

AN INVERSE PROBLEM FOR HELMHOLTZ'S EQUATION

A.G. RAMM

Mathematics Department
 Kansas State University
 Manhattan, KS 66506

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ABSTRACT. The refraction coefficient in Helmholtz's equation is found from the knowledge of a family of the solutions to this equation on two lines.

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1. INTRODUCTION.

Let

$$[\nabla^2 + k^2 + k^2 v(x)]u = -\delta(x-y), \quad k > 0 \tag{1.1}$$

where $x = (x_1, x_3)$, $y = (y_1, y_3)$, $v = v(x_1, x_3)$, $u = u(x_1, x_3, y_1, y_3, k)$.

Assume that

$$v(x) = 0 \text{ for } x_1 \geq a \text{ or } x_1 \leq -a, \text{ or } x_3 \geq 0 \text{ or } x_3 \leq -R, v \in L^2 \tag{1.2}$$

Here $R > 0$ is an arbitrary large fixed number. Write (1.1) as

$$u = g + k^2 \int g v u dz, \quad g := (i/4)H_0^{(1)}(k|x-y|) \tag{1.3}$$

where the integral is taken over the support of v and $H_0^{(1)}$ is the Hankel function.

The problem is: find $v(k)$ from the knowledge of $u(-a, x_3, a, y_3, k)$ for all $-\infty < x_3, y_3 < \infty$ and $0 < k < k_0$, where $k_0 > 0$ is an arbitrary small number.

2. SOLUTION.

Let $L_a = \{x: x_1 = a, x_3 \in \mathbb{R}^1\}$, $\mathbb{R}^1 = (-\infty, \infty)$. We use the method given in [1], [2]. It follows from (1.3) that

$$f(x_3, y_3, k) := k^{-2}(u-g) = \int g v g dz + o(k) \text{ as } k \rightarrow 0, x \in L_{-a}, y \in L_a. \tag{2.1}$$

Let us take the Fourier transform of (2.1) in x_3 and y_3 , define

$$\tilde{f}(\lambda, \mu) := (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-i\lambda x_3 - i\mu y_3) f(x_3, y_3) dx_3 dy_3, \text{ and use the formula} \tag{2.2}$$

$$(2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-i\lambda x_3) g(x, z) dx_3 = i(4\pi)^{-1} \exp\{-i\lambda z_3 + i(a+z_1)(k^2 - \lambda^2)^{\frac{1}{2}}\} / (k^2 - \lambda^2)^{\frac{1}{2}}$$

where $x = (-a, x_3)$, the radical $(k^2 + i0 - \lambda^2)^{\frac{1}{2}} > 0$ for $\lambda^2 < k^2$ and is defined by analytic continuation for all complex λ on the complex λ -plane with the cut $(-k, k)$, $k > 0$, so that

$$(k^2 - \lambda^2)^{\frac{1}{2}} = i (\lambda^2 - k^2)^{\frac{1}{2}} \quad \text{if} \quad k^2 < \lambda^2. \quad (2.3)$$

The result is

$$\tilde{f}(\lambda, \mu) = \int dz v(z) h(\lambda, \mu, z, k) + o(k) \quad (2.4)$$

where for $k^2 > \lambda^2$, $k^2 > \mu^2$, and $r(\lambda) := (k^2 - \lambda^2)^{\frac{1}{2}}$ one has

$$h := -(16\pi^2)^{-1} \exp\{-i(\lambda + \mu)z_3 + i(a + z_1)r(\lambda) + i(a - z_1)r(\mu)\} r^{-1}(\lambda) r^{-1}(\mu) \quad (2.5)$$

and for $k^2 < \mu^2$ and $k^2 < \lambda^2$ one uses (2.3).

In the Born approximation one drops the term $o(k)$ in (2.4) and solves the resulting linear integral equation for $v(z)$ [2].

