

A COMMUTATIVITY THEOREM FOR LEFT s -UNITAL RINGS

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ABSTRACT. In this paper we generalize some well-known commutativity theorems for associative rings as follows: Let R be a left s -unital ring. If there exist non-negative integers $m > 1$, $k > 0$, and $n > 0$ such that for any x, y in R , $[x^k y - x^n y^m, x] = 0$, then R is commutative.

KEY WORDS AND PHRASES. Associative ring, s -unital ring, ring with unity, commutativity of rings.

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1. INTRODUCTION.

Throughout this paper, R denotes an associative ring (may be without unity), $Z(R)$ represents the center of R , N the set of all nilpotent elements of R , N' the set of all zero divisors of R , and $C(R)$ the commutator ideal of R . For any $x, y \in R$, we write $[x, y] = xy - yx$.

As stated in Hirano and Kobayashi [1] and Quadri and Khan [2], a ring R is called left (resp. right) s -unital if $x \in Rx$ (resp. $x \in xR$) for each $x \in R$. Further, R is called s -unital if it is both left as well as right s -unital, that is $x \in Rx \cap xR$, for every $x \in R$. If R is s -unital (resp. left or right s -unital), then for any finite subset F of R , there exists an element $e \in R$ such that $ex = e = xe$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e will be called a pseudo-identity (resp. pseudo left identity or pseudo right identity) of F in R .

The famous Jacobson theorem stated that any ring R in which for every $x \in R$ there exists a positive integer $n = n(x) > 1$ such that $x^n = x$ is commutative, has been generalized as follows: if for each pair $x, y \in R$ there exists a positive integer $n = n(x, y) > 1$ such that $(xy)^n = xy$, then R is commutative. Recently, Ashraf and Quadri [3] investigated the commutativity of the rings satisfying the following condition: For all $x, y \in R$ there is a fixed integer $n > 1$ such that $x^n y^n = xy$. In fact, Ashraf and Quadri [3] have generalized the above results as follows: Let R be a ring with unity 1 in which $[xy - x^n y^m, x] = 0$, for all x, y in R and fixed integers $m > 1$, $n > 1$. Then R is commutative.

The objective of this paper is to generalize the above mentioned results. Indeed, we prove the following:

THEOREM 1.1. Let R be a left s -unital ring with the property that

- (P) "there exist positive integers $m > 1$, $k > 0$, and $n > 0$ such that $[x^k y - x^n y^m, x] = 0$ for all $x, y \in R$ ".

Then R is commutative.

We notice that the property (P) of the above theorem can be rewritten as follows:

$$x^k [x, y] = x^n [x, y^m]. \quad (1.1)$$

Thus for any integer $t > 1$, we have

$$\begin{aligned} x^{tk} [x, y] &= x^{(t-1)k} (x^k [x, y]) \\ &= x^{(t-1)k} (x^n [x, y^m]) \\ &= x^{(t-2)k} (x^n x^k [x, y^m]) \\ &= x^{(t-2)k} (x^{2n} [x, y^{m^2}]) \\ &= \dots \end{aligned}$$

By repeating the above process and using (1.1), we get

$$x^{tk} [x, y] = x^{tn} [x, y^{m^t}]. \quad (1.2)$$

2. PRELIMINARY LEMMAS.

In preparation for the proof of the above theorem we start by stating without proof the following well-known Lemmas.

LEMMA 2.1 (Bell [4, Lemma]). Suppose x and y are elements of a ring R with unity 1, satisfying $x^m y = 0$ and $(1+x)^m y = 0$ for some positive integer m . Then $y = 0$.

LEMMA 2.2. (Bell [5, Lemma 3]). Let x and y be in R . If $[x, y]$ commutes with x , then $[x^k, y] = k x^{k-1} [x, y]$ for all positive integers k .

LEMMA 2.3 ([2, Lemma 3]). Let R be a ring with unity 1. If $(1 - y^k)x = 0$, then $(1 - y^{km}) x = 0$, for any positive integers m and k .

LEMMA 2.4 ([1, Proposition 2]). Let f be a polynomial in non-commuting indeterminates x_1, x_2, \dots, x_n with integer coefficients. Then the following statements are equivalent:

- 1) For any ring R satisfying $f = 0$, $C(R)$ is a nil ideal.
- 2) Every semiprime ring satisfying $f = 0$ is commutative.
- 3) For every prime p , $(GF(p))_2$ fails to satisfy $f = 0$.

3. MAIN RESULTS.

The following lemmas will be used in the proof our main theorem.

LEMMA 3.1. Let R be a left s -unital ring satisfying $[x^k y - x^n y^m, x] = 0$, for each $x, y \in R$ and any non-negative integers k, n and $m > 1$. Then R is s -unital.

PROOF. Let $u \in N$. Then for any $x \in R$, and $t > 1$, we have $x^{tk} [x, u] = x^{tn} [x, u^{m^t}]$.

