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ON THE CARDINALITY OF SOLUTIONS OF MULTILINEAR DIFFERENTIAL EQUATIONS AND APPLICATIONS

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ABSTRACT. We study the existnece and cardinality of solutions of multilinear differential equations giving upper bounds on the number of solutions.

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1. INTRODUCTION.

Let $n(i)$, $i = 1,2,\ldots,m$ be positive integers such that $n(1) \geq n(2) \geq \ldots \geq n(m)$ and let $L_i = \sum_{j=0}^{n(i)} C_{ij}D^j$, $i = 1,2,\ldots,m$ be regular linear differential operators defined on $C^{n(1)}(I)$, where $I = [a,b]$ usually (but not necessarily). The coefficient functions C_{ij}, $i = 1,2,\ldots,m$, $j = 0,1,2,\ldots,n(i)$ are never vanishing real and continuous on I.

Using some ideas from [1] and [3] we study the branching of solutions $u \in C^{n(1)}(I)$ to the multilinear equation

$$Mu = (L_1u)(L_2u)\ldots(L_mu) = 0 \tag{1.1}$$

Equation (1.1) is related with the null set $N(M)$

$$N(M) = \{u \in C^{n(1)}(I) : Mu = 0\} \tag{1.2}$$

which can be infinite dimensional.

We give necessary and sufficient conditions for a $(m-1)$-tuple $(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})$ to be a multiple ordinary branching of a solution to (1.1) where $\alpha_e \in I$, $e = 1,2,\ldots,m-1$.

We also study the existence and cardinality of solutions to the initial value problem

$$D^{n(1)}u(z) = z_i, \ i = 1,2,\ldots,n(1)-1 \tag{1.3}$$

where $z, z_i \in I$, giving upper bounds on the number of solutions with n multiple branchings.

Multilinear equations have a rather extensive literature [3], [4], [6]. A few special cases of applications (e.g., pursuit problems and bending of beams) may be formulated in the form (1,1).

Finally we study the problem

$$\frac{dM}{d\lambda} - \lambda M = 0 \tag{1.4}$$

when it assumes the form (1.1) for some function λ.

2. BASIC THEOREMS.

DEFINITION 1. Let $B_1, B_2, \ldots$ and B_m denote bases for $N(L_1), N(L_2), \ldots$ and $N(L_m)$ respectively where

$$B_i = \{U_{1i}, U_{2i}, \ldots, U_{n(i)i}\} \quad \text{with} \quad \dim(B_i) = n(i),\ i = 1,2,\ldots,m$$

and let

$$E_j = (B_j \cap C^{n(1)}(I)) - B_{j-1} \quad \text{with} \quad \dim(E_j) = \bar{n}(j) < n(j),\ j = 2,3,\ldots,m.$$

Obviously $N(L_1) \cup N(L_2) \cup \ldots \cup N(L_m) \subset N(M)$. We will seek solutions $u \in N(M)$ of the form

$$u_{\alpha_1\alpha_2\ldots\alpha_{m-1}}(x) = u(x) = \left\{\begin{array}{lll} \sum_{j=1}^{n(1)} c_{1j}u_{1j}(x) = u_{\alpha_1}(x) & & \alpha \le x \le \alpha_1 \\ \vdots & & \\ \sum_{j=1}^{n(e)} c_{ej}u_{ej} & \alpha_{e-1} \le x \le \alpha_e & \\ & & \\ \sum_{j=1}^{n(e+1)} c_{e+1j}u_{e+1j} & \alpha_e \le x \le \alpha_{e+1} & (2.1) \\ \vdots & & \\ \sum_{j=1}^{n(m)} c_{mj}u_{mj} = u_{\alpha_m}(x) & & \alpha_{m-1} \le x \le b \end{array}\right\} (2.2)$$

for $\alpha_e \in I$, $e = 1,2,\ldots,m-1$ and $\alpha_e \notin N(L_e) \cup N(L_{e+1})$. A function of the form (2.1) in $N(M)$ will be said to have <u>a single ordinary branching at</u> $x = \alpha_e$, on $[\alpha_{e-1}, \alpha_{e+1}]$. A function of the form (2.2) will be said to have <u>a multiple ordinary branching at</u> $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ on $I = [a,b]$ with $\alpha_e \le \alpha_{e+1}$, $e = 1,\ldots,m-2$.

