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SOLUTION TO A PARABOLIC EQUATION WITH INTEGRAL TYPE BOUNDARY CONDITION

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ABSTRACT In this paper we study the existence, and continuous dependence of the solution $\vartheta = \vartheta(x,t)$ on a Hölder space $H^{2+\gamma,1+\gamma/2}(\overline{Q}_\tau)(\overline{Q}_\tau = [0,1] \times [0,\tau], \quad 0 < \gamma < 1)$ of a linear parabolic equation, prescribing $\vartheta(x,0) = f(x), \vartheta_x(1,\tau) = g(\tau)$ the integral type condition $\int_0^b \vartheta(x,\tau)dx = E(\tau)$.

KEY WORD AND PHRASES. Parabolic equation, integral boundary condition.
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1. INTRODUCTION.

Consider the problem of finding $\vartheta = \vartheta(x,\tau)$ such that

$$\vartheta_\tau = (r(x,\tau)\vartheta_x)_x, \quad 0 < x < 1, \quad 0 < \tau \leq \mathcal{T}, \tag{1.1}$$

$$\vartheta_x(1,\tau) = g(\tau), \qquad 0 \leq \tau \leq \mathcal{T}, \tag{1.2}$$

$$\vartheta(x,0) = f(x), \qquad 0 \leq x \leq 1, \tag{1.3}$$

$$\int_0^b \vartheta(x,\tau)dx = E(\tau), \qquad 0 \leq \tau \leq \mathcal{T}, \tag{1.4}$$

with $E(0) = \int_0^b f(x)dx$, for b fixed with $0 < b < 1$ and $r(x,\tau) \geq r_0 > 0$ on $[0,1] \times [0,\mathcal{T}]$.

In Cannon, Yanpin Lin [1] it is proved a result on existence, uniqueness and continuous dependence for this problem. In this paper we give conditions for which the solution of (1.1)-(1.4) belongs to a Hölder space and we prove that this solution depends continuously upon the data with respect to the corresponding Hölder norms. Similar problems are considered in [2,3,5,6,8,9,10].

Notice that function ϑ satisfies (1.1)-(1.4) if and only if $u(x,t) = \vartheta(x,\tau)$, with $t = \int_0^\tau \frac{ds}{r(b,s)}$, satisfies

$$\begin{aligned} u_t &= u_{xx} + \left[\left(\frac{a(x,t)-a(b,t)}{a(b,t)}\right)u_x\right]_x \\ &= \left(\frac{a(x,t)}{a(b,t)}u_x\right)_x, \quad 0 < x < 1, \quad 0 < t \le T, & (1.5) \\ u_x(1,t) &= \tilde{g}(t), \qquad 0 < t \le T, & (1.6) \\ u(x,0) &= f(x), \qquad 0 \le x \le 1, & (1.7) \\ \int_0^b u(x,t)dx &= \tilde{E}(t), \qquad 0 \le t \le T, & (1.8) \end{aligned}$$

where $\tilde{E}(t) = E(\tau)$, $\tilde{g}(t) = g(\tau)$, $a(x,t) = r(x,\tau)$, $T = \int_0^T \frac{ds}{r(b,s)}$, and $\tilde{E}(0) = E(0) = \int_a^b f(x)dx$.

(A) and (B) will denote problems (1.1)-(1.4) and (1.5)-(1.8), respectively. The results on existence, uniqueness and continuous dependence will be based on a standard fixed point argument for a contraction defined on a subset of an appropriate functional space. We shall follow Ladyzenskaja et al. [11] to define the spaces of Hölder continuous functions:

Let $Q_T = (0,1) \times (0,T)$, $\overline{Q}_T = [0,1] \times [0,T]$. For $M > 0$, $k = 0,1,2$ and $0 < \gamma < 1$, $H^{k+\gamma}[0,M]$ shall denote the spaces of functions $h = h(t)$ in $[0,M]$, with $\|h\|_M^{(k+\gamma)} < \infty$; where

$$\|h\|_M^{(k+\gamma)} = \sum_{n=0}^{k} \|h^{(n)}\|_M + \|h^{(k)}\|_M^{(\gamma)},$$

$$\|h\|_M = \sup_{t\in[0,M]} |h(t)|,$$

$$\|h\|_M^{(\gamma)} = |h(0)| + \sup_{t,t'\in[0,M]} \frac{|h(t)-h(t')|}{|t-t'|^\gamma},$$

where $h^{(n)}$ denotes the derivative of h of order n.

