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UNSTABLE PERIODIC WAVE SOLUTIONS OF NERVE AXION DIFFUSION EQUATIONS

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ABSTRACT. Unstable periodic solutions of systems of parabolic equations are studied. Special attention is given to the existence and stability of solutions.

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1. INTRODUCTION.

Diffusion systems of partial differential equations are of great importance in biosciences. In this paper, unstable periodic solutions of systems of the form

$$u_t = u_{xx} + F(u,w),$$

$$w_t = G(u,w), \tag{1.1}$$

are studied. Equations of this type arise in neurophysiology in the study of nerve impulses on nerve axon, see [1,2]. Other classes of diffusion equations are also involved in biology, see for example [3-9].

2. EXISTENCE OF SOLUTIONS

It is known that for $G(u,w) = \varepsilon u$, if $\varepsilon > 0$ is sufficiently small, equation (1.1) has two types of wave solutions, namely, pulse travelling wave solutions and periodic travelling wave solutions. A travelling wave solution is a solution of equation (1.1) of the form

$$[u(x,t),\ w(x,t)] = [\phi(z;c),\ \psi(x;c)],\ z = x + ct,$$

hence $[\phi(z;c), \psi(z;c)]$ satisfies the ordinary differential equation

$$\frac{d^2\phi}{dz^2} - c\frac{d\phi}{dz} + F(\phi,\psi) = 0, \tag{2.1}$$

$$-c\frac{d\psi}{dz} + G(\phi,\psi) = 0.$$

A pulse travelling wave solution is a non-constant solution of (2.1) satisfying

$$\lim_{|z|\to\infty} [\phi(z;c),\ \psi(z;c)] = [0,0],$$

and a periodic travelling wave solution is a periodic solution of (2.1).

In [10], Evans showed that equation (1.1) has two pulse travelling solutions with different propagation speeds c_1 and c_2. On the existence of periodic travelling wave solutions, Hastings [11] showed that equation (2.1) with $G(u,w) = \varepsilon u$ has a non-constant periodic solution if $\varepsilon > 0$ is sufficiently small and the speed c is limited to a certain range. Rinzel and Keller [12] studied the case in which $F(u,w)$ is a function of u only given by

$$F(u,w) = \begin{cases} u & \text{for} \quad u \le a, \\ u-1 & \text{for} \quad a < u, \end{cases}$$

where $0 < a < 1/2$. Under this assumption, equation (2.1) has a non-constant periodic solution if c is limited in the range $c_1 < c < c_2$ and the period $p(c)$ is a smooth function of c. They demonstrated the behavior of the function $p(c)$ under the two cases when a is not very small and when a is very small. Dai [13] proved the existence and uniqueness of solutions for a general case and studied stability of the solution.

3. STABILITY ANALYSIS.

Stability of periodic travelling wave solutions is related to the eigenvalues of a matrix in the following theorem. Let $A(z;\lambda,c)$ be the matrix

$$A(z;\lambda,c) = \begin{bmatrix} 0 & 1 & 0 \\ \lambda - F_1[\phi(z;c),\ \psi(z;c)] & c & -F_2[\phi(z;c),\ \psi(z;c)] \\ \dfrac{G_1[\phi(z;c),\ \psi(z;c)]}{c} & 0 & \dfrac{G_2[\phi(z;c),\ \psi(z{:}c)] - \lambda}{c} \end{bmatrix}$$

where F_i and G_i denote the partial derivatives as usual, and let $X(z;\lambda,c)$ be a matrix satisfying the differential equation

$$\frac{d}{dz} X = A X$$

with the initial condition $X(0;\lambda,c) = I$.

THEOREM 3.1. Suppose the functions F and G in equation (1.1) staisfy (a) $F(0,0) = 0$, (b) $G(0,0) = 0$ and (c) the matrix $X(p(c);\lambda,c)$ has an eigenvalue of modulus 1, for some complex number λ with $\operatorname{Re}\lambda > 0$, then a periodic travelling wave solution $[\phi(z;c),\ \psi(z;c)]$ is unstable.

