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AN INVERSE PROBLEM FOR HELMHOLTZ'S EQUATION

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ABSTRACT. The refraction coefficient in Helmholtz's equation is found from the knowledge of a family of the solutions to this equation on two lines.

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1. INTRODUCTION.

Let

$$[\nabla^2+k^2+k^2v(x)]u = -\delta(x-y) \ , \quad k > 0 \tag{1.1}$$

where $x = (x_1,x_3)$, $y = (y_1,y_3)$, $v = v(x_1,x_3)$, $u = u(x_1,x_3,y_1,y_3,k)$.

Assume that

$$v(x) = 0 \text{ for } x_1 \geqq a \ \text{ or } \ x_1 \leqq -a, \text{ or } x_3 \geqq 0 \ \text{ or } \ x_3 \leqq -R, \ v\varepsilon L^2 \tag{1.2}$$

Here $R > 0$ is an arbitrary large fixed number. Write (1.1) as

$$u = g + k^2 \int gvudz, \quad g := (i/4)H_0^{(1)}(k|x-y|) \tag{1.3}$$

where the integral is taken over the support of v and $H_0^{(1)}$ is the Hankel function.

The problem is: find $v(k)$ *from the knowledge of* $u(-a,x_3,a,y_3,k)$ *for all* $-\infty < x_3,y_3 < \infty$ *and* $0 < k < k_0$, *where* $k_0 > 0$ *is an arbitrary small number.*

2. SOLUTION.

Let $L_a = \{x: \ x_1 = a, \ x_3 \ \varepsilon R^1\}$, $R^1 = (-\infty,\infty)$. We use the method given in [1], [2]. It follows from (1.3) that

$$f(x_3,y_3,k) := k^{-2}(u-g) = \int gvgdz + o(k) \text{ as } k\to 0, \ x\varepsilon L_{-a}, \ y\varepsilon L_a. \tag{2.1}$$

Let us take the Fourier transform of (2.1) in x_3 and y_3, define $\tilde{f}(\lambda,\mu) := (2\pi)^{-2} \iint_{-\infty}^{\infty} \exp(-i\lambda x_3-i\mu y_3)f(x_3,y_3)dx_3dy_3$, and use the formula

$$(2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-i\lambda x_3)g(x,z)dx_3 = i(4\pi)^{-1}\exp\left\{-i\lambda z_3+i(a+z_1)(k^2-\lambda^2)^{\frac{1}{2}}\right\}/(k^2-\lambda^2)^{\frac{1}{2}} \tag{2.2}$$

where $x = (-a,x_3)$, the radical $(k^2+io-\lambda^2)^{\frac{1}{2}} > 0$ for $\lambda^2 < k^2$ and is defined by analytic continuation for all complex λ on the complex λ-plane with the cut $(-k,k)$, $k > 0$, so that

$$(k^2-\lambda^2)^{\frac{1}{2}} = i\ (\lambda^2-k^2)^{\frac{1}{2}} \quad \text{if} \quad k^2 < \lambda^2 . \tag{2.3}$$

The result is

$$\tilde{f}(\lambda,\mu) = \int dz v(z) h(\lambda,\mu,z,k) + o(k) \tag{2.4}$$

where for $k^2 > \lambda^2$, $k^2 > \mu^2$, and $r(\lambda) := (k^2-\lambda^2)^{\frac{1}{2}}$ one has

$$h := -(16\pi^2)^{-1}\exp\{-i(\lambda+\mu)z_3 + i(a+z_1)r(\lambda) + i(a-z_1)r(\mu)\}r^{-1}(\lambda)r^{-1}(\mu) \tag{2.5}$$

and for $k^2 < \mu^2$ and $k^2 < \lambda^2$ one uses (2.3).

In the Born approximation one drops the term $o(k)$ in (2.4) and solves the resulting linear integral equation for $v(z)$ [2].

In the exact theory one passed to the limit $k \to 0$ in (2.4), obtains a linear integral equation for v and solves this equation analytically [2]. It is not possible to pass to the limit $k \to 0$ in (2.1) because $g(kr) = \alpha(k) + g_0 + O[(kr)^2 \ln(k/2)]$ as $k \to 0$, where $g_o := (2\pi)^{-1}\ln(r^{-1})$, $\alpha(k) := -(2\pi)^{-1}\ln(k/2)+i/4-\gamma/(2\pi)$, and $\gamma = 0.5572 \ldots.$ is Euler's constant. Thus $g(kr)$ does not have a finite limit as $k \to 0$. Nevertheless one can pass to the limit $k \to 0$ in (2.4) if $\gamma \neq 0$ or $\mu \neq 0$. The reason is that the term $\alpha(k)$ in (2.1) after the Fourier transform becomes $\alpha(k)\delta(\lambda)\delta(\mu)$, and this term, which contains the factor $\alpha(k) \to \infty$ as $k \to 0$, is zero for $\lambda \neq 0$ or $\mu \neq 0$. Another way to study the limit behavior of the solution to (2.1) is given in [2]. To give the exact theory, pass to the limit $k \to 0$ in (2.4) to get

$$\int v(z)\exp(-ipz_3 + qz_1)dz_1dz_3 = \psi(p,q) \tag{2.6}$$

where we used (2.5) and set

$$p := \lambda+\mu,\ q := |\mu| - |\lambda| , \tag{2.7}$$

$$\psi(p,q) := 16\pi^2\tilde{f}(\lambda,\mu)|\lambda|\ |\mu|\ \{\exp a(|\lambda| + |\mu|)\} \tag{2.8}$$

and the right side of (2.8) should be expressed as a function of (p,q) by formulas (2.7).

If $\mu > 0$ and $\lambda > 0$ then the point (p,q) defined by (2.7) runs through $Q_+ = \{p,q: \ |q| < p , p > 0\}$.

If $\lambda < 0$ and $\mu < 0$ then (p,q) runs through $Q_- = \{p,q: \ |q| < -p , p < 0\}$. If $\psi(p,q)$ is known in Q_+ or Q_- then $v(z)$ can be uniquely recovered from (2.6) by the analytical methods given in [2] p. 270-274, where inversion of the Fourier and Laplace transforms of compactly supported functions from a compact set is given. This inversion problem is ill-posed and its numerical implementation is not a simple matter.

One can use the same ideas to solve equation (2.4) at a fixed $k > 0$ in the Born approximation. The basic equation analogous to (2.4) for the case when $-k < \lambda$, $\mu < k$, is:

$$\int v(z) \exp\{-i(pz_3+q_1z_1)\}dz = f(p,q_1) \text{ for } -k < \mu, \lambda < k \tag{2.9}$$

where $p = \lambda+\mu$, $q_1 := r(\mu)-r(\lambda)$,

$$F(p,q_1) := -16\pi^2 \tilde{f}(\lambda,\mu)r(\lambda)r(\mu) \exp\{-ia[r(\lambda)+r(\mu)]\} \tag{2.10}$$

and the right side of (2.10) should be expressed as a function of p,q_1.

If (λ,μ) $\{\lambda,\mu$: $|\lambda| > k$ and $|\mu| > k,\lambda,\mu$ are real$\}$ then the basic equation in the Born approximation is equation (2.6) in which the right side is now given by the formula $\psi = F$, where F is defined by (2.10) and in (2.10) the radicals $r(\lambda)$ and $r(\mu)$ are computed by formula (2.3) for $\lambda^2 > k^2$ and $\mu^2 > k^2$.

Equation (2.9) can also be solved analytically with the prescribed accuracy by the methods given in [2].

The problem considered is of interest in application.

REFERENCES

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