For $u : \overline{Q}_T \to \mathbf{R}$, let

$$H_{x,\gamma}^T(u) = \sup_{\substack{x,x'\in[0,1] \\ t\in[0,T]}} \frac{|u(x,t)-u(x',t)|}{|x-x'|^\gamma}$$

$$H_{t,\gamma}^T(u) = \sup_{\substack{x\in[0,1] \\ t,t'\in[0,T]}} \frac{|u(x,t)-u(x,t')|}{|t-t'|^\gamma}$$

$$\|u\|_{Q_T} = \sup_{(x,t)\in Q_T} |u(x,t)|$$

Then $H^{\gamma,\gamma/2}(\overline{Q}_T)$ and $H^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$ will denote the space of all functions $u : \overline{Q}_T \to \mathbf{R}$ such that

$$\|u\|_T^{\gamma,\gamma/2} = \|u\|_{Q_T} + H_{x,\gamma}^T(u) + H_{t,\gamma/2}^T(u) < \infty$$

and

$$\begin{aligned}\|u\|_T^{2+\gamma,1+\gamma/2} &= \|u\|_{Q_T} + \|u_x\|_{Q_T} + \|u_{xx}\|_{Q_T} + \|u_t\|_{Q_T} \\ &+ H^T_{t,\frac{\gamma+1}{2}}(u_x) + H^T_{x,\gamma}(u_t) + H^T_{x,\gamma}(u_{xx}) < \infty,\end{aligned}$$

respectively.

$K = K(x,t)$ will denote the fundamental solution to the heat equation

$$K(x,t) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{x^2}{4t}},\ x \in \mathbf{R},\ t > 0,$$

and $\theta = \theta(x,t)$ shall be the Theta function

$$\theta(x,t) = \sum_{m=-\infty}^{\infty} K(x+2m,t), \text{(see [4])}.$$

2. EXISTENCE, UNIQUENESS AND CONTINUOUS DEPENDENCE.

DEFINITION. A function $u(x,t)$ on $\overline{Q}_T$ is called a solution of problem (B), if

1) u and u_x are continuous in $\overline{Q}_T$,
2) u_{xx} is bounded in $\overline{Q}_T$,
3) u satisfies (1.5)-(1.8).

We notice that if u is such that u_x is continuous in $\overline{Q}_T$ and satisfies (1.5)-(1.7), then u is a solution of problem (B) if and only if

$$a(b,t)\tilde{E}'(t) = a(b,t)u_x(b,t) - a(0,t)u_x(0,t) \tag{2.1}$$

or

$$E'(\tau) = r(b,\tau)\vartheta_x(b,\tau) - r(0,\tau)\vartheta_x(0,\tau), \tag{2.2}$$

for $0 \le \tau \le \mathcal{T},\ 0 \le t \le T$, provided E is differentiable.

We shall assume the following compatibility hypothesis:

H1) $\tilde{g}(0) = f'(1)$,

H2) $a(b,0)\tilde{E}'(0) = a(b,0)f'(b) - a(0,0)f'(0)$, and the regularity conditions:

R1) $\tilde{E} \in H^{1+(\frac{1+\gamma}{2})}[0,T], \tilde{g} \in H^{\frac{1+\gamma}{2}}[0,T], f \in H^{2+\gamma}$,

R2) $a, a_x, a_{xx} \in H^{\gamma,\gamma/2}(\overline{Q}_T)$ and $H^T_{x,\delta}(a_t) < \infty$ for some $\delta > 0$.

