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GENERALIZATION OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

TADAYUKI SEKINE

Department of Mathemetics
Science and Technology
Nihon University
1-8 Kanda Surugadai, Chiyoda-ku
Tokyo 101, Japan

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ABSTRACT. We introduce the subclass $\mathbf{T}_j(n,m,\alpha)$ of analytic functions with negative coefficients by the operator D^n. Coefficient inequalities and distortion theorems of functions in $\mathbf{T}_j(n,m,\alpha)$ are determind. Further, distortion theorems for fractional calculus of functions in $\mathbf{T}_j(n,m,\alpha)$ are obtained.

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1. INTRODUCTION.

Let $\mathbf{A}_j$ denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbf{N} = \{1,2,3,\dots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathbf{U} = \{z:|z| < 1\}$.
For a function f(z) in $\mathbf{A}_j$, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbf{N}). \tag{1.4}$$

With the above operator D^n, we say that a function f(z) belonging to $\mathbf{A}_j$ is in the class $\mathbf{A}_j(n,m,\alpha)$ if and only if

$$\mathrm{Re}\left(\frac{D^{n+m} f(z)}{D^n f(z)}\right) > \alpha \quad (n,m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}) \tag{1.5}$$

for some α $(0 \leqq \alpha < 1)$, and for all $z \in \mathbf{U}$.

We note that $A_1(0,1,\alpha) = S^*(\alpha)$ is the class of starlike functions of order α, $A_1(1,1,\alpha) = K(\alpha)$ is the class of convex functions of order α, and that $A_1(n,1,\alpha) = S_n(\alpha)$ is the class of functions defined by Salagean [1].

Let T_j denote the subclass of A_j consisting of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geqq 0;\ j \in N). \tag{1.6}$$

Further, we define the class $T_j(n,m,\alpha)$ by

$$T_j(n,m,\alpha) = A_j(n,m,\alpha) \cap T_j. \tag{1.7}$$

Then we observe that $T_1(0,1,\alpha) = T^*(\alpha)$ is the subclass of starlike functions of order α (Silverman [2]), $T_1(1,1,\alpha) = C(\alpha)$ is the subclass of convex functions of order α (Silverman [2]), and that $T_j(0,1,\alpha)$ and $T_j(1,1,\alpha)$ are the classes defined by Chatterjea [3].

2. DISTORTION THEOREMS.

We begin with the statement and the proof of the following result.

LEMMA 1. Let the function $f(z)$ be defined by (1.6) with $j = 1$. Then $f(z) \in T_1(n,m,\alpha)$ if and only if

$$\sum_{k=2}^{\infty} k^n(k^m - \alpha)a_k \leqq 1 - \alpha \tag{2.1}$$

for $n \in N_0$, $m \in N_0$, and $0 \leqq \alpha < 1$. The result is sharp.

PROOF. Assume that the inequality (2.1) holds and let $|z| = 1$. Then we have

$$\left|\frac{D^{n+m}f(z)}{D^n f(z)} - 1\right| \leqq \frac{\sum_{k=2}^{\infty} k^n(k^m - 1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} k^n a_k|z|^{k-1}}$$

$$= \frac{\sum_{k=2}^{\infty} k^n(k^m - 1)a_k}{1 - \sum_{k=2}^{\infty} k^n a_k}$$

$$\leqq 1 - \alpha \tag{2.2}$$

which implies (1.5). Thus it follows from this fact that $f(z) \in T_1(n,m,\alpha)$.

Conversely, assume that the function $f(z)$ is in the class $T_1(n,m,\alpha)$. Then

$$\operatorname{Re}\left\{\frac{D^{n+m}f(z)}{D^n f(z)}\right\} = \operatorname{Re}\left\{\frac{1 - \sum_{k=2}^{\infty} k^{n+m}a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k^n a_k z^{k-1}}\right\}$$

$$> \alpha \tag{2.3}$$

for $z \in U$. Choose values of z on the real axis so that $D^{n+m}f(z)/D^n f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we obtain

$$1 - \sum_{k=2}^{\infty} k^{n+m} a_k \geqq \alpha\left(1 - \sum_{k=2}^{\infty} k^n a_k\right) \tag{2.4}$$

which gives (2.1). The result is sharp with the extremal function $f(z)$ defined by

$$f(z) = z - \frac{1-\alpha}{k^n(k^m - \alpha)} z^k \quad (k \geqq 2) \tag{2.5}$$

REMARK 1. In view of Lemma 1, $T_1(n,m,\alpha)$ when $n \in N_0$ and $m \in N$ is the subclass of $T^*(\alpha)$ introduced by Silverman [2], and $T_1(n,m,\alpha)$ when $n \in N$ and $m \in N$ is the subclass of $C(\alpha)$ introduced by Silverman [2].

