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NONSEPARATED MANIFOLDS AND COMPLETELY UNSTABLE FLOWS

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ABSTRACT. We define an order structure on a nonseparated n-manifold. Here, a nonseparated manifold denotes any topological space that is locally Euclidean and has a countable basis; the usual Hausdorff separation property is not required. Our result is that an ordered nonseparated n-manifold X can be realized as an ordered orbit space of a completely unstable continuous flow ϕ on a Hausdorff (n + 1)-manifold E.

KEY WORDS AND PHRASES. Completely unstable flows, nonseparated manifolds, order structure, orbit space.
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1. INTRODUCTION

Nonseparated manifolds arise in a very natural way in the study of ordinary differential equations and completely unstable flows. A topological space that is non-Hausdorff, locally Euclidean and has a countable basis is referred to as a nonseparated manifold. A flow ϕ on a manifold E is said to be completely unstable if it has no nonwandering points. Such systems occur very naturally. For example, on $\mathbb{R}^2$ any continuous flow without equilibria is completely unstable, and the restriction of any flow to the complement of its set of nonwandering points is completely unstable. All open manifolds admit completely unstable flows.

Let $\phi : E \times \mathbb{R}^1 \to E$ be a completely unstable c^0 flow on an (n + 1)-manifold E. The orbit space of ϕ is the set E/ϕ of all orbits of ϕ with the quotient topology (the finest topology in which the natural projection $\pi : E \to E/\phi$ is continuous). If ϕ admits local cross-sections at every point of E that are n-Euclidean, we say ϕ is locally trivial. It is known that if either E and ϕ are c^1 or $n \le 2$, then ϕ is locally trivial ([1],[2]). Moreover, if ϕ is locally trivial, completely unstable c^0 flow then E/ϕ is a nonseparated n-manifold. The ordered orbit space of ϕ is obtained from this non-separated manifold by imposing an additional structure that indicates the order in which the cross-sections of ϕ that correspond to the charts of E/ϕ are traversed by orbits of ϕ (precise definitions are given in [3]). We then have the following classification theorem which shows that completely unstable flows on manifolds can be classified completely in terms of their associated ordered orbit spaces (Theorem 3.1, [3]).

CLASSIFICATION THEOREM. If ϕ and ϕ' are locally trivial, completely unstable c^0 flows on m-manifolds M and M', respectively, then (M,ϕ) and (M',ϕ') are topologically equivalent if and only if M/ϕ is order isomorphic to M'/ϕ'.

Our interest here is in the question of realization: What nonseparated manifolds can be realized as the ordered orbit space of a completely unstable c^0 flow on some Hausdorff manifold? Some restriction on the nonseparated manifold is undoubtedly necessary. However, it appears to be a difficult problem to characterize the realizable ones. We present a preliminary result in this direction in the present paper. We first define a restricted class of nicely ordered nonseparated manifolds. We then prove that these manifolds are all realizable.

REALIZATION THEOREM. If X is nicely ordered, nonseparated n-manifold then X can be realized as the ordered orbit space of a completely unstable continuous flow (E,ϕ), where E is a Hausdorff $(n + 1)$-manifold.

Essentially the same result, in the case X is a one-dimensional simply connected variety and $E = \mathbb{R}^2$ is stated in Haefliger and Reeb [4]. It is also stated in Neumann [3] for one-dimensional manifold X.

In §2 below, we give most of the definitions and notation required in the proof of the realization theorem; the proof itself occupies §3 - §5. Finally, in §6 we prove the following corollary.

COROLLARY. Let X and E be as in the realization theorem. If $\pi_n(X) = 0$, then $\pi_n(E) = 0$ for $n \geq 1$. Moreover, if X is a one-dimensional simply connected nonseparated manifold, then E is homeomorphic to $\mathbb{R}^2$.

2. PRELIMINARIES.

DEFINITIONS AND NOTATION. Throughout what follows, E denotes a Hausdorff $(n + 1)$-manifold, $\phi : E \times \mathbb{R}^1 \to E$ denotes a continuous flow on E, and X denotes a nonseparated n-manifold with a countable basis (V_i, Ψ_i) where each V_i is homeomorphic to D^n, the compact unit n-disk. A topological space is a nonseparated manifold if it is locally Euclidean and has a countable basis, the usual Hausdorff separation axiom is not assumed. For any set $S \subseteq E$, $T \subseteq \mathbb{R}^1$, $S \cdot T = \{\phi(x,t) | x \varepsilon S, t \varepsilon T\}$; $x \cdot T = \{x\} \cdot T$; and for $x \varepsilon E$ and $t \varepsilon IR^1$, $x \cdot t = \phi_t(x) = \phi(x, t)$. The orbit of $x \varepsilon E$ is the set $\gamma(x) = x \cdot \mathbb{R}^1$. The orbit space E/ϕ is the set of all orbits of ϕ with the quotient topology. Also, throughout what follows, for any set A contained in a topological space, $\mathring{A}$ and $\bar{A}$ will denote the interior and the closure of A respectively.

