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ALMOST NONE OF THE SEQUENCES OF 0's AND 1's ARE ALMOST CONVERGENT

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Abstract. We establish that, in the sense of the Law of Large Numbers, almost none of the sequences of 0's and 1's are assigned the same value by every Banach limit.

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The result established in this note is precisely the result promised in the title. To place the theorem in perspective, however, it will be helpful to recall a few definitions and a fundamental result of probability theory.

First we recall an extremely useful extension of the usual notion of convergence. A sequence $x = (x_n)$ is said to be Cesaro summable to s provided $\lim_n n^{-1}\sum_{k=1}^{n} x_k = s$. If x is Cesaro summable to s, we write $C\text{-}\lim x = s$.

Banach limits provide the first step in developing another extension of the usual definition of convergence.

DEFINITION. *A real valued function f defined on the bounded real number sequences is a Banach limit provided*

(1) $f(ax + by) = af(x) + bf(y)$,

(2) $f(x) \geq 0$ *if* $x_n \geq 0$, $n = 1, 2, 3, \dots$,

(3) $f(x) = f(Tx)$ *where* $T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$

(4) $f(e) = 1$ *where* $e = (1, 1, \dots)$

for all bounded real sequences $x = (x_n), y = (y_n)$ *and real numbers* a, b.

The existence of Banach limits can be established by a corollary of the Hahn-Banach theorem [1]. G.G. Lorentz used these functionals to give meaning to the phrase "almost convergent to s."

DEFINITION. *A bounded real sequence x is almost convergent to s provided* $f(x) = s$ *for every Banach limit* f.

The notions of Cesaro summability and almost convergence both extend the usual concept of convergence in a non-trivial fashion. Straightforward applications of the definitions yield that $C\text{-}\lim x = \lim x = f(x)$ for every convergent sequence x and every Banach limit f. It can also be readily established (from the definitions) that the sequence $0, 1, 0, 1, \ldots$ is both Cesaro summable and almost convergent to $1/2$.

Lorentz also characterized the almost convergent sequences as being the 'uniformly' Cesaro summable sequences.

THEOREM [4]. *A bounded real sequence $x = (x_n)$ is almost convergent to s if and only if*

$$\lim_k k^{-1} \sum_{i=1}^{k} x_{n+i} = s$$

uniformly with respect to n.

An elegant proof of Lorentz's theorem which also yields the existence of Banach limits is given by G. Bennett and N. Kalton in [2]. Observe that if a sequence is almost convergent to s then it must also be Cesaro summable to s.

We now establish the framework for computing the promised probability.

We let $\Omega = \{0, 1\}^N$, Σ denote the σ -field of subsets generated by the coordinate projections and P denote the natural 'fair coin' probability measure defined on Σ.

Now let (X_n) be the sequence of {0,1}-valued random variables defined on Ω by $X_n(\omega) = \omega_n$; (X_n) is a sequence of independent identically distributed random variables, each with expected value 1/2. Observe that if we set $S_n = \sum_{k=1}^{n} X_k$, the Law of Large Numbers yeilds that

$$P[\omega \in \Omega : \lim_n S_n(w)/n = 1/2] = 1$$

or equivalently

$$P[\omega \in \Omega : \text{C-}\lim_n \omega_n = 1/2] = 1$$

This early version of the law of large numbers was known to Emile Borel [3]. In more conventional language we have established that almost all of the sequences of 0's and 1's are Cesaro summable to 1/2.

Borel's Law of Large Numbers indicates that the Cesaro method is, in the sense of measure, extremely effective on Ω. We now show that the method of almost convergence is not nearly as effective.

THEOREM. *Almost none of the sequences of 0's and 1's are almost convergent.*

PROOF: Borel's theorem together with Lorentz's criterion tells us that almost all of the ω's in Ω that are almost convergent are almost convergent to 1/2.

Lorentz's criterion also tells us that if $\omega \in \Omega$ satisfies the condition that for each $k \geq 2$ there is an n such that

$$X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) = k,$$

then ω is not almost convergent to 1/2. Alternatively, if $\omega \in \Omega$ is almost convergent to 1/2, the there is a $k \geq 2$ for which, regardless of n, we have

$$X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) < k$$

With this in mind, let $k \geq 2$ and define

$$A_k = \bigcap_{n \geq 1} [\omega \in \Omega : X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) < k].$$

Now observe that the independence of the sequence (X_n) implies that of the sequence

$$(X_{nk+1} + \cdots + X_{nk+k})_{n \geq 1};$$

correspondingly, given j

$$\begin{aligned} &P[\bigcap_{n \geq 1}^{j} \omega \in \Omega : X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) < k] \\ &= \prod_{n=1}^{j} P[\omega \in \Omega : X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) < k] \\ &=(1 - 2^{-k})^j \end{aligned}$$

since each event $[\omega \in \Omega : X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) < k]$ has probability $1 - 2^{-k}$. Since

$$A_k \subset \bigcap_{n \geq 1}^{j} [\omega \in \Omega : X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) < k]$$

it follows that $P(A_k) \leq (1 - 2^{-k})^j$ for all j, i.e.,$P(A_k) = 0$, and so $P(\bigcup_{k \geq 2} A_k) = 0$.

Now set $F = \Omega - \cup_{k \geq 2} A_k$ and note that $P(F) = 1$. By construction, if $\omega \in F$ then for each $k \geq 2$ there is an n such that

$$X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) = k,$$

or equivalently,

$$(\omega_{nk+1} + \cdots + \omega_{nk+k})/k = 1.$$

This shows us that ω is not almost convergent to 1/2. Since

$$F \subset \{\omega \in \Omega : \omega \text{ is not almost convergent to } 1/2\},$$

we have established that $P[\omega \in \Omega : \omega \text{ is almost convergent}] = 0$. ∎

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