Denote the Wronskian

$$W_e(u_{1i}, u_{2i}, \ldots, u_{n(i)i}, u_{1(i+1)}(x_0))$$

by

$$W_e(x_0),\ e = 1,2,\ldots,m-1.$$

The following theorem shows when $N(M)$ will contain functions having a multiple ordinary branching.

THEOREM 1. Assume that

$$n(e) - \bar{n}(e) + n(e+1) \ge n(1) + 1,\ e = 2,\ldots,m-1 \tag{2.3}$$

and if

(i) E_j has just one function $u_{1j}(x)$, $j = 2,\ldots,m$, then there exists $u \in N(M)$ having a multiple ordinary branching at $(\alpha_1, \ldots, \alpha_{m-1})$ if and only if

$$W_e(\alpha_e) = 0,\ e = 1,\ldots,m-1 \iff (L_i u_{1(i+1)})(\alpha_i) = 0,\ i = 1,\ldots,m-1. \tag{2.4}$$

(ii) $\dim(E_j) \neq 1$, $j = 2,\ldots,m$, then for every $(\alpha_1,\ldots,\alpha_{m-1})$ with $\alpha_e \in \text{int } I$, and $\alpha_e \geq \alpha_{e+1}$, $e = 1,\ldots,m-2$ there exists a $u \in N(M)$ having a multiple ordinary branching at $(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})$.

PROOF. It is enough to find numbers, $C_{11},\ldots,C_{1n(1)}$, $C_{21},\ldots C_{2n(2)},\ldots C_{m1}$, $\ldots,C_{mn(m)}$, so that $u \in C^{n(1)}(I)$. Therefore we must have

$$\left.\begin{aligned} \sum_{j=1}^{n(1)} c_{1j}u_{1j}^{(k)}(\alpha_1) &= \sum_{j=1}^{\bar{n}(2)} c_{2j}u_{2j}^{(k)}(\alpha_1) \\ \sum_{j=1}^{n(2)} c_{2j}u_{2j}^{(k)}(\alpha_2) &= \sum_{j=1}^{\bar{n}(3)} c_{3j}u_{3j}^{(k)}(\alpha_2), \; k = 0,1,\ldots,n(1) \\ &\;\;\vdots \\ \sum_{j=1}^{n(m-1)} c_{m-1j}u_{m-1j}^{(k)}(x) &= \sum_{j=1}^{\bar{n}(m)} c_{mj}u_{mj}^{(k)}(\alpha_{m-1}) \end{aligned}\right\} \tag{2.5}$$

CASE (i). In this case (2.5) becomes

$$\sum_{j=1}^{n(e)} c_{ej}u_{ej}^{(k)}(\alpha_e) - c_1^{n(e+1)}u_{1n(e+1)}(\alpha_e) = 0, \; e=1,2,\ldots,m-1, \; k=0,1,\ldots,n(1) \tag{2.6}$$

where $c_{1n(e+1)} \neq 0$ (we take $c_{1n(e+1)} = 1$). The homogeneous equation (2.6) has a nontrivial solution if and only if (2.4) holds.

Note that it is easy to verify that

$$\begin{aligned} W_e(\alpha_e) &= W_e(u_{1e},u_{2e},\ldots,u_{n(e)e},u_{1(e+1)}(\alpha_e) \\ &= \alpha_{en(e)}^{-1}(\alpha_e)W_e(u_{1e},u_{2e},\ldots,u_{n(e)e}(\alpha_e)(L_e u_{1(e+1)})(\alpha_e), \; e=1,2,\ldots,m-1. \end{aligned}$$

CASE (ii). If $(L_e u_{s_e(e+1)})(\alpha_e) = 0$, $e = 1,2,\ldots,m-1$ we let $c_{s_e(e+1)} = 1$ and the rest coefficients zero. We then work as in Case (i). Otherwise we write (2.5) as

$$\sum_{j=1}^{n(e)} c_{ej}u_{ej}^{(k)}(\alpha_e) - c_{1n(e+1)}u_{1n(e+1)}(\alpha_e) = \sum_{j=1}^{\bar{n}(e+1)} c_{jn(e+1)}u_{jn(e+1)}(\alpha_e),$$
$$e = 1,2,\ldots,m-1, \; k = 0,1,\ldots,n(1).$$

Note now, that the rank of the coefficients matrix on the left hand side is $(n(1)+1)$ and thus we have a unique solution for the coefficients on the left hand side for any choice of the coefficients on the right hand side and for any $\alpha_e \in I$, $e = 1,2,\ldots,m-1$.