PROOF. With the change of variables,

$$z = x + ct,$$

$$t = t,$$

$$[u(x,t),\ w(x,t)] = [\bar{u}(z,t),\ \bar{w}(z,t)],$$

equation (1.1) becomes

$$\bar{u}_t = \bar{u}_{zz} - c\,\bar{u}_z + F(\bar{u},\bar{w}), \tag{3.1}$$

$$\bar{w}_t = -c\,\bar{w}_z + G(\bar{u},\bar{w}).$$

The linearized perturbation equation of the above system with respect to the solution $[\phi(z;c),\ \psi(z;c)]$ is

$$\bar{U}_t = \bar{U}_{zz} - c\,\bar{U}_z + F_1\,[\phi,\psi]\,\bar{U} + F_2\,[\phi,\psi]\,\bar{W},$$

$$\bar{W}_t = -\,c\,\bar{W}_z + G_1\,[\phi,\psi]\,\bar{U} + G_2\,[\phi,\psi]\,\bar{W}, \tag{3.2}$$

where $\phi = \phi(z;c)$ and $\psi = \psi(z;c)$, since $F(0,0) = G(0,0) = 0$. Equation (3.2) has a solution of the form

$$\bar{U}(z,t) = e^{\lambda t} y_1\,(z;\lambda),$$

$$\bar{W}(z,t) = e^{\lambda t} y_2\,(z;\lambda),$$

where $(y_1,\ y_2)$ satisfies the following system of linear ordinary differential equations

$$\lambda y_1 = \frac{d^2 y_1}{dz^2} - c\,\frac{dy_1}{dz} + F_1\,[\phi,\psi]\,y_1 + F_2\,[\phi,\psi]\,y_2,$$

$$\lambda y_2 = -\,c\,\frac{dy_2}{dz} + G_1\,[\phi,\psi]\,y_1 + G_2\,[\phi,\psi]\,y_2, \tag{3.3}$$

where $\phi = \phi(z;c)$ and $\psi = \psi(z;c)$. Note that if equation (3.3) has a solution which is bounded for all z in $(-\infty,\infty)$ for a number λ with $\mathrm{Re}(\lambda) > 0$, then equation (3.2) has a solution $[\bar{U}(z,t),\ \bar{W}(z,t)]$ which grows exponentially, and hence, the travelling wave solution $[\phi(z;c),\ \psi(z;c)]$ is unstable.

Using Floquet's theory, we can show that equation (3.3) has a bounded non-trivial solution if and only if one of the eigenvalues of $X(p(c);\lambda,c)$ is a modulus 1. Equation (3.3) can be rewritten as

$$\frac{d}{dz}\left(\frac{dy_1}{dz}\right) = \Big(\lambda - F_1[\phi,\psi]\Big)\ y_1 + c\,\frac{dy_1}{dz} - F_2\,[\phi,\psi]\,y_2,$$

$$c\,\frac{dy_2}{dz} = G_1\,[\phi,\psi]\,y_1 + (G_2[\phi,\psi] - \lambda)\,y_2,$$

and so can be represented by the matrix differential equation

$$\frac{d}{dz}\,\underline{v} = A(z;\lambda,c)\,\underline{v},$$

where

$$\underline{v} = \begin{bmatrix} y_1 \\ \frac{dy_1}{dz} \\ y_2 \end{bmatrix}$$

and the matrix A is as defined before. Now, since the coefficient matrix $A(z;\lambda,c)$ is a $p(c)$-periodic function of z, Floquet's theory yields that equation (3.3) has a bounded non-trivial solution if and only if one of the eigenvalues of the matrix $X(p(c);\lambda,c)$ defined before is of modulus 1. The proof is now complete.

