Internat. J. Math. & Math. Sci.
Vol. 9 No. 4 (1986) 785-789

ON A GENERALIZATION OF THE CORONA PROBLEM

GRAZIANO GENTILI and DANIELE C. STRUPPA

Scuola Normale Superiore
Piazza dei Cavalieri, 7
56100 Pisa, Italy

(Received May 7, 1985)

ABSTRACT. Let $g, f_1, \ldots, f_m \in H^\infty(\Delta)$. We provide conditions on $f_1, \ldots, f_m$ in order that $|g(z)| \leq |f_1(z)| + \ldots + |f_m(z)|$, for all z in Δ, imply that g, or g^2, belong to the ideal generated by $f_1, \ldots, f_m$ in H^∞.

KEY WORDS AND PHRASES. Corona problem, congenial functions.
1980 AMS SUBJECT CLASSIFICATION CODE. 30D55, 30D50.

1. INTRODUCTION.

Let $H(\Delta)=H$ be the space of all holomorphic functions on $\Delta=\{z\in\mathbb{C}: |z|<1\}$, and let $H^\infty(\Delta)=H^\infty$ be the subspace of all bounded functions of $H(\Delta)$. Let $f_1, \ldots, f_m$ be functions in H^∞ and let $g\in H^\infty$ satisfy the following condition:

$$|g(z)| \leq |f_1(z)| + \ldots + |f_m(z)| \qquad (\text{any } z\in\Delta). \tag{1.1}$$

As a generalization of the corona problem (which was first solved by Carleson [1]) it is natural to ask if (1.1) implies that g belongs to the ideal $I_{H^\infty}(f_1, \ldots, f_m)$ generated in H^∞ by $f_1, \ldots, f_m$, i.e. if (1.1) implies the existence of $g_1, \ldots, g_m$ in H^∞ such that, on Δ,

$$g = f_1g_1 + \ldots + f_mg_m. \tag{1.2}$$

Rao, [2], has shown that the answer to this question is negative in general. On the other hand Wolff (see [3], th. 2.3) has proved that (1.1) implies that g^3 belongs to $I_{H^\infty}(f_1, \ldots, f_m)$. The question whether (1.1) implies the existence of $g_1, \ldots g_m$ in H^∞ such that

$$g^2 = f_1g_1 + \ldots + f_mg_m \tag{1.3}$$

is still open, as Garnett has pointed out ([4], problem 8.20).

In this work we obtain some results on this generalized corona problem, making use of techniques which appear in the theory of A_p spaces, the spaces of entire functions with growth conditions introduced by Hörmander [5].

With the same aim of Berenstein and Taylor [6] in A_p, we introduce in H^∞ the notion of jointly invertible functions (definition 3) and prove that if $f_1, \ldots, f_m$ are jointly invertible, condition (1.1) implies that g belongs to $I_{H^\infty}(f_1, \ldots f_m)$ (proposition 5). We also prove that if the ideal $I_{H^\infty}(f_1, \ldots, f_m)$ contains a weakly invertible function having simple interpolating zeroes (see [3]), then again (1.1) implies that g belongs to $I_{H^\infty}(f_1,\dots,f_m)$ (theorem 6).

Finally, in the same spirit of Kelleher and Taylor [7] we introduce the notion of congeniality for m-tuples of functions in H^∞, and give a partial answer to the problem posed by Garnett ([4]): we prove that if $(f_1,\dots,f_m)\in(H^\infty)^m$ is congenial, then (1.1) implies $g^2\in I_{H^\infty}(f_1,\dots,f_m)$ (theorem 8).

2. WEAK INVERTIBILITY.

We first study some conditions under which (1.1) implies that $g\in I_{H^\infty}(f_1,\dots,f_m)$.

DEFINITION 1. A function f in $H^\infty(\Delta)$ is called weakly invertible if there exists a Blaschke product B such that $f(z)=B(z)\tilde{f}(z)$ (z in Δ) with $\tilde{f}$ invertible in H^∞.

The reason for this definition is the following simple criterion of divisibility for functions in H^∞.

PROPOSITION 2. Let $f\in H^\infty$. Then f is weakly invertible if, and only if, for all $g\in H^\infty$ the fact that $g/f\in H$ implies $g/f\in H^\infty$.