The next theorem characterizes the conditions with the coefficients in (2.2) must satisfy in order that multiple branching can occur at $(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})$ with $\alpha_e \leq \alpha_{e+1}$, $e = 1,\ldots,m-2$ and $\alpha_e \in I$.

THEOREM 2. The following are equivalent:

$$u \in N(M) \text{ on } [c,d] \subset I \text{ and } u \text{ is as in (2.2).} \tag{2.7}$$

$$(L_e(\sum_{j=1}^{n(e+1)} c_{e+1j}u_{e+1j}))(\alpha_e) = 0, \; e = 1,\ldots,m-2. \tag{2.8}$$

$$D^{k_e}(L_{e+1}[\sum_{j=1}^{n(e)} c_{ej}u_{ej}])(\alpha_e) = 0, \; k_e = 0,1,\ldots,n(e+1) - n(e) \tag{2.9}$$

In particular, (2.8) with $c_{e+1j} \neq 0$ for at least one $u_{e+1j} \in E_j$ and (2.9) with $c_{ej} \neq 0$ for at least on $u_{ej} \in B_e - E_{e+1}$ are both necessary and sufficient conditions for $U \in N(M)$ to have a multiple branching at $(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})$ on $[c,d]$.

PROOF. If $B_e \cap E_{e+1} \neq 0$, $e = 1,2,\ldots,m-2$ the result is trivially true. Otherwise as in Theorem 1, we have that $u \in N(M)$ if and only if

$$\sum_{j=1}^{n(e)} c_{ej}u_{ej}^{(k)}(\alpha_e) = \sum_{j=1}^{n(e+1)} c_{e+1j}u_{e+1j}^{(k)}(\alpha_e),\ k = 0,1,\ldots,n(1),\quad e = 1,2,\ldots,m-1.$$

The above can be written in the form

$$\sum_{j=1}^{n(e)} c'_{ej}u_{ej}^{(k)}(\alpha_e) = \sum_{j=1}^{n(e+1)} c_{e+1j}u_{e+1j}^{(k)}(\alpha_e),\quad k = 0,1,\ldots,n(1),\quad e = 1,2,\ldots,m-1 \tag{2.10}$$

where $\{u_{e+1j}\}_{j=1}^{\bar{n}(e+1)} = E_{e+1}$ and at least one $c_{e+1j} \neq 0$. Here $c'_{ej} = c_{ej} - c_{e+1j}$ if $u_{e+1j} \in B_e \cap E_{e+1}$, $c'_{ej} = c_{ej}$ otherwise.

Now set $c'_{e(n(e)+1)} = -1$ and $u_{e(n(e)+1)}(x) = \sum_{j=1}^{\bar{n}(e+1)} c_{e+1j}u_{e+1j}^{(k)}(x)$ and (2.10) can be written

$$\sum_{j=1}^{n(e)+1} c'_{ej}u_{ej}^{(k)}(\alpha_e) = 0,\ k = 0,1,\ldots,n(1),\ e = 1,2,\ldots,m-1. \tag{2.11}$$

Now, (2.11) has a nontrivial solution for c'_{ej} if and only if

$$W_e(u_{1i},u_{2i},\ldots,u_{n(e)i},u_{n(e)+1i})(\alpha_e) = 0,$$

but

$$W_e(u_{1i},u_{2i},\ldots,u_{n(e)i},u_{n(e)+1i})(\alpha_e) = a_{en(e)}^{-1}(\alpha_e)W_e(u_{1i},u_{2i},\ldots,u_{n(e)i})(\alpha_e)L_eu_{e(n(e)+1)}(\alpha_e),$$

i.e., if and only if (2.8) holds and at least one $c_{p+1j} \neq 0$.