In the exact theory one passes to the limit $k \rightarrow 0$ in (2.4), obtains a linear integral equation for v and solves this equation analytically [2]. It is not possible to pass to the limit $k \rightarrow 0$ in (2.1) because $g(kr) = \alpha(k) + g_0 + 0[(kr)^2 \ln(k/2)]$ as $k \rightarrow 0$, where $g_0 := (2\pi)^{-1} \ln(r^{-1})$, $\alpha(k) := -(2\pi)^{-1} \ln(k/2) + i/4 - \gamma/(2\pi)$, and $\gamma = 0.5572 \dots$ is Euler's constant. Thus $g(kr)$ does not have a finite limit as $k \rightarrow 0$. Nevertheless one can pass to the limit $k \rightarrow 0$ in (2.4) if $\gamma \neq 0$ or $\mu \neq 0$. The reason is that the term $\alpha(k)$ in (2.1) after the Fourier transform becomes $\alpha(k)\delta(\lambda)\delta(\mu)$, and this term, which contains the factor $\alpha(k) \rightarrow \infty$ as $k \rightarrow 0$, is zero for $\lambda \neq 0$ or $\mu \neq 0$. Another way to study the limit behavior of the solution to (2.1) is given in [2]. To give the exact theory, pass to the limit $k \rightarrow 0$ in (2.4) to get

$$\int v(z) \exp(-ipz_3 + qz_1) dz_1 dz_3 = \psi(p, q) \quad (2.6)$$

where we used (2.5) and set

$$p := \lambda + \mu, \quad q := |\mu| - |\lambda|, \quad (2.7)$$

$$\psi(p, q) := 16\pi^2 \tilde{f}(\lambda, \mu) |\lambda| |\mu| \{\exp a(|\lambda| + |\mu|)\} \quad (2.8)$$

and the right side of (2.8) should be expressed as a function of (p, q) by formulas (2.7).

If $\mu > 0$ and $\lambda > 0$ then the point (p, q) defined by (2.7) runs through $Q_+ = \{p, q: |q| < p, p > 0\}$.

If $\lambda < 0$ and $\mu < 0$ then (p, q) runs through $Q_- = \{p, q: |q| < -p, p < 0\}$. If $\psi(p, q)$ is known in Q_+ or Q_- then $v(z)$ can be uniquely recovered from (2.6) by the analytical methods given in [2] p. 270-274, where inversion of the Fourier and Laplace transforms of compactly supported functions from a compact set is given. This inversion problem is ill-posed and its numerical implementation is not a simple matter.

One can use the same ideas to solve equation (2.4) at a fixed $k > 0$ in the Born approximation. The basic equation analogous to (2.4) for the case when $-k < \lambda$, $\mu < k$, is:

$$\int v(z) \exp\{-i(pz_3 + q_1 z_1)\} dz = f(p, q_1) \text{ for } -k < \mu, \lambda < k \quad (2.9)$$

where $p = \lambda + \mu$, $q_1 := r(\mu) - r(\lambda)$,

$$F(p, q_1) := -16\pi^2 \tilde{f}(\lambda, \mu) r(\lambda) r(\mu) \exp\{-ia[r(\lambda) + r(\mu)]\} \quad (2.10)$$

and the right side of (2.10) should be expressed as a function of p, q_1 .

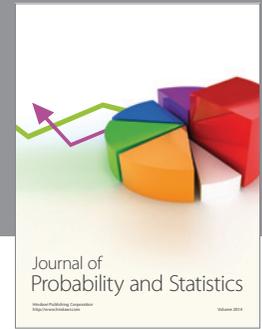
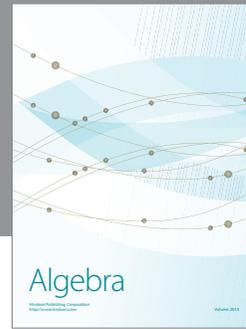
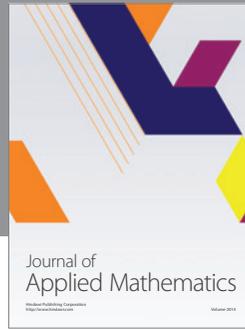
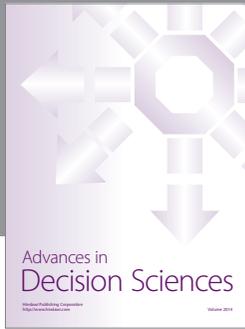
If $(\lambda, \mu) \{ \lambda, \mu: |\lambda| > k \text{ and } |\mu| > k, \lambda, \mu \text{ are real} \}$ then the basic equation in the Born approximation is equation (2.6) in which the right side is now given by the formula $\psi = F$, where F is defined by (2.10) and in (2.10) the radicals $r(\lambda)$ and $r(\mu)$ are computed by formula (2.3) for $\lambda^2 > k^2$ and $\mu^2 > k^2$.

Equation (2.9) can also be solved analytically with the prescribed accuracy by the methods given in [2].

The problem considered is of interest in application.

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