In the following lemma, it is shown that under the special case $\lambda = 0$, one eigenvalue of $X(p(c);0,c)$ is unity and the product of the other two eigenvalues is greater than one.

LEMMA 3.1. Suppose (a) $G_2(u,w) \geq 0$ for all u and w and (b) $\lambda = 0$, let $\mu_i(\lambda,c)$, $i = 1, 2, 3$, denote the eigenvalues of $X(p(c);\lambda,c)$, then one eigenvalue, say

$$\mu_1(0,c) = 1,$$

and

$$\mu_2(0,c)\mu_3(0,c) > 1.$$

PROOF. Differentiation of equation (2.1) leads to

$$\frac{d}{dz}\left(\frac{d^2\phi}{dz^2}\right) = c\,\frac{d}{dz}\left(\frac{d\phi}{dz}\right) - F_1\,[\phi,\psi]\,\frac{d\phi}{dz} - F_2\,[\phi,\psi]\,\frac{d\psi}{dz}$$

$$c\,\frac{d}{dz}\left(\frac{d\psi}{dz}\right) = G_1\,[\phi,\psi]\,\frac{d\phi}{dz} + G_2\,[\phi,\psi]\,\frac{d\psi}{dz}, \tag{3.4}$$

where $\phi = \phi(z;c)$ and $\psi = \psi(z;c)$. Therefore the vector

$$\underline{W} = \begin{bmatrix} \phi \\ \frac{d\phi}{dz} \\ \psi \end{bmatrix}$$

satisfies the matrix equation

$$\frac{d}{dz}\,\underline{w_z} = A\,(z;0,c)\,\underline{w_z},$$

that is,

$$\frac{d}{dz}\begin{bmatrix} \frac{d\phi}{dz} \\ \frac{d^2\phi}{dz^2} \\ \frac{d\psi}{dz} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -F_1[\phi(z;c),\ \psi(z;c)] & c & -F_2[\phi(z;c),\ \psi(z;c)] \\ \frac{G_1[\phi(z;c),\ \psi(z;c)]}{c} & 0 & \frac{G_2[\phi(z;c),\ \psi(z;c)]}{c} \end{bmatrix}\begin{bmatrix} \frac{d\phi}{dz} \\ \frac{d^2\phi}{dz^2} \\ \frac{d\psi}{dz} \end{bmatrix}.$$

We know that (see for example, Sanchez)

$$\underline{w}_z (z;c) = X(z;0,c)\, \underline{w}_z (0;c)$$

and since $\underline{w}_z (z;c)$ is a $p(c)$ - periodic function of z, it follows that

$$\underline{w}_z (0;c) = \underline{w}_z (p(c);c) = X(p(c);0,c)\, \underline{w}_z (0;c). \tag{3.5}$$

Thus there is an eigenvalue, say

$$\mu_1 (0,c) = 1.$$

Further, by Jacobi's formula,

$$\det \{X(z;\lambda,c)\} = \{\det X(0;\lambda,c)\} \exp \int_0^z \operatorname{tr} \{A(\xi;\lambda,c)\}\, d\xi$$

$$= (1) \exp \int_0^z \left(c + \frac{G_2[\phi,\psi] - \lambda}{c}\right) d\xi .$$

In particular,

$$\det \{X(p(c);0,c)\} = \exp [c\, p(c)] \exp \int_0^{p(c)} \frac{G_2[\phi,\psi]}{c}\, d\xi$$
$$> 1$$

since $c > 0$, $p(c) > 0$ and $G_2 (u,w) \geq 0$ for all u,w.
But $\det \{X (p(c);0,c)\} = \mu_1 (0,c)\, \mu_2 (0,c)\, \mu_3 (0,c)$ and

$$\mu_1 (0,c) = 1, \quad \text{hence} \quad \mu_2 (0,c)\, \mu_3 (0,c) > 1.$$

Note that under the assumptions of Lemma 3.1, either $|\mu_2 (\lambda,c)| > 1$ or $|\mu_3 (\lambda,c| > 1$ for λ sufficiently small. In the next theorem, we will see that if $L(c)$ is decreasing, i.e. $L'(c) < 0$, then $\mu_1 (\lambda,c)$ is increasing at $\lambda = 0$, i.e. $\frac{\partial}{\partial\lambda} \mu_1(\lambda,c) \big|_{\lambda=0} > 0$.