On the other hand, u has a nontrivial branching at $(\alpha_1,\ldots,\alpha_{m-1})$ if and only if (2.10) has a nontrivial solution for the coefficients on the right hand side. As before we set $c_{e(\bar{n}(e+1)+1)} = -1$ and

$$u_{e(\bar{n}(e+1)+1)}(x) = \sum_{j=1}^{n(e)} c'_{ej}u_{ej}^{(k)}(x)$$

and (2.10) can now be written as

$$\sum_{j=1}^{\bar{n}(e+1)+1} c_{e+1j}u_{e+1j}^{(k)}(\alpha_e) = 0,\ k = 0,1,2,\ldots,n(1),\ e = 1,\ldots,m-1 \tag{2.12}$$

or

$$A_e\bar{d}_e = \bar{0} \tag{2.13}$$

in matrix form, where A_e is the coefficient matrix in (2.12) and $\bar{d}_e$ the unknown vector. There will exist a nontrivial solution $\bar{d}_e \neq \bar{0}$, $e = 1,2,\ldots,m-1$ if and only if the rank of A_e, $e = 1,2,\ldots,m-1 \leq \bar{n}(e+1)$. But the $\bar{n}(e+1) \times \bar{n}(e+1)$ principle submatrix of A_e is the Wronskian matrix evaluated at α_e. Hence the rank of $A_e \geq \bar{n}(e+1)$. Therefore (2.13) will have a nontrivial solution if and only if the rank of A_e is $\bar{n}(e+1)$. Now elementary row operations on A_e show that this is equivalent to (2.9).

We now show that $N(M)$ may contain infinitely many linearly independent functions.

THEOREM 3. Assume that either Case (i) holds in theorem for infinitely many $(\alpha_{1i},\alpha_{2i},\ldots,\alpha_{m-1i})$, $i = 1,2,\ldots$ or Case (ii) holds. In either case, there is a sequence $\{u_{\alpha_{1i}\alpha_{2i}\ldots\alpha_{m-1i}}\}_{i=1}^{\infty} \subset N(M)$ such that $u_{\alpha_{1i}\alpha_{2i}\ldots\alpha_{m-1i}}$ has a multiple branching at $(\alpha_{1i},\alpha_{2i}\ldots\alpha_{m-1i})$ with $\alpha_{ei} < \alpha_{e+1i}$, $e = 1,\ldots,m-2$, $i = 1,2,\ldots$ and the set $\{u_{\alpha_{1i}\alpha_{2i}\ldots\alpha_{m-1i}}\}_{i=1}^{n}$ is linearly independent on I for every n.

PROOF. We proceed by induction. We may assume without loss of generality that $\alpha_{ei} < \alpha_{ei+1}$, $i = 1,2,\ldots$, $e = 1,2,\ldots,m-2$. Choose u_{e+1j}'s $\subset E_{e+1}$ then

$$L_p\left(\sum_{j=1}^{\bar{n}(e+1)} c_{e+1j}u_{e+1j}\right)(x) \neq 0,\ x \in [\alpha_{e1},\alpha_{(e+1)}].$$

Hence $u_{\alpha_{e1}}(x) \neq 0$ on $[\alpha_{e1},\alpha_{(e+1)1}]$, so

$$u_{\alpha_{11}\alpha_{21}\ldots\alpha_{m-11}}(x) \neq 0$$

on $I = [a,b]$.

Now suppose that $u_{\alpha_{1i}\alpha_{2i}\ldots\alpha_{m-1i}}$, $i = 1,2,\ldots,n$ are linearly independent. Suppose that there exist constants d_k, $i = 1,2,\ldots,n+1$:

$$\sum_{i=1}^{n+1} d_i u_{\alpha_{1i}\alpha_{2i}\ldots\alpha_{m-1}}(x) = 0$$

if $d_{n+1} = 0$ then $d_i = 0$, $i = 1,2,\ldots,n$ and $\{u_{\alpha_{1i}\alpha_{2i}\ldots\alpha_{m-1}}\}_{i=1}^{n+1}$ is linearly independent. If $d_{n+1} \neq 0$

$$u_{\alpha_{1n+1}\alpha_{2n+1}\ldots\alpha_{m-1n+1}}(x) = d_{n+1}^{-1}\sum_{i=1}^{n} d_i u_{\alpha_{1i}\alpha_{2i}\ldots\alpha_{m-1i}}$$

