ON THE NON-EXISTENCE OF
SOME INTERPOLATORY POLYNOMIALS

C.H. ANDERSON and J. PRASAD

Department of Mathematics
California State University
Los Angeles, California 90032 U.S.A.

ABSTRACT. Here we prove that if x_k, $k = 1, 2, \ldots, n + 2$ are the zeros of $(1 - x^2)T_n(x)$ where $T_n(x)$ is the Tchebycheff polynomial of first kind of degree n, α_j, β_j, $j = 1, 2, \ldots, n + 2$ and γ_j, $j = 2, 3, \ldots, n + 1$ are any real numbers there does not exist a unique polynomial $Q_{3n+3}(x)$ of degree $\leq 3n + 3$ satisfying the conditions: $Q_{3n+3}(x_j) = \alpha_j$, $Q'_{3n+3}(x_j) = \beta_j$, $j = 1, 2, \ldots, n + 2$ and $Q'''_{3n+3}(x_j) = \gamma_j$, $j = 2, 3, \ldots, n + 1$. Similar result is also obtained by choosing the roots of $(1 - x^2)P_n(x)$ as the nodes of interpolation where $P_n(x)$ is the Legendre polynomial of degree n.

1. INTRODUCTION.

In [1] R.B. Saxena considered an interesting problem of $(0,1,3)$ interpolation by taking the roots of $(1 - x^2)P_{n-2}(x)$, where $P_{n-2}(x)$ is the Legendre polynomial of degree $n - 2$, as the nodes of interpolation. By $(0,1,3)$ interpolation, Saxena meant that for the collections $\{\alpha_j\}_1^n, \{\beta_j\}_2^{n-1}$, and $\{\gamma_j\}_1^n$ of real numbers and the zeros x_j of $(1 - x^2)P_{n-2}(x)$ arranged so that

$$-1 = x_n < x_{n-1} < \ldots < x_2 < x_1 = 1$$

a polynomial $R_n(x)$ of degree $\leq 3n - 3$ can be constructed so that

$$R_n(x_j) = \alpha_j; \quad j = 1, 2, \ldots, n,$$

$$R'_n(x_j) = \beta_j; \quad j = 2, 3, \ldots, n - 1,$$

and

$$R'''_n(x_j) = \gamma_j; \quad j = 1, 2, \ldots, n.$$

Saxena proved that such a polynomial exists uniquely if n is even and for n odd there does not exist a unique polynomial $R_n(x)$ satisfying the above conditions. Later Varma [2] obtained the following result in this direction:

THEOREM 1 (VARMA). Given a positive integer n and real numbers $\alpha_k (k = 1, 2, \ldots, n + 2)$, $\beta_k, \gamma_k (k = 2, 3, \ldots, n + 1)$ there is, in general no polynomial $F_{3n+1}(x)$ of degree $\leq 3n + 1$ such that $F_{3n+1}(x_k) = \alpha_k$; $k = 1, 2, \ldots, n + 2$, $F'_{3n+1}(x_k) = \beta_k$;

$k = 2,3,\ldots,n + 1$ and $F'''_{3n+1}(x_k) = \gamma_k$; $k = 2,3,\ldots,n + 1$ provided $x'_k s$ are the
zeros of $(1 - x^2)T_n(x)$ where $T_n(x)$ is Tchebycheff polynomial of first kind and if
there exists such a polynomial then there is an infinity of them.

2. MAIN RESULTS.

In connection with the above results we shall prove the following.

THEOREM 2. For any positive integer n, with $1 = \xi_1 > \xi_2 > \ldots > \xi_{n+1} > \xi_{n+2}$
$= -1$ the zeros of $(1 - x^2)P_n(x)$ where $P_n(x)$ is the Legendre polynomial of degree
n, there is in general no polynomial $R_{3n+1}(x)$ of degree $\leq 3n + 1$ such that, for
arbitrary real numbers $\{\alpha_j\}_1^{n+2}$, $\{\beta_j\}_2^{n+1}$ and $\{\gamma_j\}_2^{n+1}$ the conditions:

$$R_{3n+1}(\xi_j) = \alpha_j; \quad j = 1,2,\ldots,n + 1, n + 2, \tag{2.1}$$

$$R'_{3n+1}(\xi_j) = \beta_j; \quad j = 2,3,\ldots,n + 1 \tag{2.2}$$

and

$$R'''_{3n+1}(\xi_j) = \gamma_j; \quad j = 2,3,\ldots,n + 1 \tag{2.3}$$

are satisfied. If there does exist such a polynomial then there are infinitely many
of them.

We also prove the following result for Tchebycheff nodes:

THEOREM 3. For any positive integer n, with $1 = x_1 > x_2 > \ldots > x_n > x_{n+1}$
$> x_{n+2} = -1$ the zeros of $\omega_n(x) = (1 - x^2)T_n(x)$, there is in general no polynomial
$Q_{3n+3}(x)$ of degree $\leq 3n + 3$ such that for arbitrary real numbers $\{\alpha_j\}_1^{n+2}$, $\{\beta_j\}_1^{n+2}$
and $\{\gamma_j\}_2^{n+1}$ the conditions:

$$Q_{3n+3}(x_j) = \alpha_j; \quad j = 1,2,\ldots,n + 1, n + 2, \tag{2.4}$$

$$Q'_{3n+3}(x_j) = \beta_j; \quad j = 1,2,\ldots,n + 1, n + 2 \tag{2.5}$$

and

$$Q'''_{3n+3}(x_j) = \gamma_j; \quad j = 2,3,\ldots,n + 1 \tag{2.6}$$

are satisfied. If there does exist such a polynomial then there are infinitely many
of them.