THEOREM 3.2. Suppose (a) $p'(c) < 0$, then $\frac{\partial}{\partial\lambda} \mu_1(\lambda,c) \big|_{\lambda=0} > 0$, and hence if (b) the assumptions in Lemma 3.1 also hold, then $\mu_1 (\lambda,c) > 1$ for λ sufficiently small.

PROOF: We claim that the following equality

$$\frac{\partial}{\partial\lambda} \mu_1(\lambda,c) \Big|_{\lambda=0} = -p'(c)$$

actually holds.

Recall the vector $\underline{w}$ (z;c), namely,

$$\underline{w} = \begin{bmatrix} \phi \\ \frac{d\phi}{dz} \\ \psi \end{bmatrix}$$

which satisfies the periodicity

$$\underline{w}\,(p(c);c) = \underline{w}\,(0;c).$$

Differentiation of the above equation with respect to c leads to

$$\underline{w_z}\,(p(c);c)\,p'(c) + \underline{w_c}\,(p(c);c) = \underline{w_c}\,(0;c). \tag{3.6}$$

Let $\underline{v} = \underline{v}^* = [y_1^*\,(z;\lambda,c),\ y_2^*\,(z;\lambda,c)]$ be a solution of equation (3.3) satisfying the initial condition

$$\underline{v}\,(0;\lambda,c) = \underline{w_z}\,(0;c) + \lambda\,\underline{w_c}\,(0;c), \tag{3.7}$$

where $\underline{v}\,(z;\lambda,c)$ is the vector defined before. We have observed before that $[\frac{d\phi}{dz}\,(z;c), \frac{d\psi}{dz}]\,(z;c)$, which satisfies equation (3.4), is a solution of equation (3.3) under $\lambda=0$. In view of the condition (3.7) and by uniqueness of solutions, we have

$$\underline{v}^*\,(z;0,c) = \underline{w_z}\,(z;c). \tag{3.8}$$

Differentiation of equation (3.3) with respect to λ leads to

$$y_1 + \lambda\frac{\partial y_1}{\partial\lambda} = \frac{d^2}{dz^2}\left(\frac{\partial y_1}{\partial\lambda}\right) - c\frac{d}{dz}\left(\frac{\partial y_1}{\partial\lambda}\right) + F_1[\phi,\psi]\frac{\partial y_1}{\partial\lambda} + F_2\,[\phi,\psi]\frac{\partial y_2}{\partial\lambda},$$

$$y_2 + \lambda\frac{\partial y_2}{\partial\lambda} = -c\frac{d}{dz}\left(\frac{\partial y_2}{\partial\lambda}\right) + G_1\,[\phi,\psi]\frac{\partial y_1}{\partial\lambda} + G_2\,[\phi,\psi]\frac{\partial y_2}{\partial\lambda}. \tag{3.9}$$

Under $\lambda = 0$, and replacing $[y_1, y_2]$ by $[y_1^*, y_2^*]$, equation (3.9) by equality (3.8) becomes

$$\frac{d\phi}{dz}\,(z;c) = \frac{d^2}{dz^2}\left(\frac{\partial y_1^*}{\partial\lambda}\right) - c\frac{d}{dz}\left(\frac{\partial y_1^*}{\partial\lambda}\right) + F_1[\phi,\psi]\frac{\partial y_1^*}{\partial\lambda} + F_2[\phi,\psi]\frac{\partial y_2^*}{\partial\lambda},$$