A set $U \subseteq E$ is said to be wandering (with respect to ϕ) if there exists $t_0 \varepsilon \mathbb{R}^1$ such that $U \cdot t \cap U = \emptyset$ for each t with $|t| \geq t_0$. A point $x \varepsilon E$ is nonwandering if it has no wandering neighborhood. Equivalently, $x \varepsilon E$ is nonwandering if $x \varepsilon J^+(x)$, here $J^+(x)$ denotes the set of limits of sequences $\{x_n \cdot t_n\}$, where $\{x_n\}$ converges to x and t_n tends to ∞. The (closed ϕ - invariant) set of all nonwandering points of ϕ is denoted as $\Omega(\phi)$. A flow ϕ is said to be completely unstable if $\Omega(\phi) = \emptyset$. A cross-section of ϕ is a set $S \subseteq E$ for which the mapping $h: S \times \mathbb{R}^1 \to E$ defined by $h(s,t) = s \cdot t$ is a homeomorphism of $S \times \mathbb{R}^1$ onto a subset of E.

3. STATEMENT OF THE REALIZATION THEOREM.

In order to state our main result, we first need to define the following order structure.

DEFINITION 3.1. Let X be a nonseparated n-manifold with a countable atlas $< V_i, \Psi_i >$ where each V_i is homeomorphic to D^n (the compact unit n-disk), and $\{\overset{o}{V}_i\}_{i\geq 1}$ forms an open cover for X. We say that X is nicely ordered if there exists a collection of continuous functions $h_{ij}: V_i \cap V_j \to \{-1,1\}$ satisfying:

(a) $h_{ij}(x) = -h_{ji}(x)$ for every $x \in V_i \cap V_j$;

(b) If $x \in V_i \cap V_j \cap V_k$ with $h_{ij}(x) = +1(-1)$ and $h_{jk}(x) = +1(-1)$, then $h_{ik}(x) = +1(-1)$;

(c) If $\{x^n\}$ is a sequence in $V_i \cap V_j \cap V_k$ $(i<j<k)$ converging to $x \in V_i$, and $x \notin V_j$, then $x \notin V_k$.

The order structure defined as above is a generalization of the order structure on a nonseparated 1-manifold, as given by Neumann in [5]. However, the property (c) of the order structure as above is slightly more restrictive than the property (3) of the order structure given by Neumann (see §3.5 below), and thus the phrase "nicely ordered" is used.

Our main result is the following Realization Theorem.

REALIZATION THEOREM. 3.2. If X is a nicely ordered, nonseparated n-manifold, then X can be realized as the ordered orbit space of a completely unstable continuous flow (E,ϕ), where E is a Hausdorff $(n + 1)$ - manifold.

REMARKS 3.3. (α) This result in the case X is a one-dimensional simply connected variety and $E = \mathbb{R}^2$ is stated in Haefliger and Reeb [4]. It is also stated in Neumann [3] for one-dimensional manifold X.

(β) Properties (a) and (b) of the order structure defined in 3.1 above will be used implicitly throughout the proof of the realization theorem.

OUTLINE 3.4. We shall prove the realization theorem by induction on the number of charts in X in the following two steps.

(1) We first show that X can be realized as a base space of a bundle $\mathcal{B} = < E,p,X >$, where E is a Hausdorff $(n + 1)$ - manifold.

(2) We then define a flow ϕ on E, show that it is completely unstable and finally show that X is the orbit space of the dynamical system (E,ϕ).

The first step, that is to show the existence of the bundle $\mathcal{B} = < E,p,X >$, is the major step in the proof of the realization theorem.

DISCUSSION 3.5. We would like to point out that the direct generalization of the order structure given by Neumann for nonseparated 1-manifold in [5] would be: (a) and (b) same as in the definition 3.1 above and replace (c) by a less restrictive condition (c') as follows:

(c') If $\{x^n\}$ is a sequence in $V_i \cap V_j \cap V_k$ such that

(α) $h_{ij}(x^n) = 1$ and $h_{jk}(x^n) = 1$ for each n, and

(β) $x^n \to x \varepsilon V_i$ with $x \notin V_j$, then $x \notin V_k$.

Moreover, if ϕ is a completely unstable c^0 flow on the (n + 1) - manifold E and admits cross-sections that are locally Euclidean, then E/ϕ can be ordered in this sense: choose a covering system $\{S_i\}_{i\geq 1}$ of cross-sections for the dynamical system (E,ϕ) (see 4.2, 4.3 of [3]). Set $V_i = p(S_i)$ for each i. Then $\{\overset{o}{V}_i\}_{i\geq 1}$ forms an open cover of E/ϕ. Let $f_{ij}: V_i \cap V_j \to \mathbb{R}^1$ be defined as in the proof of the classification theorem (Theorem 3.1, [3]). Set $h_{ij}(x) = \mathrm{sgn}(f_{ij}(x))$, $x \varepsilon V_i \cap V_j$. Using the properties of f_{ij} (see [3]), it is now immediate that h_{ij} satisfy the properties (a), (b) and (c') above.