for all $x \in I$ in particular for each $x \in (\alpha_{e-1i},\alpha_{e+1i})$, but

$$L_e u_{\alpha_{en+1}}(x) = 0,\ x \in (\alpha_{en-1},\alpha_{en+1})$$

whereas

$$\left(d_{n+1}^{-1}\sum_{i=1}^{n} d_i u_{ei}\right) \in \text{span } E_{e+1}$$

when $x \in (\alpha_{en},\alpha_{en+1})$, so $L_e u_{\alpha_{en+1}}(x) \neq 0$ for some $x \in (\alpha_{en},\alpha_{en+1})$ a contradiction.

DEFINITION 2. Define the set S_i by setting

$$S_i = \{x \in I/(L_i u)(x) = 0\}.$$

Then since $L_i u$, $i = 1,2,\ldots,m$ are continuous functions on I the S_i's, $i = 1,2,3,\ldots,m$ are closed sets and $S_1 \cup S_2 \cup \ldots \cup S_m = I$. In particular, any point $\alpha_e \in [\alpha_{e-1},\alpha_{e+1}]$ at which an ordinary branching occurs on $[\alpha_{e-1},\alpha_{e+1}]$, $e = 1,2,\ldots,m-1$ must belong to $S_{e-1} \cap S_{e+1}$ together with any limit point of the set of points at which ordinary branching occurs since $S_{e-1} \cap S_{e+1}$ is closed.

We show that $S_{e-1} \cap S_{e+1}$ is nowhere dense in $[\alpha_{e-1},\alpha_{e+1}]$, $e = 1,2,\ldots,m-1$.

THEOREM 4. Assume that $u \in N(M)$ as in (2.2) and $B_e \cap E_{e+1} = \emptyset$. Then $S_{e-1} \cap S_{e+1}$ is nowhere dense in $[\alpha_{e-1},\alpha_{e+1}]$, $e = 1,2,\ldots,m-1$.

PROOF. Suppose that $S_{e-1} \cap S_{e+1}$, $e = 1,2,\dots,m-1$ contains a maximal closed interval $[\alpha'_{e-1},\alpha'_{e+1}]$ with $|\alpha'_{e+1} - \alpha'_{e-1}| \neq 0$, $e = 1,2,\dots,m-1$. Then

$$u(x) = \sum_{j=1}^{n(e+1)} c_{ej}u_{ej} \quad \text{for} \quad x \in [\alpha'_{e-1},\alpha'_{e+1}].$$

Now let $(\alpha''_{e-1},\alpha''_{e+1}) \subset [\alpha'_{e-1},\alpha'_{e+1}]$. Then by Case (i) in Theorem 1 there exist constants $c^{(1)}_{ej}$, $c^{(2)}_{ej}$, such that

$$u(\alpha''_{e-1},\alpha''_{e+1j}x) = \begin{cases} \sum_{j=1}^{n(e)} c^{(1)}_{ej}u_{ej}(x) & \alpha_{e-1} \le x \le \alpha''_{e-1} \\ \sum_{j=1}^{n(e+1)} c_{ej}u_{ej}(x) & \alpha''_{e-1} \le x \le \alpha''_{e+1} \\ \sum_{j=1}^{n(e)} c^{(2)}_{ej}u_{ej}(x) & \alpha''_{e+1} \le x \le \alpha_{e+1} \end{cases}$$

belongs to $N(M)$ since

$$L_e\Big(\sum_{j=1}^{n(e+1)} c_{ej}u_{ej}\Big)(z) = 0$$

at $z = \alpha''_{e-1},\alpha''_{e+1}$. But

$$L_e\Big(\sum_{j=1}^{n(e+1)} c_{ej}u_{ej}\Big)(x) = 0$$

on $[\alpha''_{e-1},\alpha''_{e+1}]$. Hence $U(\alpha''_{e-1},\alpha''_{e+1j}x) \in N(L_e)$. Since $\sum_{j=1}^{n(e+1)} c_{ej}u_{ej}(x) \notin N(L_e)$, $e = 1,2,\dots,m-1$ the proof of Theorem 3 shows that the set $B_e \cup \{u(\alpha''_{e-1},\alpha''_{e+1j}x)\}$ is linearly independent. But this contradicts $d(L_e) = n(e)$, $e = 1,2,\dots,m$.