Let $V_T = \{\varphi \in H^{(\frac{1+\gamma}{2})}[0,T] : \varphi(0) = f'(0)\}$. We define a nonlinear operator $\mathcal{F} : V_T \to V_T$ as follows: For $\varphi \in V_T$, let u^φ be the unique solution in $H^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$ of (1.5)-(1.7), with $u_x(0,t) = \varphi(t)$, (cf [11], Theorem 5.3 p. 320). Then we define

$$\mathcal{F}\varphi(t) = \frac{a(b,t)}{a(0,t)}(u_x^\varphi(b,t) - \tilde{E}'(t)).$$

Since $u^\varphi \in H^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$ and (H2) holds, we have $\mathcal{F}\varphi \in V_T$, furthermore, if φ is a fixed point of $\mathcal{F}$ then u^φ is a solution of problem (B) and conversely.

LEMMA 2.1. There exists $\epsilon > 0$ not depending on $f, \tilde{g}, \tilde{E}$, such that if $0 < T^* < \epsilon$ then

a) $\|\mathcal{F}\varphi - \mathcal{F}\psi\|_{T^*} \leq \frac{1}{2}\|\varphi - \psi\|_{T^*}, \quad \varphi, \psi \in V_T,$

b) $\|\mathcal{F}\varphi - \mathcal{F}\psi\|_{T^*}^{(\frac{1+\lambda}{2})} \leq \frac{1}{2}\|\varphi - \psi\|_{T^*}^{(\frac{1+\lambda}{2})}, \varphi, \psi \in V_T.$

PROOF. Let $T^* \leq T$, φ and ψ in V_{T^*}, $h = \varphi - \psi$ and $w = u^\varphi - u^\psi$. Then

$$
\begin{aligned}
w(x,t) &= -2\int_0^t \theta(x, t-\tau)h(\tau)d\tau + \\
&\quad \int_0^t\int_0^1 \{\theta(x-\xi, t-\tau) + \theta(x+\xi, t-\tau)\}F(\xi,\tau)d\xi\, d\tau, \qquad (2.3)
\end{aligned}
$$

with $F(x,t) = (\frac{a(x,t)-a(b,t)}{a(b,t)}w_x)_x$ (cf [4] p. 339).

It follows that for $t \in [0, T^*]$,

$$
\begin{aligned}
w_x(b,t) &= -2\int_0^t \theta_x(b, t-\tau)h(\tau)d\tau \\
&\quad + \int_0^t\int_0^1 \theta_x(b+\xi, t-\tau)F(\xi,\tau)d\xi\, d\tau + \int_0^t\int_0^1 \theta_x(b-\xi, t-\tau)F(\xi,\tau)d\xi d\tau \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

We clearly have

$$|I_1| \leq 2\|\varphi - \psi\|_{T^*}\int_0^{T^*} |\theta_x(b,\tau)|d\tau \leq C_1 T^*\|h\|_{T^*}.$$

Since term by term differentiation of the series in I_2 is possible, then we have

$$
\begin{aligned}
I_2 &= \int_0^t\int_0^1 \theta_x(b+\xi, t-\tau)\left(\frac{a(\xi,\tau)-a(b,\tau)}{a(b,\tau)}w_\xi(\xi,\tau)\right)_\xi d\xi d\tau \\
&= -\int_0^t \theta_x(b, t-\tau)\left(\frac{a(0,\tau)-a(b,\tau)}{a(b,\tau)}\right)w_\xi(0,\tau)d\tau \\
&\quad -\int_0^t\int_0^1 \theta_{xx}(b+\xi, t-\tau)\left(\frac{a(\xi,\tau)-a(b,\tau)}{a(b,\tau)}\right)w_\xi(\xi,\tau)d\xi\, d\tau.
\end{aligned}
$$

Condition (R2) implies that equation (1.5) (satisfied by w) can be differentiated (see [7, Sec. 3.5]) and then w_x satisfies a linear parabolic equation. Thus, by the weak maximun principle it follows that

$$\|w_x\|_{\overline{Q}_{T^*}} \leq e^{MT^*}\|\varphi - \psi\|_{T^*} = e^{MT^*}\|h\|_{T^*}, \text{ where } M = \sup_{\overline{Q}_T}\left|\left(\frac{a(x,t)}{a(b,t)}\right)_{xx}\right|,$$

(cf. [7, Th. 2.3.8]).