4. EXISTENCE OF A BUNDLE $\mathcal{B}$ = < E, p, X >.

In the setting of the existence theorem (Theorem 3.2 of [6]), to show the existence of a bundle $\mathcal{B}$=<E, p, X >, we seek the coordinate transformations $\{g_{ij}\}$ in the space X, with the structure group the group τ of all translations of $\mathbb{R}^1$. In particular, we seek the maps:

$$g_{ij}: V_i \cap V_j \to \tau \text{ satisfying:}$$

(a) $g_{ij}(x) \circ g_{jk}(x) = g_{ik}(x)$ for each $x \varepsilon V_i \cap V_j \cap V_k$ (compatibility condition);

(b) If $\{x^n\}$ is a sequence in $V_i \cap V_j$ such that $\{x^n\}$ converges to both $x_i \varepsilon V_i$ and $x_j \varepsilon V_j$ with $x_i \neq x_j$, and $h_{ij}(x^n) = +1(-1)$ for all n, then
$g_{ij}(x^n)(t) \to +\infty\ (-\infty)$ as $n \to \infty$ (for every $t \varepsilon \mathbb{R}^1$).

We define g_{ij} in terms of the translations f_{ij} as follows:

$$\begin{aligned} g_{ij}: V_i \cap V_j &\to \tau \\ x &\to g_{ij}(x): \mathbb{R}^1 \to \mathbb{R}^1 \text{ such that} \\ g_{ij}(x)(t) &= t + f_{ij}(x);\ x \varepsilon V_i \cap V_j \text{ and } t \varepsilon \mathbb{R}^1. \quad (*) \end{aligned}$$

Where

$$f_{ij}: V_i \cap V_j \to \mathrm{IR}^1$$

are to be defined so as to satisfy:

(A) $f_{ij}(x) + f_{jk}(x) = f_{ik}(x)$ for each $x \varepsilon V_i \cap V_j \cap V_k$; and

(B) If $\{x^n\}$ is a sequence in X such that $\{x^n\}$ converges to both

$x_i \in \overset{\circ}{V}_i$ and $x_j \in \overset{\circ}{V}_j$ with $x_i \neq x_j$, and $h_{ij}(x^n) = +1(-1)$ for all n, then $f_{ij}(x^n) \to +\infty(-\infty)$ as $n \to \infty$.

If we assume that f_{ij} satisfying (A) and (B) exist, then g_{ij} defined by (*) trivially satisfy (b). For (a), fix $x \in V_i \cap V_j \cap V_k$ and $t \in \mathbb{R}^1$.
Then using (A) for f_{ij}, we have

$g_{ij}(x) \circ g_{jk}(x)(t) = g_{ij}(x)(t + f_{jk}(x)) = t + f_{jk}(x) + f_{ij}(x) = t + f_{ik}(x) = g_{ik}(x)(t)$ as desired.

Thus, to show the existence of the coordinate transformations $\{g_{ij}\}$, we need to show the existence of the translations $\{f_{ij}\}$ satisfying (A) and (B) above. We show the existence of $\{f_{ij}\}$ by induction on the number of charts in X. Note that since each chart is Hausdorff and X is not, X can not have a single chart.

REMARK 4.1. One should note that the existence theorem (Theorem 3.2 of [6]) not only gives the existence of a bundle $B = \langle E, p, X \rangle$ but also its uniqueness up to bundle equivalence.

NOTATION 4.2. In What follows, B_1 denotes the set of all non-Hausdorff points of X and $V_{ij} (i \neq j)$ denotes the set of all those points $x_i \in B_1 \cap V_i$ such that there exists a sequence $\{x^n\}$ in $V_i \cap V_j$ with $\{x^n\}$ converging to x_i and also to another point $x_j \in V_j$ with $x_i \neq x_j$. Note that V_{ij} is the set of all those non-Hausdorff points in V_i that can not be separated from some point in V_j.

PROPOSITION 4.3 For any $i \neq j$, the set V_{ij} is a closed subset of the metric space V_i.

PROOF: Let $\{y_k\}$ be a sequence in V_{ij} such that $\{y_k\}$ converges to $y \in V_i$. We want to show that $y \in V_{ij}$. Without loss of generality, let $y_k \in B_{\frac{1}{2^k}}(y)$ for each k, where $B_{\frac{1}{2^k}}(y)$ is an open ball in the metric space V_i Moreover, for each k, let $\{x_k^n\}$ be a sequence in $V_i \cap V_j$ such that $\{x_k^n\}$ converges to y_k and also to another point $y_k' \in V_j$, with $y_k \neq y_k'$. Since V_j is compact, the sequence $\{y_k'\}$ has a convergent subsequence $\{y_{k_\ell}'\} \to y' \in V_j$. As above, let $y_{k_\ell}' \in B'_{\frac{1}{2^\ell}}(y')$ for each ℓ, where $B'_{\frac{1}{2^\ell}}(y')$ is an open ball in V_j. By induction, there exists $N_m > N_{m-1}$ in z^+ such that $x_{k_m}^{N_m} \in B_{\frac{1}{2^m}}(y) \cap B'_{\frac{1}{2^m}}(y')$. Now the sequence $\{x_{k_m}^{N_m}\}$ is in $V_i \cap V_j$ and obviously converges to both y and y'. Moreover, it can be easily seen that $y \neq y'$. Hence $y \in V_{ij}$ as desired.

