We were able to give a positive answer and provide an example.

KEY WORDS AND PHRASES. Hypervaluation,Valuation,Totally ordered semigroup,Ring
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1. PRELIMINARIES

In what follows,all semigroups are supposed to have a unit element 1 and a zero (absorbent) element 0,such that $a.0=0.a=0$ for all elements a in the semigroup. In any semigroup we can adjoint a zero element if it does not already have one,without changing its structure. We remark that in each semigroup 1 and 0 are unique.

DEFINITION 1. We say that a semigroup S is ordered if it is supplied with an order $<$ such that:

1. For a,b,c in S, $a<b \Longrightarrow ca \leq cb$ and $ac \leq bc$.
2. $0<1$ (hence $0=0c \leq 1c = c$ for all c in S)

If the order is total S is called totally ordered.

DEFINITION 2. An hypervaluation on a ring R is a function from R onto a totally ordered semigroup S, satisfying the following conditions: For all a,b in R.

1. $|a| = 0 \Longleftrightarrow a=0$
2. $|a| = |-a|$
3. $|ab| = |a|\,|b|$
4. $|a+b| \leq \mathrm{Max}\,\{|a|,|b|\}$

Notice that if the semigroup S does not have any zero divisors then the ring R does not have any either. For if $a,b \in R$ with $a \neq 0, b \neq 0$,and $ab=0$,then $0=|0| =|ab|=|a|\,|b|$,while $|a| \neq 0$ and $|b| \neq 0$. But this is impossible since S is assumed with no zero divisors. Also we easily see that a cancellative semigroup has no zero divisors,however the converse is not true in general as we shall see in what follows.

2. CONSTRUCTION OF A NON-CANCELLATIVE, TOTALLY ORDERED SEMIGROUP WITHOUT ZERO DIVISORS

We begin with an arbitrary given totally ordered semigroup $(S_1,\cdot,>)=\{0_1,a,b,\dots\}$ where 0_1 its absorbent (zero) element. Consider now the set $S_2=S_1 \cup \{0_2\}$ that we get if we adjoint a new element 0_2 to the set S_1. Define an operation $*$ on S_2 by setting $a*b=a.b$ if a,b are in S_1, and $0_2*a=a*0_2=0_2$ for all a in S_2. In particular $0_2*0_1=0_1*0_2=$ $=0_2$. We observe then that:

- $(S_2, *)$ is a semigroup and 0_2 is its zero (absorbent) element (self evident)
- $(S_2, *)$ has no zero divisors. Indeed if $a,b\in S_2$ with $a\neq 0_2$ $b\neq 0_2$ then $a,b\in S_1$ and by definition $a*b=ab\in S_1$ and hence $ab\neq 0_2$.
- $(S_2, *)$ is non cancellative. Indeed we can take a,b in S_1 with $a\neq b$. Then $0_1*a=0_1a$ $=0_1=0_1b=0_1*b$. Thus $0_1*a=0_1*b$ but $a\neq b$.
- Finally we define a total order $\succ$ on S_2 by setting $a\succ 0_2$ for all a in S_1, and for a,b, in S_1, $a\succ b \iff a>b$. It is obvious that this is well defined, and that $(S_2,*,\succ)$ becomes a totally ordered semigroup.

3. A PROPOSITION

Notation: In what follows, we will denote by S_1 an arbitrary given totally ordered semigroup, and by S_2 the corresponding totally ordered non cancellative semigroup without zero divisors, obtained from S_1, by adjoining a new absorbent element 0_2, as it was done in section 2.

PROPOSITION: Let I be a two sided ideal of a (not necessarilly commutative) integral domain R. If R/I can be hypervaluated by S_1, then R can be hypervaluated by S_2.

PROOF: Let $|\dots|: R/I\to S_1=\{0_1,a,b,\dots\}$ be a valuation from R/I onto S_1. We define the function $\|\ \|: R\to S_2$ by setting: For a in R, $\|a\|=0_2$ if $a=0$ and $\|a\|=|a+I|$ if $a\neq 0$. This implies that if a is in I, with $a\neq 0$, then $\|a\|=0_1$.

We see that $\|\dots\|$ thus defined, satisfies the four properties of hypervaluation: Indeed properties (1) and (2) of definition 2 are obviously satisfied. That (3) holds for all a,b in R is immediate if at least one of them equals to zero. So we may assume $a\neq 0, b\neq 0$, and thus $ab\neq 0$ since R is an integral domain. Then $\|ab\|=|ab+I|=|(a+I)(b+I)|=$ $=|a+I|\,|b+I|=\|a\| * \|b\|$.