PROOF. Suppose f is weakly invertible: then there exists a Blaschke product B such that $f(z)=B(z)\tilde{f}(z)$, with $\tilde{f}$ invertible in H^∞. Since g/f is holomorphic and since B contains exactly the zeroes of f, it follows that $g/B\in H$; however, since B is a Blaschke product, $g/B\in H$ implies, [8], that $g/B\in H^\infty$. Since $1/\tilde{f}\in H^\infty$ one has $g/f=(g/B)(1/\tilde{f})$, i.e. $g/f\in H^\infty$. Conversely, suppose that for all $g\in H^\infty$ such that $g/f\in H$, it follows $g/f\in H^\infty$. Write $f(z)=B(z)\tilde{f}(z)$, where B is the Blaschke product of all the zeroes of f (see [8]). Then B/f is holomorphic on Δ and therefore $1/\tilde{f}$ must belong to H^∞. ¶

An extension of the notion of weak invertibility to m-tuples of functions in H^∞ is given by the following definition, analogous to the one given by Berenstein and Taylor for the spaces A_p in [6].

DEFINITION 3. The functions $f_1,\dots,f_m\in H^\infty$ are called jointly invertible if the ideal generated by $f_1,\dots,f_m$ in H^∞ coincides with $I_{loc}(f_1,\dots,f_m)=\{g\in H^\infty(\Delta)$: for any $z\in\Delta$, there exists a neighborhood U of z and $\lambda_1,\dots,\lambda_m$ in H(U) such that $g=\lambda_1f_1+..+\lambda_mf_m$ on U$\}$.

In view of Cartan's theorem B, it follows immediately that $f_1,\dots,f_m$ are jointly invertible if, and only if, $I_{H^\infty}(f_1,\dots,f_m)=I_H(f_1,\dots,f_m)$, the latter being the ideal generated by $f_1,\dots,f_m$ in $H(\Delta)$. As a consequence of the corona theorem, all m-tuples $f_1,\dots,f_m$ in H^∞ for which there exists $\delta>0$ such that $|f_1(z)|+\dots+|f_m(z)|\geq\delta$ for all z in Δ, are jointly invertible ($I_H=I_{H^\infty}=H^\infty$). More generally one has:

PROPOSITION 4. Let $b\in H^\infty$ be weakly invertible, and let $f_1(z)=b(z)\tilde{f}_1(z),\dots,f_m(z)=$ $=b(z)\tilde{f}_m(z)$, for $\tilde{f}_1,\dots,\tilde{f}_m$ in H^∞ such that $|\tilde{f}_1(z)|+\dots+|\tilde{f}_m(z)|\geq\delta>0$ for some δ and all z in Δ. Then $f_1,\dots,f_m$ are jointly invertible.

PROOF. Let $g\in H^\infty$ belong to $I_H(f_1,\dots,f_m)$. There exist $\lambda_1,\dots,\lambda_m$ in $H(\Delta)$ such that

$$g(z) = \lambda_1(z)f_1(z)+\dots\lambda_m(z)f_m(z) \qquad (\text{all } z\in\Delta) \tag{2.1}$$

i.e., for all z in Δ,

$$g(z) = b(z)[\lambda_1(z)\tilde{f}_1(z)+\dots+\lambda_m(z)\tilde{f}_m(z)]. \tag{2.2}$$

Since b is invertible, and $g/b\in H$, it follows that $\tilde{g}=g/b=\lambda_1\tilde{f}_1+\dots+\lambda_m\tilde{f}_m\in H^\infty$. By the corona theorem, then, it follows that there are $h_1,\dots,h_m$ in H^∞ such that

$$\tilde{g}(z) = h_1(z)\tilde{f}_1(z)+\dots+h_m(z)\tilde{f}_m(z), \tag{2.3}$$

therefore

$$g(z) = \tilde{g}(z)b(z) = h_1(z)f_1(z)+\dots+h_m(z)f_m(z) \tag{2.4}$$

and the assertion is proved.

Let now $f_1,\dots,f_m, g\in H^\infty(\Delta)$, and suppose that (1.1) holds. It is well known, [2], that in general (1.1) does not imply that $g\in I_{H^\infty}(f_1,\dots,f_m)$. However, (1.1) certainly implies that $g\in I_{loc}(f_1,\dots,f_m)$ and hence

PROPOSITION 5. Let $f_1,\dots,f_m$ be jointly invertible. Then if g satisfies condition (1.1), it follows that $g\in I_{H^\infty}(f_1,\dots,f_m)$.

A different situation in which (1.1) implies that $g\in I_{H^\infty}(f_1,\dots,f_m)$ occurs when at least one of the f_j's, say f_1, is weakly invertible and has simple zeroes which form an interpolating sequence ([3]); this happens, for example, when f_1 is an interpolating Blaschke product with simple zeroes ([3]). Indeed, following an analogous result proved in [7] for the space of entire functions of exponential type, one has:

THEOREM 6. Let $f_1,\dots,f_m\in H^\infty$, and suppose f_1 is weakly invertible with simple, interpolating zeroes. Then if $g\in H^\infty$ satisfies condition (1.1) it follows that g belongs to $I_{H^\infty}(f_1,\dots,f_m)$.