EXISTENCE OF f_{ij} FOR TWO CHARTS 4.4. If X has only two charts, say V_1 and V_2, then $\{f_{ij}\}$ $(1 \le i, j \le 2)$ satisfying (A) and (B) above exist for these two charts.

Let $X' = (V_1 \cap V_2) \cup V_{12}$ (disjoint union). Note that X' is a metric subspace of the metric space V_1. Define $f': X' \to [0,1]$ by

$$f'(x) = \frac{1}{1 + d(x, V_{12})} \; ; \; x \in X'.$$

Then f' is a continuous function, and since V_{12} is a closed set (§4.3), $f'(x) = 1$ if and only if $x \in V_{12}$. Define $f'_{12}: X' \to [0,\infty]$ by $f'_{12}(x) = \tan \frac{\pi}{2} (f'(x))$. Then f'_{12} is continuous and $f'_{12}(x) = \infty$ if and only if $x \in V_{12}$. Now set $f_{12} = f'_{12} \mid V_1 \cap V_2$, where $f'_{12}|V_1 \cap V_2$ indicates the restriction of the function f'_{12} on the set $V_1 \cap V_2$.

Finally, the set f_{12}, $f_{21} = -f_{12}$, and $f_{ii} = 0 (i = 1,2)$ is the desired set of f_{ij} $(1 \le i, j \le 2)$, satisfying (A) and (B) above. This completes the construction of f_{ij} in the case X has only two charts.

REMARK 4.5. In the construction of f_{12} above, observe that $f_{12}(x) > 0$ for every $x \in V_1 \cap V_2$. In the rest of the proof, we would construct f_{ij} so as to satisfy (A) and (B) above and also the following added property:

(C) If $j > i$, then $f_{ij}(x) > 0$ for every $x \in V_i \cap V_j$.

INDUCTION STEP 4.6. Suppose that we can define $\{f_{ij}\}$ $(1 \le i, j \le n)$ satisfying (A), (B), and (C)above in the case X has n-charts say $V_1, V_2, V_3, \ldots, V_n$, we show that $\{f_{ij}\}$ satisfying (A), (B), and (C) above can be defined in the case X has $(n + 1)$ - charts $V_1, V_2, \ldots, V_n, V_{n+1}$.

In order to show the existence of $\{f_{ij}\}$ in the case X has $(n + 1)$ - charts, we first need to show the existence of $\{f_{ij}\}$ in the case X has only three charts, which in turn requires the following lemma:

LEMMA 4.7. Let A and B be closed subsets of a metric space Y. If $g: A \to [0,1]$ is a continuous map such that $g(x) = 1$ if and only if $x \in A \cap B$, then g can be extended to a continuous map $\bar{g}: Y \to [0,1]$ such that $\bar{g}(x) = 1$ if and only if $x \in B$.

PROOF: Define $g_{A \cup B}: A \cup B \to [0,1]$ by

$$g_{A \cup B}(x) = \begin{cases} g(x) & \text{if } x \in A; \\ 1 & \text{if } x \in B. \end{cases}$$

It is obvious that $g_{A \cup B}$ is a well-defined map that extends g. Also, it is continuous by glueing lemma ([7], page 50). Moreover, $g_{A \cup B}(x) = 1$ if and only if $x \in B$.

In order to extend $g_{A \cup B}$ to the whole of Y, we observe that $A \cup B$ is a closed subspace of the metric space Y. Therefore, there exists a continuous function $u: Y \to [0,1]$ such that $u(x) = 1$ if and only if $x \in A \cup B$. Moreover, by Tietze Extension Theorem, there exists a continuous extension $g': Y \to [0,1]$ of $g_{A \cup B}$ such that $g'(x) = 1$ if $x \in B$.

Finally, define $\bar{g}$: $Y \to [0,1]$ by $\bar{g}(x) = u(x) \cdot g'(x)$ for $x \in Y$. It can be easily seen that $\bar{g}$ is the desired map. This completes the proof of the lemma.

EXISTENCE OF f_{ij} FOR THREE CHARTS 4.8. If X has only three charts, say V_1, V_2, and V_3, then $\{f_{ij}\}$ $(1 \le i, j \le 3)$ satisfying (A), (B) and (C) above can be defined for these three charts.

Let f'_{12}: $(V_1 \cap V_2) \cup V_{12} \to [0, \infty]$ be the function as obtained in the case of two charts (cf. §4.4). Define f'_{23}: $(V_2 \cap V_3) \cup V_{23} \to [0, \infty]$ analogous to f'_{12}. Here, V_{23} is a set as defined in §4.2.