Finally (4) is also satisfied. For if $a,b,\in R$, if at least one of them equals to zero the proof is immediate. Suppose now $a,b\neq 0$. Then we could have $a+b=0$ or $a+b\neq 0$. If $a+b=0$ then $\|a+b\|=0_2\prec\|a\|$, $\|b\|\preceq \text{Max}\{\|a\|, \|b\|\}$. If $a+b\neq 0$ then $\|a\|=|a+I|$, $\|b\|=|b+I|$ and $\|a+b\|=|a+b+I|=|(a+I)+(b+I)|\leq \text{Max}\{|a+I|,|b+I|\}=\text{Max}\{\|a\|, \|b\|\}$

This completes the proof.

4. COFFIS THEOREM FOR HYPERVALUABILITY OF A RING

DEFINITION 3. Ler R be a ring. For any element a in R we call the set of left annihilators of a to be the set $\{x\in R|x.a=0\}$ and we denote this set by $A_1(a)$. In an analogous way we define the set of right annihilators of a denoted by $A_r(a)$.

THEOREM 1: (Coffi - Nikestia): Let R be a ring with a unit element 1. R can be hypervaluated by a totally ordered semigroup S if and only if it satisfies the following conditional:

1. For all $a \in R$, $A_1(a)=A_r(a)$ and we denote this set by $A(a)$.
2. For all $a,b \in R$, $A(a.b)=A(b.a)$
3. The class $C=\{A(a),\ A \in R\}$ is totally ordered by inclusion.

In particular,R posesses an hypervaluation $|\dots|$ such that $|a| \to A(a)$ is a one-to-one correspondence between S and C.

We remark that Coffi in his construction supposes the semigroup to be commutative. The ring R is not supposed to be necessarilly commutative,but with an identity element 1. The details can be found in Coffi [1] . The idea is the following:For each a in R,its "value" $|a|$ is A(a). So $|\dots|:R \to C=S$. Moreover S is totally ordered by the total order defined as follows: For a,b in R $|a| \le |b|$ if $A(a) \supseteq A(b)$.

5. OUR MAIN THEOREM

THEOREM: There exists a totally ordered non cancellative semigroup S without zero divisors,and a ring R that can be hypervaluated by this semigroup.

PROOF: We choose an integral domain R (not necessarily commutative) such that R/I (for some two-sided ideal I of R)be a ring satisfying the conditions of Coffi's theorem. Then by Coffi's theorem R/I can be hypervaluated by a totally ordered semigroup S_1.

From S_1 we obtain a totally ordered,non cancellative semigroup S_2 without zero divisors,as we did in section 2.

By our Proposition 1, we can hypervaluate R by S_2 that has the desired properties. This concludes the proof of our theorem.

6. A CONCRETE EXAMPLE

We provide in this paragraph a concrete example of a Ring hypervaluated by a totally ordered,non cancellative semigroup S_2 without zero divisors.

Let Z be the ring of integers and (16) the ideal in Z generated by 16. It suffices to show that the ring Z/(16) satisfies the conditions of Coffi's theorem and thus can be hypervaluated by a totally ordered semigroup S_1. Because then,by our Proposition of section 3,Z can be hypervaluated by a semigroup S_2,having the desired properties. Indeed since Z/(16) is commutative,conditions 1 and 2 are obviously satisfied. Now if $a,b,x \in Z$ and $\bar{a},\bar{b},\bar{x} \in Z/(16)$ their corresponding equivalence classes,$\bar{x}$ is then an annihilator of $\bar{a}$ in Z/(16) if and only if $x.a \in (16)$ i.e.iff 16 divides xa. Let (a,b) denote the least common multiple of two elements a,b in Z.

Thus: If $(a,16)=1$ then $A(\bar{a})=\{\bar{16}\} = \{\bar{0}\}$

If $(a,16)=2$ then $A(\bar{a})=\{\bar{8},\bar{16}\}$

If $(a,16)=4$ then $A(\bar{a})=\{\bar{4},\bar{8},\bar{12},\bar{16}\}$

If $(a,16)=8$ then $A(\bar{a})=\{\bar{2},\bar{4},\bar{6},\bar{8},\bar{10},\bar{12},\bar{14},\bar{16}\}$

If in general $(a,16)=(b,16)$ then $A(\bar{a})=A(\bar{b})$,if $(a,16)>(b,16)$ then $A(\bar{a})\supset A(\bar{b})$. Condition 3 of Coffi's theorem is also therefore satisfied.

REFERENCES

1. COFFI-NIKESTIA J.B. "Valuation des anneau avec diviseurs de zero au moyen des demi-groupes totalement ordonnes. Proprietes des anneaux valuables". Compte-Rendus, Acad.Sci.Paris 254 III,1962.