We now assume that $n(1) = N(2) = \dots = n(m)$ for simplicity (the other cases can be dealt analogously) and consider the following problem: given $(z_0,z_1,\dots,z_{n(1)-1}) \in \mathbb{R}^{n(1)}$ and $z \in I$ find u such that

$$M_u = (L_1u)(L_2u)\dots(L_mu) = 0$$

$$D^{n(1)}u(z) = z_i, \ i = 0,1,\dots,n(1)-1. \tag{2.14}$$

if $N(L_e) \neq N(L_{e+1})$, $e = 1,2,\dots,m-1$, then we have at least m solutions, the unique solutions belonging to $N(L_e)$, $e = 1,2,\dots,m-1$. In addition according to Theorems 1 and 2 we may have solutions with one or many multiple ordinary branchings.

In the event that L_e, $e = 1,2,\dots,m$ have constant coefficients we proceed as follows: let s_{je}, $j = 1,2,\dots,n(1)$, $e = 1,2,\dots,m$ denote the solutions of the characteristic equation L_e and assume $u \in N(L_e)$ on some subinterval $I(z)$ of I containing z, then the restriction $\bar{u}$ of u on $I(z)$ can be written

$$\bar{u}(x) = \sum_{j=1}^{n(1)} c_{je}e^{s_{je}x}$$

where $\{e^{s_{je}x}\}_{j=1}^{n(1)}$ spans $N(L_e)$ and c_{je} are uniquely determined by (2.14). By (2.9) we must have

$$L_{e+1}(\bar{u}(\alpha_e)) = 0, \quad e = 1,2,\dots,m-1.$$

It follows that

$$\sum_{j=1}^{n(1)} d_{je} e^{t_{je}\alpha_e} = 0 \tag{2.15}$$

where

$$d_{je} = c_{je} \sum_{i=0}^{n(1)} c_{ie+1} t_{ie}^{i}, \ i = 1,2,\ldots,n(1) \tag{2.16}$$

$$t_{ie} = s_{ie} - s_{n(1)e}, \ e = 1,2,\ldots,m. \tag{2.17}$$

Note that each one of the equations in (2.15) can have at most $n(1)-1$ real solutions if the d_{je}'s and t_{je}'s are all real [7].

Denote by $\alpha_{p1},\alpha_{p2},\ldots,\alpha_{pn(1)-1}$ the solutions obtained in the $p^{\underline{th}}$ equation in (2.15), $p = 1,2,\ldots,m-1$ and assume that (the other cases can be dealt analogously)

$$\begin{array}{c} \alpha_{11} \le \alpha_{12} \le \alpha_{13} \le \cdots \le \alpha_{1n(1)-1} \le \\ \alpha_{21} \le \alpha_{22} \le \alpha_{23} \le \cdots \le \alpha_{2n(1)-1} \le \\ \vdots \\ \alpha_{m-11} \le \alpha_{m-12} \le \alpha_{m-13} \le \cdots \le \alpha_{m-1n(1)-1} \end{array} \tag{2.18}$$

Inequality (2.18) shows that we can have at most $(n(1)-1)^{m-1}$ ordinary multiple branchings, e.g. $(\alpha_{11},\alpha_{21},\ldots,\alpha_{m-11})$ is one of them. We have thus proved.

THEOREM 5. If L_i, $i = 1,2,\ldots,m$ have constant coefficients, then there exists a solution $u \in N(M)$ (u as in (2.2)) to the intial value problem (2.14) having a multiple ordinary branching $(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})$ with $\alpha_e \in I$, $e = 1,2,\ldots,m-1$ if and only if α_e is a root of the exponential polynomial (2.15), where the d_{je}'s and t_{je}'s are all real and they are given by (2.16) and (2.17).

Moreover if (2.18) holds there are at most $(n(1)-1)^{m-1}$ solutions $u \in N(M)$ (u as in (2.2)).