In order to define f'_{13}: $(V_1 \cap V_3) \cup V_{13} \to [0, \infty]$, let $A_{123} = V_1 \cap V_2 \cap V_3$ and define V_{123} to be the set of all those points $x_1 \in B_1 \cap V_1$, such that there exists a sequence $\{x^n\}$ in A_{123} with $\{x^n\}$ converging to x_1 and also to another point x_3 in V_3 with $x_1 \ne x_3$, (B_1 is the set as defined in §4.2 above). Observe that $V_{123} \subseteq V_{13}$. We claim:

THEOREM 4.9. If $x_1 \in V_{123}$ then either $x_1 \in V_{12}$ or $x_1 \in V_{23}$.

PROOF: If $x_1 \in V_{123}$ then there exists a sequence $\{x^n\}$ in A_{123} such that $\{x^n\}$ converges to x_1 and also to another point x_3 in V_3 with $x_1 \ne x_3$. Since V_2 is compact, $\{x^n\}$ has a convergent subsequence $\{x^k\} \to x_2 \in V_2$. If $x_2 = x_1 \ne x_3$, then $x_1 \in V_{23}$, otherwise $x_1 \in V_{12}$.

We now define f'_{13}: $A_{123} \cup V_{123} \to [0, \infty]$ by

$$f'_{13}(x) = f'_{12}(x) + f'_{23}(x); \; x \in A_{123} \cup V_{123}. \tag{4.2}$$

Then f'_{13} is a well-defined continuous map. Since both f'_{12} and f'_{23} are finite on A_{123}, it follows from 4.9 above that $f'_{13}(x) = \infty$ if and only if $x \in V_{123}$. We want to extend f'_{13} continuously to f'_{13}: $(V_1 \cap V_3) \cup V_{13} \to [0, \infty]$ such that $f'_{13}(x) = \infty$ if and only if $x \in V_{13}$. (Note that the extension of f'_{13} is also denoted as f'_{13}).

In the setting of the lemma 4.7 above, we have $Y = (V_1 \cap V_3) \cup V_{13}$, $A = A_{123} \cup V_{123}$ and $B = V_{13}$. Assuming A to be a closed subset (proved below) of Y, define g: $A \to [0,1]$ by $g(x) = \frac{2}{\pi}$ arc tan $(f'_{13}(x))$ where f'_{13} is defined by (4.2) above. Let $\bar{g}$: $Y \to [0,1]$ be an extension of g as obtained in the lemma 4.7. Define f'_{13}: $Y \to [0, \infty]$ by $f'_{13} = \tan \frac{\pi}{2} (\bar{g}(x))$. Note that $f'_{13}(x) = \infty$ if and only if $x \in B = V_{13}$. To complete the definition of f'_{13}, we still need to show:

THEOREM 4.10 The set $A = A_{123} \cup V_{123}$ is a closed subset of $Y = (V_1 \cap V_3) \cup V_{13}$.

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PROOF: As in proposition 4.3, it can be seen that V_{123} is a closed subset of Y. Thus, to complete the proof, it suffices to show that the closure of A_{123} in Y is contained in A.

Let $\{x^n\}$ be a sequence in A_{123} such that $\{x^n\} \to x_1 \in Y$. Since $Y = (V_1 \cap V_3) \cup V_{13}$ (disjoint union), either $x_1 \in V_1 \cap V_3$ or $x_1 \in V_{13}$. Let us first consider the case $x_1 \in V_1 \cap V_3$. Since V_2 is compact, $\{x^n\}$ has a convergent subsequence $\{x^k\} \to x_2 \in V_2$. We claim that $x_2 \in V_1$ and thus $x_2 = x_1$. If not, then $x_1 \notin$

V_2, Thus by property (c) of the definition of order structure, we have $x_1 \notin V_3$, a contradiction. Hence, in this case, $x_1 \notin A_{123} \subseteq A$.

If $x_1 \in V_{13}$, then $x_1 \notin V_3$. Also, since V_3 is compact, therefore the sequence $\{x^n\}$ admits a convergent subsequence $\{x^k\} \to x_3 \in V_3$. Hence $x_1 \in V_{123} \subseteq A$, as desired.

Finally, $f_{13} = f'_{13}|V_1 \cap V_3$, $f_{12} = f'_{12}|V_1 \cap V_2$, $f_{23} = f'_{23}|V_2 \; V_3$, $f_{ji} = -f_{ij}(1 \le i,\ j \le 3)$ and $f_{ii} = 0(i = 1,2,3)$ is the desired set of $\{f_{ij}\}$ satisfying (A), (B) and (C) above. Here, $f'_{ij}|V_i \cap V_j$ denotes the restriction of f'_{ij} on $V_i \cap V_j(1 \le i,\ j \le 3)$. This completes the construction of f_{ij} in the case X has three charts.

We now return to our induction step. We want to define $\{f_{ij}\}$ $(1 \le i,\ j \le n+1)$ satisfying (A), (B) and (C) in the case X has (n + 1) charts $V_1, V_2, \dots V_{n+1}$, knowing that $\{f_{ij}\}$ $(1 \le i,\ j \le n)$ satisfying (A), (B) and (C) have already been defined in the case X has n charts $V_1, V_2, \dots, V_n$. For convenience sake, we will use the following notation in the rest of the proof.