For sufficiently large t , we have $x^{tk}[x,u] = x^{tn}[x,u^m]^t = 0$, since u is nilpotent and $u^m = 0$.

Since, R is a left s -unital ring, we have $u = eu$ for some $e \in R$. But $e^{tk}[e,u] = 0$ which gives $u = ue$. For arbitrary $x \in R$, there exists $e' \in R$ such that $e'x = x$. Further, for some $e'' \in R$, we have $e''e' = e'$. Thus $e''x = x$ and $(x - xe'')^2 = 0$, that is $(x - xe'') \in N$. Since $e'(x - xe'') = x - xe''$, we have $x - xe'' = (x - xe'')e' = 0$ which implies $x = xe''$. Hence R is s -unital.

LEMMA 3.2. Let R be a ring with unity 1 which satisfies the property (P). Then every nilpotent element of R is central.

PROOF. Let u be a nilpotent element of R . Then by (1.2) for any $x \in R$ and a positive integer $t > 1$ we have $x^{tk}[x,u] = x^{tn}[x,u^m]^t$. But $u \in N$, then $u^m = 0$, for sufficiently large t , and hence $x^{tk}[x,u] = 0$ for each $x \in R$. By Lemma 2.1 this yields $[x,u] = 0$, which forces $N \subseteq Z(R)$. Thus every nilpotent element of R is central.

LEMMA 3.3. Let R be a ring with unity 1 which satisfies the property (P), then $C(R) \subseteq Z(R)$.

PROOF. Now, R satisfies $[x^k y - x^n y^m, x] = 0$ for all $x, y \in R$, which is a polynomial identity with relatively prime integral coefficients. Let $x = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we find that no ring of 2×2 matrices over $GF(p)$, p a prime, satisfies the above polynomial identity. Hence by Lemma 2.4, the commutator ideal $C(R)$ of R is nil. Therefore $C(R) \subseteq Z(R)$.

In view of Lemma 3.3 it is guaranteed that the conclusion of Lemma 2.2 holds for each pair of elements x, y in a ring R with unity 1 which satisfies the property (P).

LEMMA 3.4. Let R be a ring with unity 1 , satisfying (P), then R is commutative.

PROOF. Since R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i each of which as a homomorphic image of R satisfies the property (P) placed on R , R itself can be assumed to be a subdirectly irreducible ring. Let S be the intersection of all its non-zero ideals, then $S \neq (0)$.

Let $k = n = 0$, in (1.1). Then we have $[x, y] = [x, y^m]$ or $[x, y - y^m] = 0$ for all $x, y \in R$. This forces commutativity of R by Herstein [6, Theorem 18]. Next, we assume $k = n = 1$ in (1.1). Then replacing x by $(x + 1)$, we obtain $[x, y] = [x, y^m]$, for every $x, y \in R$, and again by [6, Theorem 18] R is commutative. If $(k, n) = (1, 0)$, then $x[x, y] = [x, y^m]$ and hence by replacing x by $(x + 1)$ we have $[x, y] = 0$, for all $x, y \in R$. Therefore R is commutative. If $(k, n) = (0, 1)$, then $[x, y] = x[x, y^m]$, and hence by replacing x by $x + 1$ we have $[x, y^m] = 0$, for all $x, y \in R$. Thus $[x, y] = x[x, y^m] = 0$ for all $x, y \in R$. Thus R is commutative.

Next, we suppose that $k > 1$, and $n > 1$. Let $q = 2^m - 2$ be a positive integer. Then by (1.1) we have

$$\begin{aligned} q x^k [x, y] &= 2^m x^k [x, y] - 2 x^k [x, y] \\ &= 2^m x^n [x, y^m] - x^k [x, 2y] \\ &= x^n [x, (2y)^m] - x^n [x, (2y)^m] \\ &= 0, \end{aligned}$$

that is, $qx^k [x, y] = 0$. By replacing x by $(x + 1)$ and using Lemma 2.1, this yields $q[x, y] = 0$ for all $x, y \in R$. Now combining Lemma 3.3 with Lemma 2.2, we get $[x^q, y] = q x^{q-1} [x, y] = 0$ which yields

$$x^q \in Z(R) \text{ for all } x, y \in R. \quad (3.1)$$

Replacing y by y^m in (1.1), we get

$$x^k [x, y^m] = x^n [x, (y^m)^m]. \quad (3.2)$$

By applying Lemma 3.3 and Lemma 2.2, we obtain

$$\begin{aligned} x^k [x, y^m] &= [x, y^m] x^k \\ &= m y^{m-1} [x, y] x^k \\ &= m y^{m-1} x^k [x, y] \\ &= m y^{m-1} x^n [x, y^m] \\ &= m y^{m-1} [x, y^m] x^n \end{aligned}$$

and, using similar techniques, we get

$$\begin{aligned} x^n [x, (y^m)^m] &= [x, (y^m)^m] x^n \\ &= m (y^m)^{m-1} [x, y^m] x^n \\ &= m y^{m^2 - m} [x, y^m] x^n \\ &= m y^{m-1} y^{(m-1)^2} [x, y^m] x^n. \end{aligned}$$