$$\frac{d\psi}{dz}\,(z;c) = -\,c\frac{d}{dz}\left(\frac{\partial y_2^*}{\partial\lambda}\right) + G_1[\phi,\psi]\frac{\partial y_1^*}{\partial\lambda} + G_2[\phi,\psi]\frac{\partial y_2^*}{\partial\lambda}, \tag{3.10}$$

where $\frac{\partial y_i^*}{\partial\lambda} = \frac{\partial y_i^*}{\partial\lambda}\,(z;0,c)$ now. On the other hand, differentiating equation (2.1) with respect to c, we get

$$\frac{d^2}{dz^2}\left(\frac{\partial \phi}{\partial c}\right) - \frac{d\phi}{dz} - c\,\frac{d}{dz}\left(\frac{\partial \phi}{\partial c}\right) + F_1[\phi,\psi]\,\frac{\partial \phi}{\partial c} + F_2[\phi,\psi]\,\frac{\partial \psi}{\partial c} = 0,$$

$$-\frac{d\psi}{dz} - c\,\frac{d}{dz}\left(\frac{\partial \psi}{\partial c}\right) + G_1[\phi,\psi]\,\frac{\partial \phi}{\partial c} + G_2[\phi,\psi]\,\frac{\partial \psi}{\partial c} = 0, \qquad (3.11)$$

where $\phi = \phi\,(z;c)$ and $\psi = \psi\,(z;c)$. Therefore both $[\frac{\partial y_1^*}{\partial \lambda}(z;0;c), \frac{\partial y_2^*}{\partial \lambda}(z;0,c)]$ and $[\frac{\partial \phi}{\partial c}(z;c), \frac{\partial \psi}{\partial c}(z;c)]$ satisfy the same differential equation. In addition, differentiation of the initial condition (3.7) yields

$$\underline{v_\lambda}\,(0;\lambda,c) = \underline{w_c}\,(0;c),$$

in particular,

$$v_\lambda\,(0;0,c) = \underline{w_c}\,(0;c)$$

and hence the equality

$$\underline{v_\lambda^*}\,(z;0,c) = \underline{w_c}\,(z;c),\ 0 \le z \le p(c). \qquad (3.12)$$

The equalities (3.8) and (3.12) together give

$$\underline{v}^*\,(z;\lambda,c) = \underline{w_z}\,(z;c) + \lambda\,\underline{w_c}\,(z;c) + 0(\lambda^2), \qquad (3.13)$$

$$0 \le z \le p(c), \quad \text{as} \quad \lambda \to 0.$$

Knowing $\underline{v}^*\,(z;\lambda,c) = X\,(z;\lambda,c)\,\underline{v}^*\,(0;\lambda,c)$, by equation (3.13) for $z = p(c)$ and also $z = 0$, we get

$$\underline{w_z}\,(p(c);c) + \lambda\,\underline{w_c}\,(p(c);c) + 0\,(\lambda^2)$$

$$= X\,(p(c);\lambda,c)\,[\underline{w_z}\,(o;c) + \lambda\,\underline{w_c}\,(0;c)]. \qquad (3.14)$$

Substitution of the equation (3.6) containing p'(c) into the left hand side of equation (3.14) and periodicity lead to

$$X\,(p(c);\lambda,c)\,[w_z\,(0;c) + \lambda\,\underline{w_c}\,(0;c)]$$

$$= [1 - \lambda\,p'(c)][w_z\,(0;c) + \lambda\,w_c\,(0;c)] + 0\,(\lambda^2).$$

Hence the eigenvalue $\mu_1(\lambda,c)$ satisfies

$$\frac{\partial}{\partial \lambda}\,\mu_1(\lambda,c)\,\Big|_{\lambda=0} = -p'(c).$$

The proof is now complete.

On the other hand, under certain conditions, two eigenvalues have modulus less than one and one has modulus greater than one.