With the aid of Lemma 1, we prove

THEOREM 1. Let the function $f(z)$ be defined by (1.6). Then $f(z) \in T_j(n,m,\alpha)$ if and only if

$$\sum_{k=j+1}^{\infty} k^n(k^m - \alpha) a_k \leqq 1 - \alpha \tag{2.6}$$

for $n \in N_0$, $m \in N_0$ and $0 \leqq \alpha < 1$. The result is sharp for the function

$$f(z) = z - \frac{1-\alpha}{k^n(k^m - \alpha)} z^k \quad (k \geqq j+1). \tag{2.7}$$

PROOF. Putting $a_k = 0$ $(k = 2,3,4,\ldots,j)$ in Lemma 1, we can prove the assertion of Theorem 1.

COROLLARY 1. Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\alpha)$. Then

$$a_k \leqq \frac{1-\alpha}{k^n(k^m - \alpha)} \quad (k \geqq j+1). \tag{2.8}$$

The equality in (2.8) is attained for the function $f(z)$ given by (2.7).

COROLLARY 2. $T_j(n+1,m,\alpha) \subset T_j(n,m,\alpha)$ and $T_j(n,m+1,\alpha) \subset T_j(n,m,\alpha)$.

REMARK 2. Taking $(j,n,m) = (1,0,1)$ and $(j,n,m) = (1,1,1)$ in Theorem 1, we have the corresponding results by Silverman [2]. Taking $(j,n,m) = (j,0,1)$ and $(j,n,m) = (1,1,1)$ in Theorem 1, we have the corresponding results by Chatterjea [3].

THEOREM 2. Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\alpha)$. Then

$$|D^i f(z)| \geqq |z| - \frac{1-\alpha}{(j+1)^{n-i}\{(j+1)^m - \alpha\}} |z|^{j+1} \tag{2.9}$$

and

$$|D^i f(z)| \leqq |z| + \frac{1-\alpha}{(j+1)^{n-i}\{(j+1)^m - \alpha\}} |z|^{j+1} \tag{2.10}$$

for $z \in U$, where $0 \leqq i \leqq n$. The equalities in (2.9) and (2.10) are attained for the function f(z) given by

$$f(z) = z - \frac{1-\alpha}{(j+1)^n\{(j+1)^m - \alpha\}} z^{j+1} \tag{2.11}$$

PROOF. Note that $f(z) \in T_j(n,m,\alpha)$ if and only if $D^i f(z) \in T_j(n-i,m,\alpha)$, and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k. \tag{2.12}$$

Using Theorem 1, we know that

$$(j+1)^{n-i}\{(j+1)^m - \alpha\} \sum_{k=j+1}^{\infty} k^i a_k \leqq 1 - \alpha, \tag{2.13}$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i a_k \leqq \frac{1-\alpha}{(j+1)^{n-i}\{(j+1)^m - \alpha\}}. \tag{2.14}$$

It follows from (2.12) and (2.14) that

$$|D^i f(z)| \geqq |z| - \frac{1-\alpha}{(j+1)^{n-i}\{(j+1)^m - \alpha\}} |z|^{j+1} \tag{2.15}$$

and

$$|D^i f(z)| \leqq |z| + \frac{1-\alpha}{(j+1)^{n-i}\{(j+1)^m - \alpha\}} |z|^{j+1}. \tag{2.16}$$

Finally, we note that the equalities in (2.9) and (2.10) are attained for the function f(z) defined by

$$D^i f(z) = z - \frac{1-\alpha}{(j+1)^{n-i}\{(j+1)^m - \alpha\}} z^{j+1}. \tag{2.17}$$

This completes the proof of Theorem 2.