THEOREM 6. Assume that the hypotheses of Theorem 5 are satisfied. Then there are at most

$$(m-1)(n(1)-1)(n(1)-2)^{n-1} \tag{2.19}$$

solutions u (u as in (2.2)) to the initial value problem having exactly n multiple branchings $(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})$ in I where any $m-2$ of the α_e's are fixed $e = 1,2,\ldots,m-1$.

Moreover in this case if there are no solutions with $n+1$ multiple branchings then the total number of solutions to the problem

$$Mu = 0$$

is bounded by

$$(m-1)(n(1)-1) \sum_{j=0}^{n-1} (n(1)-2)^{j}. \tag{2.20}$$

PROOF. Without loss of generality we can assume that α_{11} denote the first point at which a branching occurs and $u \in N(L_1)$ on some subinterval $I(z) = [z,\alpha_{11}]$. Then $u \in N(L_2)$ on $[\alpha_{11},\alpha_{11}+\epsilon]$, for some $\epsilon > 0$. There are at most $m-1$ possible values for α_{11}. Suppose $w > \alpha_{11}$ is the next point at which a multiple branching of u occurs. Then $u \in N(L_2)$ on $[\alpha_{11},w]$. Hence there exist uniquely determined $c_{j2}(\alpha_{11})$, $j = 1,2,\ldots,n(1)$ such that

$$u(x) = \sum_{j=1}^{n(1)} d_{j2}(\alpha_{11})u_{j2}(x)$$

on $[\alpha_{11},w]$ where $\{u_{j2}\}_{j=1}^{n(1)}$ span $N(L_2)$. By Theorem 2,

$$[L_1(\sum_{j=1}^{n(1)} d_j(\alpha_{11})u_{j2})](v) = 0$$

at $v = \alpha_{11}$ and $v = w$. Hence there are $m-2$ possible w's with $w > \alpha_{11}$. This argument applies again for the next branching. Since this argument can be applied in any of the $m-1$ rows in (2.18), this proves (2.19).

Finally (2.20) can easily be proved if we use (2.19) for $j = 0,1,2,\ldots,n$ and add the results.

REMARK 1. (a) We can assume in Theorem 6 that any h points $h \in \{1,2,\ldots,m-1\}$ are fixed from $(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})$ then proceeding as in Theorem 6 we can prove that the corresponding relations for (2.19) and (2.20) are respectively

$$(m-1-(h-1))(m(1)-1)^h(n(1)-2)^{h-1} \tag{2.21}$$

and

$$(m-1-(h-1))(n(1)-1)^h\sum_{j=0}^{n-1}(n(1)-2)^h \tag{2.22}$$

(b) Up till now we obtained the cardinality results in Theorems 5, 6 and in (a) above by assuming that (2.18) is true and u as in (2.2). But (2.2) can be written in $(m-1)!$ different ways by interchanging the role of the L_i's, $i = 1,2,\ldots,m$. Therefore in general all the cardinality results obtained up till now can be multiplied by $(m-1)!$

(c) If the L_i, $i = 1,2,\ldots,m$ are nonconstant but continuous (as in the Introduction) we can restate Theorem 5 and (2.2). However the conclusions and the proofs are going to be exactly analogous.

We now provide examples for Theorems 4 and 6 and (1.4).

3. APPLICATIONS.

EXAMPLE 1. Let $m = 2$ and consider the function f defined by

$$f(x) = \begin{cases} x^8 \ln\frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$u_1(x) = e^{f(x)},\ u_2(x) = e^{2f(x)},\ L_1u = u' - f'(x)u,\ L_2u = u' - 2f'(x)u.$$

Then $u_1 \in N(L_1)$, $u_2 \in N(L_2)$ and a $u \in N(M)$ can be written as

$$u(x) = \begin{cases} ce^{f(x)}, & -\epsilon \le x \le 0 \\ de^{2F(x)}, & 0 \le x \le \epsilon \end{cases}, \quad \epsilon > 0$$

or

$$u(x) = \begin{cases} ce^{2f(x)}, & -\epsilon \le x \le 0 \\ de^{f(x)}, & 0 \le x \le \epsilon \end{cases}, \quad \epsilon > 0$$

That is, 0 is a limit point of branching points of u.