NOTATION 4.11. Any extension of f'_{ij} will be denoted as f'_{ij}. For any i, j and k, A_{ijk} denotes the set $V_i \cap V_j \cap V_k$, and V_{ijk} denotes the set of all points x_i in $B_1 \cap V_i$ (B_1 is the set of all non-Hausdorff points in X) such that there exists a sequence $\{x^n\}$ in A_{ijk} with $\{x^n\}$ converging to x_i and also to another point x_k in V_k with $x_i \neq x_k$. Moreover, for any $i \neq j$, A_{ij} denotes the set $V_i \cap V_j$.

REMARK 4.12. For any i<j<k, the set $A_{ijk} \cup V_{ijk}$ is a closed set in $A_{ik} \cup V_{ik}$ (cf. §4.10) and it would be denoted as B_{ijk}. This remark would be used implicitly throughout what follows.

We now start defining f_{ij} for (n + 1) charts. Define $f'_{n\ n+1}: A_{n\ n+1} \cup V_{n\ n+1} \to [0,\infty]$ as in the case of two charts (cf. §4.4). Next, define $f'_{n-1\ n+1}: B_{n-1\ n\ n+1} \to [0, \infty]$ by

$$f'_{n-1\ n+1}(x) = f'_{n-1\ n}(x) + f'_{n\ n+1}(x);\ x \in B_{n-1\ n\ n+1} \tag{4.3}$$

as in § 4.9 for three charts. Here, $f'_{n-1\ n}$ has been defined at the induction step. Using lemma 4.7, extend $f'_{n-1\ n+1}$ to $f'_{n-1\ n+1}: A_{n-1\ n+1} \cup V_{n-1\ n+1} \to [0,\infty]$ as was done in the case of three charts. We next define the function $f'_{n-2\ n+1}$ as follows. Define

$$f'_{n-2\ n+1}(x) = f'_{n-2\ n}(x) + f'_{n\ n+1}(x); \quad x \in B_{n-2\ n\ n+1}, \text{ and} \tag{4.4}$$

$$f'_{n-2\ n+1}(x) = f'_{n-2\ n-1}(x) + f'_{n-1\ n+1}(x); \quad x \in B_{n-2\ n-1\ n+1}, \tag{4.5}$$

where $f'_{n-2\ n}$ and $f'_{n-2\ n-1}$ have been defined at the induction step and $f'_{n-1\ n+1}$ is obtained above. Using the induction hypothesis and (4.3) above, it can be easily seen that $f'_{n-2\ n+1}$ is well defined, that is, $f'_{n-2\ n+1}$ defined by (4.4) coincides with $f'_{n-2\ n+1}$ defined by (4.5) on the intersection $(B_{n-2\ n\ n+1} \cap B_{n-2\ n-1\ n+1})$. Finally, using lemma 4.7 with $Y = A_{n-2\ n+1} \cup V_{n-2\ n+1}$, $A = B_{n-2\ n\ n+1} \cup B_{n-2\ n-1\ n+1}$, and $B = V_{n-2\ n+1}$, extend $f'_{n-2\ n+1}$ to $f'_{n-2\ n+1}: A_{n-2\ n+1} \cup V_{n-2\ n+1} \to [0,\infty]$ as was done in the case of three charts.

Continuing this process, we obtain $f'_{n-3\ n+1}$, $f'_{n-4\ n+1}$, ... , and $f'_{2\ n+1}$ inductively. Finally, define $f'_{1\ n+1}$ as follows:

$$f'_{1\ n+1} = f'_{1\ n} + f'_{n\ n+1} \qquad \text{on } B_{1\ n\ n+1},$$

$$f'_{1\ n+1} = f'_{1\ n-1} + f'_{n-1\ n+1} \qquad \text{on } B_{1\ n-1\ n+1},$$

$$\vdots$$

$$f'_{1\ n+1} = f'_{13} + f'_{3\ n+1} \qquad \text{on } B_{13\ n+1}, \text{ and}$$
$$f'_{1\ n+1} = f'_{12} + f'_{2\ n+1} \qquad \text{on } B_{12\ n+1},$$

where $f'_{i\ n+1}$ $(i = 2,3, \dots ,n)$ are the functions obtained above and f'_{1j} $(j = 2,3, \dots,n)$ are the functions that have been defined at the induction step. Using the induction hypothesis and the definition of the functions $f'_{i\ n+1}$ $(2 \le i \le n)$, it can be seen that $f'_{1\ n+1}$ is well defined. Finally, using lemma 4.7 with $Y = A_{1\ n+1} \cup V_{1\ n+1}$, $A = \bigcup_{i=2}^{n}(B_{1i\ n+1})$ and $B = V_{1\ n+1}$, extend $f'_{1\ n+1}$ to $f'_{1\ n+1} : A_{1\ n+1} \cup V_{1\ n+1} \to [0,\infty]$ as was done in the case of three charts.