Thus (3.2) gives

$$m y^{m-1} (1 - y^{(m-1)^2}) [x, y^m] x^n = 0. \quad (3.3)$$

Again the usual argument of replacing x by $(x + 1)$ in (3.3) and applying Lemma 2.1

yields $m y^{m-1} (1 - y^{(m-1)^2}) [x, y^m] = 0$. Then by Lemma 3.3 and Lemma 2.3 we have

$$m y^{(m-1)} (1 - y^{q(m-1)^2}) [x, y^m] = 0. \quad (3.4)$$

Next, we claim that $N' \subseteq Z(R)$. Let $a \in N'$, then by (3.1) $a^{q(m-1)^2} \in N' \cap Z(R)$,

and $S a^{q(m-1)^2} = (0)$. Since by (3.4), $m a^{(m-1)} (1 - a^{q(m-1)^2}) [x, a^m] = 0$, that

is, $(1 - a^{q(m-1)^2}) m a^{m-1} [x, a^m] = 0$.

Now, if $m a^{m-1} [x, a^m] \neq 0$, then $(1 - a^{q(m-1)})^2 \in N'$, and so $S(1 - a^{q(m-1)})^2 = 0$ which leads to the contradiction that $S = (0)$. Hence $m a^{m-1} [x, a^m] = 0$. From (1.1) and using Lemma 2.2 repeatedly we get

$$\begin{aligned} x^{2k} [x, a] &= x^k (x^k [x, a^m]) \\ &= x^k (x^n [x, a^m]) \\ &= x^n (x^k [x, a^m]) \\ &= x^{2n} [x, (a^m)^m] \\ &= x^{2n} m (a^m)^{m-1} [x, a^m] \\ &= x^{2n} m a^{m-1} a^{(m-1)^2} [x, a^m] \\ &= x^{2n} a^{(m-1)^2} m a^{m-1} [x, a^m] \\ &= 0. \end{aligned}$$

This implies that $x^{2k} [x, a] = 0$, and so the usual argument of replacing x by $(x + 1)$ and using Lemma 2.1 gives $[x, a] = 0$, and hence,

$$N' \subseteq Z(R). \tag{3.5}$$

Now, for any $x \in R$, x^q and x^{qm} are in $Z(R)$. Then by (1.1) for any $y \in R$, we have

$$\begin{aligned} (x^q - x^{qm}) x^k [x, y] &= x^q (x^k [x, y]) - x^{qm} (x^k [x, y]) \\ &= x^k (x^q [x, y]) - x^{qm} x^n [x, y^m] \\ &= x^k [x, x^q y] - x^n [x, (x^q y)^m] \\ &= x^k [x, x^q y] - x^k [x, x^q y]. \end{aligned}$$

Therefore $(x^q - x^{qm}) x^k [x, y] = 0$, and hence

$$(x - x^{qm-q+1}) x^{k+q-1} [x, y] = 0. \tag{3.6}$$

If R is not commutative then by [6, Theorem 18], there exists an element $x \in R$ such that $(x - x^t) \notin Z(R)$, where $t = qm - q + 1$. This also reveals $x \notin Z(R)$. Thus neither $(x - x^t)$ nor x is a zero divisor, and so $(x - x^t) x^{k+q-1} \notin N'$. Hence (3.6) forces that $[x, y] = 0$, for all $x, y \in R$. Thus $x \in Z(R)$ which is a contradiction. Hence R is commutative.

PROOF OF THE THEOREM. Let R be a left s -unital ring satisfying (P), then by Lemma 3.1, R is s -unital. Therefore, in view of [1, Proposition 1] and Lemma 3.4, R is commutative, if R with 1 satisfying (P) is commutative.

COROLLARY 3.1 ([3, Theorem]). Let R be a ring with unity 1 in which $[xy - x^n y^m, x] = 0$ for all $x, y \in R$ and fixed integers $m > 1, n > 1$. Then R is commutative.

PROOF. Actually, R satisfies the polynomial identity $x[x, y] = x^n [x, y^m]$ for all $x, y \in R$ and fixed integers $m > 1, n > 1$. Put $k = 1$ in (1.1), then R is commutative by Lemma 3.4.

COROLLARY 3.2 (Hirano, Kobayashi, and Tominaga [7, Theorem]). Let m, k be fixed non-negative integers. Suppose that R satisfies the polynomial identity

$$x^k [x, y] = [x, y^m].$$

(a) If R is a left s -unital, then R is commutative except when $(m, k) = (1, 0)$.

(b) If R is a right s -unital, then R is commutative except when $(m, k) = (1, 0)$, and $m = 0, k > 0$.

REMARK 3.1. ([7]). In case $k > 0$ and $m = 0$ in Corollary 3.2(b), R need not be commutative. For, let K be a field. Then the non-commutative ring

$R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in K \right\}$ has a right identity element and satisfies the polynomial identity $x[x, y] = 0$.

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