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RESEARCH NOTES

FUNCTIONAL DIFFERENTIAL EQUATIONS — A RECIPROCITY PRINCIPLE

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ABSTRACT. The functional differential equations proposed for solution here are mainly ordinary differential equations with fairly general argument deviations. Included among them are equations with involutions and some with reflections of the argument. Solutions will be obtained by quadratures in terms of implicitly defined functions. They have a wide range of applicability from the stability theory of differential-difference equations to electrodynamics and biological models.

KEY WORDS AND PHRASES. Functional differential equations, involutions, reflections, reciprocity principle.
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1. INTRODUCTION.

Several authors have explored the subject of F.D.E.'s whose defining equations are involutions as in [1-5] and effective procedures are now known for dealing with them. In [6], Wiener and Aftabizadeh gave existence and uniqueness theorems for boundary-value problems problems for reflection of the argument in some cases of linear and nonlinear equations. In some instances, however, the solutions are given in abstract terms or are made to depend on the solutions of equations of high-order. We seek more concrete results in some instances and more general in others.

2. MAIN EQUATION AND RESULTS.

The central theme of this paper and its major motivation was provided by the equation

$$y' = y(g(x)), \quad \text{for given} \quad g(x) \tag{2.1}$$

This equation was suggested for solution by W. R. Utz in [7].

We now state our main

THEOREM (2.1). Let $g(x)$ be continuous, 1-1 and onto, then the solution of (2.1) can be found by quadratures. We write (2.1) in parametric form as

$$U : \left\{ \begin{array}{l} V = uX \\ \\ u = V \circ g \circ X \end{array} \right\} \tag{2.2}$$

where $V = \frac{d\dot{V}}{dt}$, $X = \frac{d\bar{X}}{dt}$ and u,V have replaced y' and y respectively in a new variable, t. The symbol, $\circ$, is used for composition of functions.

Let $V = A_1(t)$, $X = B_1(t)$. Then from the above system, U, we have

$$u = A_1 \circ g \circ B_1 \tag{2.3}$$

and also

$$\frac{d}{dt}(A_1) = (A_1 \circ g \circ B_1 \frac{d}{dt}(B_1)) \tag{2.4}$$

The differential equation

$$y' = h(y(x)) \tag{2.5}$$

is solvable by separation or variables in a familiar way where h is an arbitrary function. We write it as a system

$$K : \left\{ \begin{array}{l} \dot{V} = u\dot{X} \\ \\ u = h \circ V \circ X \end{array} \right\} \tag{2.6}$$

In system K substitute as before $u = A(t)$, $V = B(t)$. From whence we get

$$X = B^{-1} \circ h^{-1} \circ A \tag{2.7}$$

and

$$\frac{d}{dt}(B) = A \frac{d}{dt}(B^{-1} \circ h^{-1} \circ A) \tag{2.8}$$

equate corresponding parts of (2.4), (2.8) so that when (2.8) holds, (2.4) must also hold. In addition, we require that the solutions of the differential equations (2.1) and (2.5) coincide. This gives us the set of equations

$$A_1 = B, \quad A_1 \circ g \circ B_1 = A, \quad B_1 = B^{-1} \circ h^{-1} \circ A, \quad A_1 \circ B^{-1} = B \circ A^{-1} \circ h \circ B \tag{2.9}$$

The first three equations refer to the insides of corresponding pairs of parentheses in (2.4) and (2.8) and the last one ensures that the solutions are identical for (2.1) and (2.5).

Eliminating the functions A_1 and B_1 from (2.9) gives

$$B \circ g \circ B^{-1} = h \tag{2.10}$$

A reciprocity relation exists between g,h which is the reason for the title of this paper. The solution process by steps is as follows: (1) Choose B; (2) find h from (2.10); (3) find A by (2.6). With the values of h and B, (4) find B_1 from the equation, $B_1 = B^{-1} \circ h^{-1} \circ A$ and find the solution of (2.1) from its value, $Q = A_1 \circ B_1^{-1} = B \circ B_1^{-1}$. This completes the proof.

3. THE NESTING EQUATION.

We turn now to the equation

$$y' = k(y(g(x))) \tag{3.1}$$

where k,g are both functions with inverses. This last equation has been called a nesting equation. In the same way as before we have the

THEOREM 3.1. The equation (3.1) is solvable by quadratures.

(Sketch of) Proof: Rewrite (3.1) in the form

$$\left\{\begin{array}{l} \dot{V} = u\dot{X} \\ \\ u = k\circ V\circ g\circ X \end{array}\right\} \tag{3.2}$$

Choose parametric functions

$$u = C, \quad x = D \tag{3.3}$$

Comparison with (2.6) gives the set of equations

$$k^{-1}\circ C\circ D^{-1}\circ g^{-1} = B, \quad C = A, \quad D = B^{-1}\circ h^{-1}\circ A, \quad k^{-1}\circ C\circ D^{-1}\circ g^{-1}\circ D^{-1} = B\circ A^{-1}\circ h\circ B \tag{3.4}$$

Elimination of C,D gives us

$$k^{-1}\circ h\circ B\circ g^{-1} = B \tag{3.5}$$

Now k,g are given and so (3.5) tells us how to choose, h, once B is chosen arbitrarily.

4. LINEAR EQUATIONS.

Consider the equation

$$y'(x) = A(x)y + B(x)y(g(x)) \tag{4.1}$$

for which we have the following

THEOREM 4.1. Let $A(x)$, $B(x)$ be integrable in finite terms and let $g(x)$ be one-to-one and onto. Then (4.1) is solvable by quadratures.

(Sketch of) Proof: Multiply both sides of (4.1) by $\exp(-\int A(x))$ as usual to bring it to the form

$$(CY)' = Dy(g(x)) \tag{4.2}$$

Then with a change of variables

$$Cy = w \tag{4.3}$$

we get an equation of the form

$$V'(x) = P(x)V(g(x)) \tag{4.4}$$

which we write in parameter form

$$\left\{\begin{array}{l} \dot{V} = u\dot{X} \\ \\ u = (P\circ X)(V\circ g\circ X) \end{array}\right\} \tag{4.5}$$

Again set

$$u = A, \quad X = g^{-1}\circ B\circ A \tag{4.6}$$

and by substitution in (4.5) we get

$$V = Q\circ A^{-1}\circ B^{-1} \tag{4.7}$$

for some Q. Proceed as before.

In spite of the attractive result in section 2 I was unable to go past the scalar case.

REFERENCES

1. WIENER, J. Differential Equations with Involutions, Differencial'nye Uravnenija 6 (1969), 1131-1137.

2. WIENER, J. Differential Equations in Partial Derivatives with Involutions, Differencial'nye Uravnenija 7 (1970), 1320-1322.

3. WIENER, J. Differential Equations with Period Transformations of thė Argument, Izv. Uyss. Ucebm. Zaued. Radiofizika 3 (1973), 481-484.

4. WIENER, J. Investigation of Some Functional Differential Equations with a Regular Singular Point, Differencial'nye Uravnenija 10 (1974), 1892-1894.

5. SHAH, S. M. and WIENER, J. Reducible Functional Differential Equations, International Journal of Mathematics and Mathematical Sciences, Vol. 8 No. 1 (1985), 1-27.

6. WIENER, J. and AFTABIZADEH, A. R. Boundary Value Problems for Differential Equations with Reflection of the Argument, International Journal of Mathematics and Mathematical Sciences, Vol. 8 No. 1 (1985), 151-163.

7. UTZ, W. R. The Equation $f'(x) = af(g(x))$, Bulletin American Mathematical Society, 71 (1965), 138.