THEOREM 3.3. Suppose (a) F_2 (u,w) is a non-zero constant and (b) G_1 (u,w) and G_2 (u,w) are constant, then for λ sufficiently large, two eigenvalues of X (p(c);λ,c) have modulus < 1 and one has modulus > 1.

PROOF: Decompose the matrix $A(z;\lambda,c)$ as follows

$$A(z;\lambda,c) = B(\lambda,c) + E(z;c)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ \lambda & c & -F_2 \\ \frac{G_1}{c} & 0 & \frac{G_2-\lambda}{c} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -F_1[\phi(z;c), \psi(z;c)] & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $s_i(\lambda,c)$, $i = 1,2,3$ be the eigenvalues of $B(\lambda,c)$ and $\underline{q_i}$ the corresponding eigenvectors. The characteristic equation of $B(\lambda,c)$ is

$$-s^3 + (\frac{G_2-\lambda}{c} + c)s^2 + (2\lambda - G_2)s + \lambda(\frac{\lambda - G_2}{c}) - \frac{F_2 G_1}{c} = 0.$$

It follows that as $\lambda \to \infty$,

$$s_1(\lambda,c) = \frac{-\lambda}{c} + O(1)$$

$$s_2(\lambda,c) = -\sqrt{\lambda} + O(1) \tag{3.15}$$

$$s_3(\lambda,c) = \sqrt{\lambda} + O(1).$$

The vectors $\underline{q_i}(\lambda,c)$ are

$$\underline{q_i}(\lambda,c) = \begin{bmatrix} 1 \\ s_i \\ \frac{s_i^2 - c\, s_i - \lambda}{-F_2} \end{bmatrix}, \quad i = 1,2,3, \tag{3.16}$$

and let $Q(\lambda,c)$ be the non-singular matrix

$$Q(\lambda,c) = [\,\underline{q_1}(\lambda,c), \underline{q_2}(\lambda,c), \underline{q_3}(\lambda,c)],$$

then

$$Q^{-1}BQ = \begin{bmatrix} s_1(\lambda,c) & 0 & 0 \\ 0 & s_2(\lambda,v) & 0 \\ 0 & 0 & s_3(\lambda,c) \end{bmatrix}.$$

Now consider the matrix

$$Y(z;\lambda,c) = Q^{-1} X(z;\lambda,c) Q$$

which has the same eigenvalues as $X(z;\lambda,c)$, in particular with $z = p(c)$, and satisfies the differential equation

$$\frac{d}{dz} Y(z;\lambda,c) = Q^{-1} A(z;\lambda,c) Q Y(z;\lambda,c)$$

$$= [Q^{-1} B(\lambda,c)Q + Q^{-1} E(z;c)Q] Y(z;\lambda,c),$$

since $\frac{d}{dz} X(z;\lambda,c) = A(z;\lambda,c) X(z;\lambda,c)$.

But $Q^{-1}BQ$ is the diagonal matrix from before and it can be shown easily using (3.15) and (3.16) that all elements of $Q^{-1}EQ$ are $o(1)$ as $\lambda \to \infty$, therefore the eigenvalues of $Y(p(c);\lambda,c)$ and hence of $X(p(c);\lambda,c)$ approach

$$\exp [s_i(\lambda,c)\, p(c)],\ i = 1,2,3 \quad \text{as } \lambda \to \infty.$$

It follows from (3.15) that as $\lambda \to \infty$, two eigenvalues of $X(p(c);\lambda,c)$ have modulus < 1 and one has modulus > 1.

To summarize, under the assumptions of both Theorem (3.2) and Theorem (3.3), at least two eigenvalues of $X(p(c);\lambda,c)$ have modulus > 1 as $\lambda \to 0+$, and two eigenvalues of $X(p(c);\lambda,c)$ have modulus < 1 as $\lambda \to \infty$. Hence one of the eigenvalues must have modulus $= 1$ for some $\lambda > 0$ and under Theorem (3.1), the travelling wave solution $(\phi(z;c), \psi(z;c))$ is unstable.