We now let $f_{ij} = f'_{ij}|V_i \cap V_j$, $f_{ji} = -f_{ij}$ and $f_{ii} = 0$ for i, j=1,2, ... , n+1. We claim that the set $\{f_{ij}\}(1 \le i, j \le n + 1)$ so obtained is the desired set of functions satisfying (A), (B) and (C). From the construction of $\{f_{ij}\}$ it is obvious that the functions $\{f_{ij}\}$ satisfy both (B) and (C). For (A), we need to show that $f_{ij}(x) + f_{jk}(x) = f_{ik}(x)$ for each $x \in V_i \cap V_j \cap V_k$ and for any i, j and k where $1 \le i, j, k \le n + 1$. In view of induction hypothesis, we only need to prove it in the case when one of the i, j or k is n + 1.

If $i = n + 1$, we need to show $f_{n+1\ j}(x) + f_{jk}(x) = f_{n+1\ k}(x)$. If $k > j$, then $f_{n+1\ j}(x) + f_{jk}(x) = -f_{j\ n+1}(x) + f_{jk}(x) = -(f_{jk}(x) + f_{k\ n+1}(x)) + f_{jk}(x) = -f_{jk}(x) - f_{k\ n+1}(x) + f_{jk}(x) = f_{n+1\ k}(x)$ as desired. If $j > k$, then $f_{n+1\ j}(x) + f_{jk}(x) = -f_{j\ n+1}(x) - f_{kj}(x) = -(f_{kj}(x) + f_{j\ n+1}(x)) = -f_{k\ n+1}(x) = f_{n+1\ k}(x)$ as desired. The cases when j or k equals (n + 1) are analogous.

This completes the induction step and hence the construction of $\{f_{ij}\}$ for i, $j \ge 1$.

Hence, by the existence theorem (Theorem 3.2, [6]), we get a bundle $\mathcal{B} = \langle E,p,X \rangle$ with the base space X and the coordinate transformations $\{g_{ij}\}$. Also, any two such bundles are equivalent. Moreover, since X is an n-manifold, E is an (n + 1) - manifold. We finally show that

THEOREM 4.13 E is a Hausdorff space

PROOF: If not, let e and e' be two nonseparated points in E. We have $\{\overset{o}{V}_k\}_{k \ge 1}$ covers X, and since for each k there exists a homeomorphism $\Psi_k: V_k \times \mathbb{R}^1 \to p^{-1}(V_k)$, ([6], page 7), each $p^{-1}(V_k)$ is Hausdorff. Consequently, there exists $j > i$, such that $e \in p^{-1}(V_i)$ and $e' \in p^{-1}(V_j)$ with $p^{-1}(V_i) \cap p^{-1}(V_j) \neq \emptyset$. Moreover, there exist $x \in \mathring{V}_1$, $x' \in \mathring{V}_j$ and $t, t' \in \mathbb{R}^1$, such that $\Psi_i(x, t) = e$ and $\Psi_j(x',t') = e'$. Let $\{V_i^n\}_{n\geq 1}$ and $\{V_j^n\}_{n\geq 1}$ be neighborhood systems at e and e', respectively, with $V_i^n \subsetneq p^{-1}(\mathring{V}_i)$ and $V_j^n \subseteq p^{-1}(\mathring{V}_j)$ for all n. Since e and e' are nonseparated points, therefore, for each n, there exists $y_n \in V_i^n \cap V_j^n$. Let $x_n = p(y_n) \in V_i \cap V_j$ for each n. Then there exists $t_n \in \mathbb{R}^1$ such that $y_n = \Psi_i(x_n,t_n) = \Psi_j(x_n,g_{ji}(x_n)(t_n))$ for each n, where g_{ij} are the coordinate transformations as constructed above.

Since y_n converges to both $e = \Psi_i(x, t)$ and $e' = \Psi_j(x',t')$ and both Ψ_i and Ψ_j are homeomorphisms, it can be easily seen that x_n converges to both x and x'; $t_n \to t$; and $g_{ji}(x_n)(t_n) \to t'$ as $n \to \infty$. Since $t_n \to t$, therefore there exists $t_o \in \mathbb{R}^1$ such that $t_n \leq t_o$ for all n. Consequently

$$g_{ji}(x_n)(t_n) \leq g_{ji}(x_n)(t_o) \text{ for all } n.$$

But from our construction of g_{ij}, we have that for any $t \in \mathbb{R}^1$, $g_{ji}(x_n)(t) \to -\infty$ as $n \to \infty$ (because $j > i$). Thus, $g_{ji}(x_n)(t_o) \to -\infty$ and consequently $g_{ji}(x_n)(t_n) \to -\infty$ as $n \to \infty$. Also, $g_{ji}(x_n)(t_n) \to t'$ (finite); a contradiction. Hence, E is Hausdorff. This completes the proof of Step 1.

5. X AS AN ORDERED ORBIT SPACE.

We now show that we can define a completely unstable continuous flow ϕ on E and that X can be realized as an ordered orbit space of the dynamical system (E, ϕ), where E is the Hausdorff manifold obtained in § 4 above.