PROOF. Choose $a_{ij}\in\mathbb{C}$, $i=2,\dots m$, $j\geq 1$, such that for $\{z_j\}=\{z\in\Delta : f_1(z)=0\}$ it is $|a_{ij}|=1$ and $a_{ij}f_i(z_j)\geq 0$. Define now $b_{ij}\in\mathbb{C}$ (i,j as before) by

$$b_{ij}=\begin{cases} 0 \text{ if } f_2(z_j)=\dots=f_m(z_j)=0 \\ \\ a_{ij}g(z_j)/(|f_2(z_j)|+\dots+|f_m(z_j)|) \quad \text{otherwise.}\end{cases}$$

By (1.1) it follows $|b_{ij}|\leq 1$ (all i,j), and since $\{z_j\}$ is interpolating, one finds $h_2,\dots,h_m$ in H^∞ such that $h_i(z_j)=b_{ij}$. Therefore the function $h=g-(h_2f_2+\dots+h_mf_m)$ belongs to H^∞ and vanishes at each z_j. The simplicity of the zeroes of f_1 shows that $f/f_1\in H$, and the invertibility of f_1 implies $h/f_1=h_1\in H^\infty$. The thesis now follows, since $g=f_1h_1+$ $+\dots+f_mh_m$. ¶

It is worthwhile noticing that the hypotheses of Proposition 5 and Theorem 6 are not comparable. Consider, indeed, the following conditions on $f_1,\dots,f_m\in H^\infty$:

(C_1) $f_1,\dots,f_m$ are jointly invertible.

(C_2) there exists j ($1\leq j\leq m$) such that f_j is invertible, with an interpolating sequence of zeroes, all of which are simple.

Then (C_1) does not imply (C_2): take m=1 and f_1 weakly invertible with non-simple zeroes. On the other hand, also (C_2) does not imply (C_1): consider f_1 invertible with simple interpolating zeroes $\{z_n\}$; let $f_2\in H^\infty$ be a function such that $f_2(z_n)=1/n$ (such a function certainly exists since $\{z_n\}$ is an interpolating sequence); now f_1 and f_2 have no common zeroes, and hence $1\in I_{loc}(f_1,f_2)$; however $1\notin I_{H^\infty}(f_1,f_2)$ since if $1=\lambda_1f_1+\lambda_2f_2$, then it is $\lambda_2(z_n)=n$, i.e. $\lambda_2\notin H^\infty$. Therefore the pair (f_1,f_2) satisfies (C_2) but not (C_1).

3. CONGENIALITY.

In this section we describe a class of m-tuples of functions in $H^\infty(\Delta)$, for which condition (1.1) implies that $g^2\in I_{H^\infty}(f_1,\dots,f_m)$.

DEFINITION 7. An m-tuple $(f_1,\dots,f_m)$ of functions in H^∞ is called congenial if, for all $i,j=1,\dots,m$,

$$(f_if_j'-f_jf_i')/\,\|f\|^2\|f'\| \quad \text{belongs to } L^\infty(\Delta),$$

where $\|f(z)\|^2=|f_1(z)|^2+\dots+|f_m(z)|^2$, $\|f'(z)\|^2=|f_1'(z)|^2+\dots+|f_m'(z)|^2$, and $f_i'=\partial f_i/\partial z$.

Notice that the class of congenial m-tuples is not empty. Indeed, one might consider pairs f_1, f_2 in H^∞ which, at their common zeroes, satisfy some simple conditions on their vanishing order easily deducible from Definition 7. For example, one can ask that $f_1(z_0)=f_2(z_0)=0$, $f_2'(z_0)\neq 0$, $f_1'(z_0)=0$. As a partial answer to problem 8.20 in [4], we prove the following