COROLLARY 3. Let the function f(z) defined by (1.6) be in the class $T_j(n,m,\alpha)$. Then

$$|f(z)| \geqq |z| - \frac{1-\alpha}{(j+1)^n\{(j+1)^m - \alpha\}} |z|^{j+1} \tag{2.18}$$

and

$$|f(z)| \leqq |z| + \frac{1-\alpha}{(j+1)^n\{(j+1)^m - \alpha\}} |z|^{j+1} \tag{2.19}$$

for $z \in U$. The equalities in (2.18) and (2.19) are attained for the function f(z) given by (2.11).

PROOF. Taking i = 0 in Theorem 2, we can easily show (2.18) and (2.19).

COROLLARY 4. Let the function f(z) defined by (1.6) be in the class $T_j(n,m,\alpha)$. Then

$$|f'(z)| \geqq 1 - \frac{1-\alpha}{(j+1)^{n-1}\{(j+1)^m - \alpha\}}|z|^j \tag{2.20}$$

and

$$|f'(z)| \leqq 1 + \frac{1-\alpha}{(j+1)^{n-1}\{(j+1)^m - \alpha\}}|z|^j \tag{2.21}$$

for $z \in U$. The equalities in (2.20) and (2.21) are attained for the function f(z) given by (2.11).

PROOF. Note that $Df(z) = zf'(z)$. Hence, making i = 1 in Thorem 2, we have the corollary.

REMARK 3. Taking (j,n,m) = (1,0,1) and (j,n,m) = (1,1,1) in Corollary 3 and Corollary 4, we have distortion theorems due to Silverman [2].

3. DISTORTION THEOREMS FOR FRACTIONAL CALCULUS.

In this section, we use the following definitions of fractional calculus by Owa [4].

DEFINITION 1. The fractional integral of order λ is defined by

$$D_z^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(z)}{(z-\xi)^{1-\lambda}}d\xi \tag{3.1}$$

where $\lambda > 0$, f(z) is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

DEFINITION 2. The fractional derivative of order λ is defined by

$$D_z^{\lambda}f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}}d\xi, \tag{3.2}$$

where $0 \leqq \lambda < 1$, f(z) is an analytic function in a simply connected region of the z-plane contining the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda}f(z) = \frac{d^n}{dz^n}D_n^{\lambda}f(z) \tag{3.3}$$

where $0 \leqq \lambda < 1$ and n $N_0 = \{0,1,2,3,\ldots\}$.

THEOREM 3. Let the function f(z) defined by (1.6) be in the class $T_j(n,m,\alpha)$. Then

$$|D_z^{-\lambda}(D^i f(z))| \geqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1 - \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-i}\{(j+1)^m-\alpha\}}|z|^j\right\} \tag{3.4}$$

and

$$|D_z^{-\lambda}(D^i f(z))| \leqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1 + \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-i}\{(j+1)^m-\alpha\}}|z|^j\right\} \qquad (3.5)$$

for $\lambda > 0$, $0 \leqq i \leqq n$, and z **U**. The equalities in (3.4) and (3.5) are attained for the function $f(z)$ given by (2.11).

PROOF. It is easy to see that

$$\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}(D^i f(z)) = z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} k^i a_k z^k. \qquad (3.6)$$

Since the function

$$\phi(k) = \frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} \quad (k \geqq j+1) \qquad (3.7)$$

is decreasing in k, we have

$$0 < \phi(k) \leqq \phi(j+1) = \frac{\Gamma(j+2)\Gamma(2+\lambda)}{\Gamma(j+2+\lambda)}. \qquad (3.8)$$

Therefore, by using (2.14) and (3.8), we can see that

$$|\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}(D^i f(z))| \geqq |z| - \phi(j+1)|z|^{j+1}\sum_{k=j+1}^{\infty} k^i a_k$$

$$\geqq |z| - \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-i}\{(j+1)^m-\alpha\}}|z|^{j+1} \qquad (3.9)$$

which implies (3.4), and that

$$|\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}(D^i f(z))| \leqq |z| + \phi(j+1)|z|^{j+1}\sum_{k=j+1}^{\infty} k^i a_k$$

$$\leqq |z| + \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-i}\{(j+1)^m-\alpha\}}|z|^{j+1} \qquad (3.10)$$

which shows (3.5). Furthermore, note that the equalities in (3.4) and (3.5) are attained for the function $f(z)$ defined by

$$D_z^{-\lambda}(D^i f(z)) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1 - \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-i}\{(j+1)^m-\alpha\}}z^j\right\} \qquad (3.11)$$

or (2.17). Thus we complete the assertion of Theorem 3.