REMARK 1. The comparison of our Theorem 2 with the above mentioned result of
Saxena shows that if we do not prescribe the third derivative at ± 1 then there does
not exist a unique polynomial regardless whether n is even or odd. In an earlier
work [3] we have shown that along with the conditions (2.1), (2.2) and (2.3) if we also
prescribe the first derivative at ± 1 a unique polynomial of degree $\leq 3n + 3$ still
does not exist. It is also evident from Theorem 3 that even if we prescribe the first
derivative at ± 1 a unique polynomial of degree $\leq 3n + 3$ does not exist although
the nodes of interpolation are different from that of [3].

REMARK 2. We shall give here the proof of Theorem 3 only. The proof of Theorem
2 can be obtained along the same lines.

PROOF OF THEOREM 3. We will show that if all of

$$\alpha_j = 0; \quad j = 1,2,\ldots,n + 1, n + 2, \tag{2.7}$$

$$\beta_j = 0; \quad j = 1,2,\ldots,n + 1, n + 2,$$

$$\gamma_j = 0; \quad j = 2,3,\ldots,n + 1$$

then there exists a polynomial $Q_{3n+3}(x)$ of degree $\leq 3n + 3$ which is not identically zero, but satisfies (2.4), (2.5) and (2.6). The desired result then follows immediately from the theory of linear equations. From the definition of $\omega_n(x)$ and conditions (2.4), (2.5) and (2.6), together with the requirements (2.7), it is clear that the desired polynomial must be of the form

$$Q_{3n+3}(x) = (1 - x^2)^2 T_n^{\,2}(x) \lambda_{n-1}(x) \tag{2.8}$$

where $\lambda_{n-1}(x)$ is an unknown polynomial of degree $\leq n - 1$. Since we have also required $Q_{3n+3}'''(x_j) = 0$; for $j = 2, 3, \ldots, n + 1$, simple calculation provides

$$(1 - x^2)\lambda_{n-1}'(x) - 3x\lambda_{n-1}(x) = cT_n(x) \tag{2.9}$$

for unknown real constant c. Letting $x = \cos\theta$ and

$$\lambda_{n-1}(x) = \sum_{k=0}^{n-1} a_k \cos k\theta$$

we obtain

$$(1 - x^2)\lambda_{n-1}'(x) = \sum_{k=1}^{n-1} a_k k \sin k\theta \sin\theta.$$

Thus (2.9) becomes

$$c \cos n\theta = \sum_{k=0}^{n-1} a_k [k \sin k\theta \sin\theta - 3 \cos k\theta \cos\theta].$$

From this, we obtain on simplification

$$2 c \cos n\theta = \sum_{k=0}^{n-1} a_k [(k - 3)\cos(k - 1)\theta - (k + 3)\cos(k + 1)\theta],$$

from which, by collecting the coefficients of $\cos k\theta$, for $k = 0, 1, \ldots, n$, we may write

$$-2a_1 - (6a_0 + a_2)\cos\theta - 4a_1 \cos 2\theta$$

$$+ \sum_{k=3}^{n-2} \{(k - 2)a_{k+1} - (k + 2)a_{k-1}\}\cos k\theta$$

$$-(n + 1)a_{n-2}\cos(n - 1)\theta - (n + 2)a_{n-1}\cos n\theta$$

$$= 2c \cos n\theta.$$

This, in turn, leads to the following system of equations

$$-2a_1 = 0$$

$$-(6a_0 + a_2) = 0,$$

$$-4a_1 = 0,$$

$$(k - 2)a_{k+1} - (k + 2)a_{k-1} = 0; \quad k = 3, 4, \ldots, n - 2,$$

$$-(n + 1)a_{n-2} = 0,$$

$$-(n + 2)a_{n-1} = 2c.$$

If n is even, then

$$a_0 = a_2 = a_4 = \ldots\ldots = a_{n-2} = 0; \quad a_1 = 0$$

but

$$a_{n-1-2j} = \frac{-2c}{n-2} \prod_{k=0}^{j} \left(\frac{n-2-2k}{n+2-2k} \right); \text{ for } j = 0,1,\dots,(n-4)/2$$

is not necessarily zero.

    If n is odd, then

$$a_1 = a_3 = a_5 = \dots\dots = a_{n-2} = 0,$$

while

$$a_{2j} = \frac{-2c}{n-2} \prod_{k=j}^{(n-1)/2} \frac{2k-1}{2k+3}; \ j = 1,2,\dots, \frac{(n-1)}{2}$$

with the special case

$$a_0 = -a_2/6$$

which are not necessarily zero. Hence regardless whether n is even or odd, in general, there does not exist a unique polynomial $Q_{3n+3}(x)$ of degree $\leq 3n+3$ satisfying (2.4), (2.5) and (2.6) and there are infinitely many if they exist.

    This completes the proof of Theorem 3. For a complete history on lacunary interpolation we refer to a paper by J. Balázs [4].

REFERENCES

1. SAXENA, R.B., On some interpolatory properties of Legendre polynomials III, Academic Bulgare Des Sciences Bull. De L'Institut De Math. 8 (1964), 63-94.

2. VARMA, A.K., Non-existence of interpolatory polynomials, Publ. Math. (Debrecan) 15 (1968), 75-77.

3. PRASAD, J. AND ANDERSON, C.H., Some remarks on non-existence of interpolation polynomials, Math. Notae Vol. 24 (1981-1982), 67-72.

4. BALÁZS, J., Súlyozott (0,2) interpolácio Ultraszférikus polinomok gyökeim, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 11 (3) (1961), 305-338.