THEOREM 8. Let $f_1,\ldots,f_m, g\in H^\infty(\Delta)$, and suppose $(f_1,\ldots,f_m)$ be congenial. If g satisfies (1.1), then $g^2\in I_{H^\infty}(f_1,\ldots,f_m)$, i.e. there are $g_1,\ldots,g_m$ in H^∞ such that (on Δ)

$$g^2(z) = f_1(z)g_1(z)+\ldots+f_m(z)g_m(z) \tag{3.1}$$

PROOF. We mainly follow the proof due to Wolff, [3], of the fact that (1.1) implies that $g^3\in I_{H^\infty}$. We can assume $\|f_j\|_\infty\leq 1$, $\|g\|_\infty\leq 1$, and $f_j, g\in H(\overline{\Delta})$ $(j=1,\ldots,m)$. Put $\psi_j=g\overline{f}_j/\|f\|^2$ (ψ_j is bounded and C^∞ on $\overline{\Delta}$) and consider the differential equation

$$\partial b_{j,k}/\partial\overline{z} = \psi_j\partial\psi_k/\partial\overline{z} = g^2 G_{j,k} \qquad (1\leq j,k\leq m) \tag{3.2}$$

for

$$G_{j,k}=\overline{f}_j\sum_\ell f_\ell(\overline{f_\ell f_k'-f_k f_\ell'})/\|f\|^6.$$

If solutions $b_{j,k}\in L^\infty$ exist, then clearly $g_j=g\psi_j+\sum_k(b_{j,k}-b_{k,j})f_k\in H^\infty$ and (3.1) holds (indeed $\overline{\partial}g_j=0$ and g_j is bounded on Δ). In order to prove that (3.2) admits a solution in L^∞ it is enough to show that $|g^2G_{j,k}|^2\log(1/|z|)dxdy$ and $\partial(g^2G_{j,k})/\partial z$ are Carleson measures for $1\leq j,k\leq m$.

As far as $|g^2G_{j,k}|^2\log(1/|z|)dxdy$ is concerned, notice that, by the congeniality of $(f_1,\ldots,f_m)$, it is

$$|g^2G_{j,k}|^2 \leq |g|^4|\overline{f}_j|^2|\sum_\ell f_\ell(\overline{f_\ell f_k'-f_k f_\ell'})|^2/\|f\|^{12} \leq c\|f'\|^2.$$

On the other hand,

$$\partial(g^2G_{j,k})/\partial z = 2gg'G_{j,k}+g^2\partial G_{j,k}/\partial z;$$

again by the congeniality of $(f_1,\ldots,f_m)$, one has

$$|gg'G_{j,k}| \leq |g||g'||\overline{f}_j||\sum_\ell f_\ell(\overline{f_\ell f_k'-f_k f_\ell'})|/\|f\|^6 \leq c(|g'|^2+\|f'\|^2)/\|f\| \leq$$
$$\leq c(|g'|^2/|g|+\|f'\|^2/\|f\|),$$

and

$$|g^2\partial G_{j,k}/\partial z| = |g|^2\cdot|f_j||\sum_\ell \overline{f}_\ell f_\ell'|\cdot|\sum_\ell f_\ell(\overline{f_\ell f_k'-f_k f_\ell'})|/\|f\|^8 +$$
$$+|g|^2|\overline{f}_j|/\|f\|^2\cdot(|\sum_\ell f_\ell'(\overline{f_\ell f_k'-f_k f_\ell'})|/\|f\|^4+2|\sum_\ell f_\ell'\overline{f}_\ell||\sum_\ell f_\ell(\overline{f_\ell f_k'-f_k f_\ell'})|/\|f\|^6) \leq$$
$$\leq c\sum_\ell|f_\ell'|^2/|f_\ell|.$$

This concludes the proof. ¶

ACKNOWLEDGMENT. The authors wish to thank Professor Carlos A. Berenstein for reading a preliminary version of this paper. They also gladly acknowledge the Ministero P.I. of the Italian Government and the University of Maryland for their financial support.

REFERENCES

1. CARLESON, L. Interpolation by Bounded Analytic Functions and the Corona Problem, Ann. of Math. (2)76 (1963), 547-559.

2. RAO, K.V.R. On a Generalized Corona Problem, J. Anal. Math. 18(1967), 277-278.

3. GARNETT, J.B. Bounded Analytic Functions, Academic Press, New York, 1981.

4. BARTH, K.F., BRANNAN, D.A. AND HAYMAN, W.K. Research Problems in Complex Analysis, Bull. London Math. Soc. 16 (1984), 490-517.

5. HORMANDER, L. Generators for Some Rings of Analytic Functions, Bull. Amer. Math. Soc. 73 (1967), 943-949.

6. BERENSTEIN, C.A. and TAYLOR, B.A. Interpolation Problems in $\mathbb{C}^n$ with Applications to Harmonic Analysis, J. Anal. Math. 38 (1980), 188-254.

7. KELLEHER, J.J. and TAYLOR, B.A. FInitely Generated Ideals in Rings of Analytic Functions, Math. Ann. 193 (1971), 225-237.

8. HOFFMAN, K. Banach Spaces of Analytic Functions, Prentice Hall Inc., Englewood Cliffs, 1962.