ALMOST NONE OF THE SEQUENCES OF 0's AND 1's ARE ALMOST CONVERGENT

JEFF CONNOR

Department of Mathematics
Ohio University
Athens, Ohio 45701

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Abstract. We establish that, in the sense of the Law of Large Numbers, almost none of the sequences of 0's and 1's are assigned the same value by every Banach limit.

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The result established in this note is precisely the result promised in the title. To place the theorem in perspective, however, it will be helpful to recall a few definitions and a fundamental result of probability theory.

First we recall an extremely useful extension of the usual notion of convergence. A sequence $x = (x_n)$ is said to be Cesaro summable to s provided $\lim_n n^{-1} \sum_{k=1}^n x_k = s$. If x is Cesaro summable to s, we write C- $\lim_n x = s$.

Banach limits provide the first step in developing another extension of the usual definition of convergence.

DEFINITION. A real valued function f defined on the bounded real number sequences is a Banach limit provided

- (1) f(ax + by) = af(x) + bf(y),
- (2) $f(x) \ge 0$ if $x_n \ge 0$, $n = 1, 2, 3, \ldots$,
- (3) f(x) = f(Tx) where $T(x_1, x_2, x_3, ...) = (x_2, x_3, ...)$
- (4) f(e) = 1 where e = (1, 1, ...)

for all bounded real sequences $x = (x_n), y = (y_n)$ and real numbers a, b.

The existence of Banach limits can be established by a corollary of the Hahn-Banach theorem [1]. G.G. Lorentz used these functionals to give meaning to the phrase "almost convergent to s."

DEFINITION. A bounded real sequence x is almost convergent to s provided f(x) = s for every Banach limit f.

The notions of Cesaro summability and almost convergence both extend the usual concept of convergence in a non-trivial fashion. Straightforward applications of the definitions yield that C- $\lim x = \lim x = f(x)$ for every convergent sequence x and every Banach limit f. It can also be

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readily established (from the definitions) that the sequence $0, 1, 0, 1, \ldots$ is both Cesaro summable and almost convergent to 1/2.

Lorentz also characterized the almost convergent sequences as being the 'uniformly' Cesaro summable sequences.

Theorem [4]. A bounded real sequence $x = (x_n)$ is almost convergent to s if and only if

$$\lim_{k} k^{-1} \sum_{i=1}^{k} x_{n+i} = s$$

uniformly with respect to n.

An elegant proof of Lorentz's theorem which also yields the existence of Banach limits is given by G. Bennett and N. Kalton in [2]. Observe that if a sequence is almost convergent to s then it must also be Cesaro summable to s.

We now establish the framework for computing the promised probability.

We let $\Omega = \{0,1\}^N$, Σ denote the σ -field of subsets generated by the coordinate projections and P denote the natural 'fair coin' probability measure defined on Σ .

Now let (X_n) be the sequence of $\{0,1\}$ -valued random variables defined on Ω by $X_n(\omega) = \omega_n$; (X_n) is a sequence of independent identically distributed random variables, each with expected value 1/2. Observe that if we set $S_n = \sum_{k=1}^n X_k$, the Law of Large Numbers yields that

$$P[\omega \in \Omega: \lim_n S_n(w)/n = 1/2] = 1$$

or equivalently

$$P[\omega \in \Omega : C - \lim_{n} \omega_n = 1/2] = 1$$

This early version of the law of large numbers was known to Emile Borel [3]. In more conventional language we have established that almost all of the sequences of 0's and 1's are Cesaro summable to 1/2.

Borel's Law of Large Numbers indicates that the Cesaro method is, in the sense of measure, extremely effective on Ω . We now show that the method of almost convergence is not nearly as effective.

THEOREM. Almost none of the sequences of 0's and 1's are almost convergent.

PROOF: Borel's theorem together with Lorentz's criterion tells us that almost all of the ω 's in Ω that are almost convergent are almost convergent to 1/2.

Lorentz's criterion also tells us that if $\omega \in \Omega$ satisfies the condition that for each $k \geq 2$ there is an n such that

$$X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) = k,$$

then ω is not almost convergent to 1/2. Alternatively, if $\omega \in \Omega$ is almost convergent to 1/2, the there is a $k \geq 2$ for which, regardless of n, we have

$$X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) < k$$

With this in mind, let $k \geq 2$ and define

$$A_k = \bigcap_{n>1} [\omega \in \Omega : X_{nk+1}(\omega) + \dots + X_{nk+k}(\omega) < k].$$

Now observe that the independence of the sequence (X_n) implies that of the sequence

$$(X_{nk+1} + \cdots + X_{nk+k})_{n \ge 1};$$

correspondingly, given j

$$P[\bigcap_{n\geq 1}^{j} \omega \in \Omega : X_{nk+1}(\omega) + \dots + X_{nk+k}(\omega) < k]$$

$$= \prod_{n=1}^{j} P[\omega \in \Omega : X_{nk+1}(\omega) + \dots + X_{nk+k}(\omega) < k]$$

$$= (1 - 2^{-k})^{j}$$

since each event $[\omega \in \Omega : X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) < k]$ has probability $1 - 2^{-k}$. Since

$$A_k \subset \bigcap_{n>1}^{j} [\omega \in \Omega : X_{nk+1}(\omega) + \dots + X_{nk+k}(\omega) < k]$$

it follows that $P(A_k) \leq (1-2^{-k})^j$ for all j, i.e., $P(A_k) = 0$, and so $P(\bigcup_{k \geq 2} A_k) = 0$.

Now set $F = \Omega - \bigcup_{k \geq 2} A_k$ and note that P(F) = 1. By construction, if $\omega \in F$ then for each $k \geq 2$ there is an n such that

$$X_{nk+1}(\omega) + \cdots + X_{nk+k}(\omega) = k$$
,

or equivalently,

$$(\omega_{nk+1} + \cdots + \omega_{nk+k})/k = 1.$$

This shows us that ω is not almost convergent to 1/2. Since

$$F\subset\{\omega\in\Omega:\omega\text{ is not almost convergent to }1/2\},$$

we have established that $P[\omega \in \Omega : \omega \text{ is almost convergent}] = 0$.

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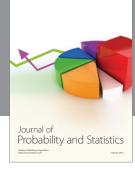
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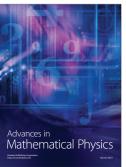


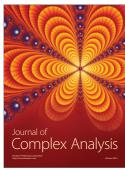




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