To define a flow $\phi: E \times \mathbb{R}^1 \to E$. Fix $(q, s) \in E \times \mathbb{R}^1$. Since $\Psi_j:V_j \times \mathbb{R}^1 \to p^{-1}(V_j)$ is a homeomorphism for each j and $\{\mathring{V}_j\}_{j\geq 1}$ cover X; therefore, there exists some $k \geq 1$ with $x \in \mathring{V}_k$ and $t \in \mathbb{R}^1$ such that $q = \Psi_k(x, t)$. Define

$$\phi(q, s) = \Psi_k(x, t + s). \tag{5.1}$$

We first show that ϕ is well defined; that is, if q also equals $\Psi_j(x',t')$ for some $j \neq k$, $x' \in V_j$ and $t' \in \mathbb{R}^1$, then $\Psi_k(x, t + s) = \Psi_j(x', t' + s)$.

Since $\Psi_k(x, t) = q = \Psi_j(x',t') = \Psi_k(x', g_{kj}(x')(t'))$ and Ψ_k is a homeomorphism; therefore, $x = x'$ and $g_{kj}(x')(t') = t$, and hence

$$\Psi_j(x',t' + s) = \Psi_k(x',g_{kj}(x')(t' + s)) = \Psi_k(x',g_{kj}(x')(t') + s) = \Psi_k(x,t + s)$$

as desired.

We next show that ϕ is a continuous flow. It is obvious that ϕ is a continuous function. Moreover, ϕ satisfies the group law for $\phi(q,0) = \Psi_k(x,t + 0) = q$; and $\phi(\phi(q,s_1),s_2) = \phi(\Psi_k(x,t + s_1), s_2) = \Psi_k(x, t + s_1 + s_2) = \phi(x, s_1 + s_2)$.

We finally show that ϕ is completely unstable and that X is the orbit space of the dynamical system (E, ϕ). To show that ϕ is completely unstable, fix $q \in E$. We want to show that q admits a wandering neighborhood. Let $q = \Psi_k(x, t_o)$ for some x

εV_k and $t_0 \in \mathbb{R}^1$. Fix $\varepsilon > 0$ and let $W_j = \Psi_k(\overset{\circ}{V}_k, (t_0 - \varepsilon, t_0 + \varepsilon))$. Then W_j is the required wandering neighborhood of q as $(W_j \cdot t) \cap W_j = \emptyset$ for all t such that $|t| > 2\varepsilon$. In order to show that X is an orbit space of (E, ϕ), fix $q \in E$. If $q = \Psi_k(x, t)$ for some $x \in V_k$ and $t \in \mathbb{R}^1$, then for any $s \in \mathbb{R}^1$, we have $\phi(q,s) = \Psi_k(x, t + s)$. Thus, $p(\phi(q, s)) = p(\Psi_k(x, t + s)) = (p \circ \Psi_k)(x, t + s) = x$. Moreover, from the construction of the bundle $B = < E, p, X >$ in the existence theorem (Theorem 3.2, [6]), it can be seen that the topology of X as a base space is equivalent to the quotient topology. Thus, X is the desired orbit space of the dynamical system (E, ϕ).

As we have mentioned in the introduction, an ordered orbit space of ϕ can now be obtained from the orbit space X by imposing an additional structure that indicates the order in which the cross-sections of ϕ that correspond to the charts of X, are traversed by orbits of ϕ (precise definition of the order structure is given in [3]).

Hence, X can be realized as an ordered orbit space of a completely unstable c^0 flow on the Hausdorff (n + 1) -manifold E. This completes the proof of the realization theorem.

6. COROLLARY.

Let X and E be as in the realization theorem. If $\pi_n(X) = 0$, then $\pi_n(E) = 0$ for $n \geq 1$. Moreover, if X is one-dimensional simply connected nonseparated manifold, then E is homeomorphic to $\mathbb{R}^2$.

PROOF. We consider the exact homotopy sequence

$$\cdots \to \pi_n(\mathbb{R}^1) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(X) \xrightarrow{\Delta} \pi_{n-1}(\mathbb{R}^1) \to \cdots \to \pi_2(X) \xrightarrow{\Delta} \pi_1(\mathbb{R}^1) \xrightarrow{i_*} \pi_1(E) \xrightarrow{p_*}$$

$\pi_1(X) \to \cdots$ of the bundle $B = < E, p, X >$. Since for each $n \geq 1$, $\pi_n(X) = 0$ and $\pi_n(\mathbb{R}^1) = 0$, it follows that $\pi_n(E) = 0$ for each $n \geq 1$. In particular, E is simply connected.

Now from the isomorphism theorem of Hurewicz ([6], page 91), we know that the first non-zero homology group and the first non-zero homotopy group have the same dimension and are isomorphic. Thus, we conclude that $H_n(E) = 0$ for each $n \geq 1$, that is, E is acyclic.

If X is a one-dimensional simply connected nonseparated manifold, then from the proof of the Realization Theorem, E is a two-dimensional Hausdorff manifold. Moreover, from above E is simply connected. Therefore, E is homeomorphic to S^2 or $\mathbb{R}^2$. But S^2 is not acyclic and hence E is homeomorphic to $\mathbb{R}^2$.

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