In the event that the characteristic equations of L_i, $i = 1,2,\dots,m$ have complex roots (2.15) may have infinite solutions to the initial value problem on $(-\infty,\infty)$ even if we have one ordinary multiple branching in $(-\infty,\infty)$.

EXAMPLE 2. Let $m = 2$, $L_1 = D^2 + 1$, $L_2 = D^2 + 4$, $u(0) = 0$, $u'(0) = 1$. Let $u \in N(L_1)$ on $[-\epsilon,\epsilon]$ for some $\epsilon > 0$. Then

$$u_1(x) = -\frac{i}{2} e^{ix} + \frac{i}{2} e^{-ix} = \sin x$$

and (2.15) due to (2.16) and (2.17) becomes

$$e^{2i\alpha} = 1$$

therefore $\alpha_n = n\pi$, $n = 0,1,2,\dots$

$$u_2(x) = \frac{1}{4i} e^{2ix} - \frac{1}{4i} e^{-2ix} = \frac{1}{2}\sin 2x, \text{ etc.}$$

EXAMPLE 3. Consider the equation

$$\frac{dM}{dx} - \lambda M = 0?.$$

Let $L_1 = (D-1)(D-2)(D-3)$, $L_2 = (D-4)(D-5)(D-6)$, $L_3 = (D-7)(D-8)(D-9)$ and $\lambda = 0$. Then

$$u_1(x) = 2e^x - 3e^{2x} + e^{3x}$$
$$u_2(x) = 5e^{4x} - 9e^{5x} + 4e^{6x}$$
$$u_3(x) = 8e^{7x} - 15e^{8x} + 7e^{9x}$$

and

$$L_2u_1(x) = 0 \rightarrow -120e^x + 72e^{2x} - 6e^{3x} = 0 \rightarrow \alpha = \ln 2,\ \ln 10$$
$$L_1u_2(x) = 0 \rightarrow 30e^{4x} - 216e^{5x} + 240e^{6x} = 0 \rightarrow \alpha = \ln(\tfrac{108\pm\sqrt{4464}}{240})$$
$$L_3u_2(x) = 0 \rightarrow -300e^{4x} + 216e^{5x} - 24e^{6x} = 0 \rightarrow \alpha = \ln(\tfrac{54\pm\sqrt{1116}}{12})$$
$$L_2u_3(x) = 0 \rightarrow 48e^{7x} - 360e^{8x} + 420e^{9x} = 0 \rightarrow \alpha = \ln(\tfrac{90\pm\sqrt{3060}}{210})$$

So we can have multiple branchings at

$$(\ln^2,\ \ln(\tfrac{54+\sqrt{1116}}{12})),$$
$$(\ln(\tfrac{108+\sqrt{4464}}{240}),\ \ln(\tfrac{54+\sqrt{1116}}{12})),$$
$$(\ln(\tfrac{108+\sqrt{4464}}{240}),\ \ln(\tfrac{54-\sqrt{1116}}{12})),$$
$$(\ln(\tfrac{108-\sqrt{4464}}{240}),\ \ln(\tfrac{54+\sqrt{1116}}{12})),$$
$$(\ln(\tfrac{108-\sqrt{4464}}{240}),\ \ln(\tfrac{54-\sqrt{1116}}{12})),$$

and

$$(\ln(\tfrac{108-\sqrt{4464}}{240}),\ \ln(\tfrac{90+\sqrt{3060}}{210})).$$

For example we can have the solution $u \in N(M)$ given by

$$u(x) = \begin{cases} 2e^{x} - 3e^{2x} + e^{3x} & , \quad -\infty < x \leq \ln 2 \\ 5e^{4x} - 9e^{5x} + 4e^{6x} & , \quad \ln 2 \leq x \leq \ln(\frac{54+\sqrt{1116}}{12}) \\ 8e^{7x} - 15e^{8x} + 7e^{9x}, & \quad \ln(\frac{54+\sqrt{1116}}{12}) \leq x < +\infty, \end{cases}$$

etc.

The above are solutions corresponding to the order (L_1, L_2, L_3). But we can obtain additional solutions corresponding to (L_1, L_3, L_2), (L_2, L_1, L_3), (L_2, L_3, L_1), (L_3, L_1, L_2) and (L_3, L_2, L_1).

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