Taking $i = 0$ in Theorem 3, we have

COROLLARY 5. Let the function $f(z)$ by (1.6) be in the class $\mathbf{T}_j(n,m,\alpha)$. Then

$$|D_z^{-\lambda}f(z)| \geqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1 - \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^n\{(j+1)^m-\alpha\}}|z|^j\right\} \tag{3.12}$$

and

$$|D_z^{-\lambda}f(z)| \leqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1 + \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^n\{(j+1)^m-\alpha\}}|z|^j\right\} \tag{3.13}$$

for $\lambda > 0$ and $z \in U$. The equalities in (3.12) and (3.13) are attained for the function f(z) given by (2.11).

Finally, we prove

THEOREM 4. Let the function f(z) defined by (1.6) be in the class $T_j(n,m,\alpha)$. Then

$$|D_z^{\lambda}(D^if(z))| \geqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 - \frac{\Gamma(j+1)\Gamma(2-\lambda)\cdot(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-i-1}\{(j+1)^m-\alpha\}}|z|^j\right\} \tag{3.14}$$

and

$$|D_z^{\lambda}(D^if(z))| \leqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 + \frac{\Gamma(j+1)\Gamma(2-\lambda)\cdot(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-i-1}\{(j+1)^m-\alpha\}}|z|^j\right\} \tag{3.15}$$

for $0 \leqq \lambda < 1$, $0 \leqq i \leqq n-1$, and $z \in U$.
The equalities in (3.14) and (3.15) are attained for the function f(z) given by (2.11).

PROOF. A simple computation gives that

$$\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}(D^if(z)) = z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)}k^ia_kz^k. \tag{3.16}$$

Note that the function

$$\psi(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \quad (k \geqq j+1) \tag{3.17}$$

is decreasing in k. It follows from this fact that

$$0 < \psi(k) \leqq \psi(j+1) = \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+2-\lambda)}. \tag{3.18}$$

Consequently, with the aid of (2.14) and (3.18), we have

$$|\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}(D^if(z))| \geqq |z| - \psi(j+1)|z|^{j+1}\sum_{k=j+1}^{\infty} k^{i+1}a_k$$

$$\geqq |z| - \frac{\Gamma(j+1)\Gamma(2-\lambda)\cdot(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-i-1}\{(j+1)^m-\alpha\}}|z|^{j+1} \tag{3.19}$$

and

$$\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}(D^if(z))| \leqq |z| + \psi(j+1)|z|^{j+1}\sum_{k=j+1}^{\infty} k^{i+1}a_k$$

$$\leqq |z| + \frac{\Gamma(j+1)\Gamma(2-\lambda)\cdot(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-i-1}\{(j+1)^m-\alpha\}}|z|^{j+1}. \tag{3.20}$$

Thus (3.14) and (3.15) follow from (3.19) and (3.20), respectively. Further, since the equalities in (3.19) and (3.20) are attained for the function f(z) defined by

$$D_z^{\lambda}(D^i f(z)) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 - \frac{\Gamma(j+1)\Gamma(2-\lambda)\cdot(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-i-1}\{(j+1)^m-\alpha\}}z^j\right\}, \tag{3.21}$$

that is, by (2.17), this completes the proof of Theorem 4.

Making i = 0 in Theorem 4, we have

COROLLARY 6. Let the function f(z) defined by (1.6) ber in the class $\mathbf{T}_j(n,m,\alpha)$. Then

$$|D_z^{\lambda} f(z)| \geqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 - \frac{\Gamma(j+1)\Gamma(2-\lambda)\cdot(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-1}\{(j+1)^m-\alpha\}}|z|^j\right\} \tag{3.22}$$

and

$$|D_z^{\lambda} f(z)| \leqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 + \frac{\Gamma(j+1)\Gamma(2-\lambda)\cdot(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-1}\{(j+1)^m-\alpha\}}|z|^j\right\} \tag{3.23}$$

for $0 \leqq \lambda < 1$ and $z \in \mathbf{U}$. the equalities in (3.22) and (3.23) are attained for the function f(z) given by (2.11).

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