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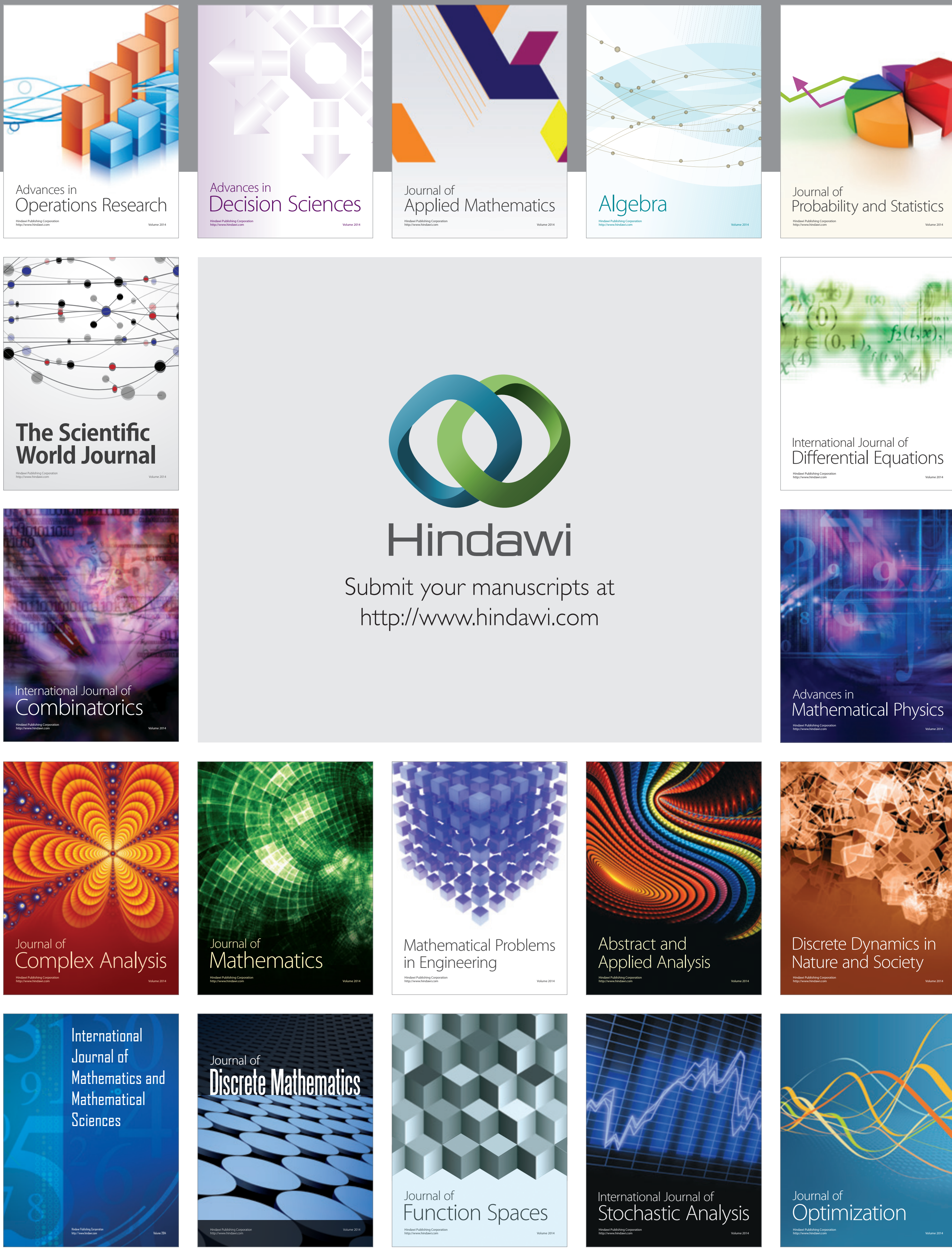
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