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EXAMPLE OF A SEQUENTIALLY INCOMPLETE REGULAR INDUCTIVE LIMIT OF BANACH SPACES

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ABSTRACT. A sequentially incomplete regular inductive limit of a sequence of Banach spaces is constructed.

KEY WORDS AND PHRASES. Regular locally convex inductive limit, sequentially complete locally convex space.

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1. INTRODUCTION.

In [1] Jorge Mujica asks: Is every regular inductive limit of Banach spaces complete? A partial answer is in [2] with an example of a regular inductive limit which is quasi-incomplete. It is conjectured in [2] that the regular inductive limit might even be sequentially incomplete. Here we prove the conjecture. On the other hand, regular inductive limits of Banach spaces always have some completeness property, e.g. they are fast complete, see [3].

2. MAIN RESULT.

Let $N = \{1, 2, \dots\}$ and R be the space of real numbers. For each $x = \{x_{i,j}\} \in R^{N\times N}$ and $n \in N$ we put $\| x \|_{i,n} = \max\{\frac{1}{i}\max\{|x_{i,j}|; j < n\}, \sup\{|x_{i,j}|; j \geq n\}\}$,
$\| x \|_n = \sup\{\| x \|_{i,n}; i \in N\}$, and $E_n = \{x \in R^{N\times N}; \| x \|_n < \infty, \lim_{i\to\infty} \| x \|_{i,n} = 0\}$.

For brevity, we also put $E_n(r) = \{x \in E_n; \| x \|_n \leq r\}$.

LEMMA 1. The map $x \mapsto \| x \|_n$: $E_n \to R$ is a norm on E_n and each functional $f_{i,j} : x \mapsto x_{i,j} : E_n \to R$ is continuous.

PROOF. $|f_{i,j}(x)| = |x_{i,j}| \leq i \| x \|_{i,n} \leq i \| x \|_n$.

LEMMA 2. Each E_n is Banach.

PROOF. Let $\{x(k); k \in N\}$ be Cauchy. Then for each $i, j \in N$, the sequence $\{f_{i,j}(x(k)); k \in N\}$ is Cauchy in R and thus converges to some $x_{i,j} \in R$. Put $x = \{x_{i,j}\}$. For any $i, k, n \in N$, we have $\lim_{m\to\infty} \| x(k) - x(m) \|_{i,n} = \| x(k) - x \|_{i,n}$, which implies $x \in E_n$ and $\lim_{k\to\infty} \| x(k) - x \|_n = 0$.

LEMMA 3. The inductive limit $E = \operatorname{ind\,lim}_{n\to\infty} E_n$ is regular.

PROOF. Let $B \subset E$ be not bounded in any E_n. Without a loss of generality we may assume that for any $n \in N$ there exists $x(n) \in B$ such that $\| x(n) \|_n > n$. This implies the existence of sequences $\{i(n)\}, \{j(n)\} \subset N$ such that: either $j(n) \geq n$ and $| x(n)_{i(n),j(n)} | > n$ or $j(n) < n$ and $| x(n)_{i(n),j(n)} | > i(n) \cdot n$.

For each $k \in N$, choose $r_k > 0$ so that $r_k^{-1} = \max\{i(n); n \leq k\}$ and denote by V the convex hull of $\cup\{E_k(r_k); k \in N\}$. Take $k, n \in N, x = \{x_{i,j}\} \in E_k(r_k)$, and distinguish three cases:

(a) $j(n) \geq k$, which implies $| x_{i(n),j(n)} | \leq \| x \|_k \leq r_k \leq 1 < \frac{1}{n} | x(n)_{i(n),j(n)} |$,

(b) $j(n) < k$ & $k \geq n$, which implies $| x_{i(n),j(n)} | \leq i(n) \cdot \| x \|_k \leq i(n) r_k \leq 1 < \frac{1}{n} | x(n)_{i(n),j(n)} |$,

(c) $j(n) < k$ & $k < n$, which implies $| x_{i(n),j(n)} | \leq i(n) \cdot \| x \|_k \leq i(n) r_k \leq i(n) < \frac{1}{n} | x(n)_{i,(n),j(n)} |$.

For any case $| x_{i(n),j(n)} | < \frac{1}{n} | x(n)_{i(n),j(n)} |$. Since $x \in E_k(r_k), k \in N$, was arbitrary, the element $\frac{1}{n}x(n)$ cannot be expressed as a convex combination of elements from $\cup\{E_k(r_k); k \in N\}$, i.e. $\frac{1}{n}x(n) \notin V$. Hence the 0-neighborhood V in E does not absorb B and B is not bounded in E.

LEMMA 4. For each $i, j, n \in N$, put $x(n)_{i,j} = j^{-1}$ if $i \leq n, x(n)_{i,j} = 0$ if $i > n$, and $x(n) = \{x(n)_{i,j}\}$. Then:

(a) For each $n \in N, x(n) \in E_1(1)$.
(b) $\{x(n)\}$ is Cauchy in E.
(c) $\{x(n)\}$ does not converge in E.

PROOF.

(a) is evident.

(b) Let V be a 0-neighborhood in E. For each $n \in N$, choose $r_n > 0$ so that $E_n(r_n) \subset V$. Further, choose $p, q \in N$ so that $pr_1 > 2$ and $qr_p > 2$. For $m, n > q$, define $y, z \in E$ by:

$$y_{i,j} = x(m)_{i,j} - x(n)_{i,j} \text{ for } j \geq p, y_{i,j} = 0 \text{ otherwise,}$$

$$z_{i,j} = x(m)_{i,j} - x(n)_{i,j} \text{ for } j < p, z_{i,j} = 0 \text{ otherwise.}$$

Since

$$| x(m)_{i,j} - x(n)_{i,j} | \leq \left\{ \begin{array}{lll} 0 & \text{for} & i \leq q \\ \frac{1}{p} & \text{for} & i > q, j \geq p \\ 1 & \text{for} & i > q, j < p \end{array} \right\},$$

We have $\| y \|_1 = \sup\{| y_{i,j} |; i \in N, j \geq p\} \leq \frac{1}{p} < \frac{1}{2}r_1$, and $\| z \|_p = \sup\{| \frac{1}{i} z_{i,j} |; i > q, j < p\} < \frac{1}{q} < \frac{1}{2}r_p$. Hence $y \in \frac{1}{2}E_1(r_1), z \in \frac{1}{2}E_p(r_p)$, and $x(m) - x(n) = y + z \in V$.

(c) Assume $x(n) \to x$ in E. Since each functional $f_{i,j}$, defined in Lemma 1, is continuous on each E_n, it is also continuous on E. Then we have $x_{i,j} = f_{i,j}(x) = \lim_{n\to\infty} f_{i,j}(x(n)) = \lim_{n\to\infty} x(n)_{i,j} = j^{-1}$. Hence $\| x \|_{i,n} = \max(\frac{1}{i}, \frac{1}{n})$ and $\lim_{i\to\infty} \| x \|_{i,n} = \frac{1}{n} \neq 0$, i.e. $x \notin E_n$ for any $n \in N$.

By combining Lemma to 1-4, we get:

THEOREM. Regular inductive limit of Banach spaces may be sequentially incomplete.

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