Then $|I_2| \leq C_2 T^*\|h\|_{T^*}$. Finally, if we write $\theta(x,t) = K(x,t) + H(x,t)$, with $H(x,t) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} K(x+2m,t)$, then

$$
\begin{aligned}
I_3 &= \int_0^t\int_0^1 H_x(b-\xi, t-\tau)F(\xi,\tau)d\xi\, d\tau \\
&\quad + \int_0^t\int_0^1 K_x(b-\xi, t-\tau)F(\xi,\tau)d\xi\, d\tau \\
&= J_1 + J_2.
\end{aligned}
$$

J_1 can be estimated just as I_2, to obtain

$$|J_1| \leq C_3 T^* \|h\|_{T^*} \text{ for } t \leq T.$$

To estimate J_2 we have to take case of the singularity of $K(x,t)$ at $(0,0)$.
Since $\left|\frac{a(\xi,\tau)-a(b,\tau)}{a(b,\tau)}\right| \leq C_4|\xi - b|$, then integrating by parts as before, we have

$$\begin{aligned} |J_2| &\leq \int_0^t |K_x(,t-\tau)(\frac{a(0,\tau)-a(b,\tau)}{a(b,\tau)})w_\xi(0,\tau)|d\tau \\ &\quad + C_5\|h\|_{T^*} \int_0^t \int_0^1 |K_{xx}(b-\xi,t-\tau)(\xi - b)|d\xi d\tau \\ &\leq C_6(T^* + T^{*1/2})\|h\|_{T^*}. \end{aligned}$$

Hence $|w_x(b,t)| \leq |I_1|+|I_2|+|I_3| \leq CT^{*1/2}\|h\|_{T^*}, t \leq T^*$, where C depends on T, b and function $a(x,t)$. From this (a) follows immediately.

Now we estimate $\|w_x(b,\cdot)\|_{T^*}^{(\frac{1+\gamma}{2})}$:

For $t < s$ we have

$$\begin{aligned} w_x(b,s) - w_x(b,t) &= -2\int_0^t \theta_x(b,\tau)(h(s-\tau) - h(t-\tau))d\tau \\ &\quad - 2\int_t^s \theta_x(b,\tau)h(s-\tau)d\tau \\ &\quad + \int_0^t \int_0^1 \theta_x(b+\xi,\tau)(F(\xi,s-\tau) - F(\xi,t-\tau))d\xi d\tau \\ &\quad + \int_t^s \int_0^1 \theta_x(b+\xi,\tau)F(\xi,s-\tau)d\xi d\tau \\ &\quad + \int_0^t \int_0^1 H_x(b-\xi,\tau)(F(\xi,s-\tau) - F(\xi,t-\tau))d\xi d\tau \\ &\quad + \int_t^s \int_0^1 H_x(b-\xi,\tau)F(\xi,s-\tau)d\xi d\tau \\ &\quad + \int_0^t \int_0^1 K_x(b-\xi,\tau)(F(\xi,s-\tau) - F(\xi,t-\tau))d\xi d\tau \\ &\quad + \int_t^s \int_0^1 K_x(b-\xi,\tau)F(\xi,s-\tau)d\xi d\tau \\ &= L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8. \end{aligned}$$

We claim that

$$|L_i| \leq M_i T^* \|h\|_{T^*}^{(\frac{1+\gamma}{2})}|s-t|^{\frac{1+\gamma}{2}}, \quad i = 1,\dots,6, \tag{2.4}$$

$$|L_7| \leq M_7 T^{*\delta/2} \|h\|_{T^*}^{(\frac{1+\gamma}{2})}|s-t|^{\frac{1+\gamma}{2}}, \tag{2.5}$$

$$|L_8| \leq M_8 T^{*\frac{1}{2}} \|h\|_{T^*}^{(\frac{1+\gamma}{2})}|s-t|^{\frac{1+\gamma}{2}}, \tag{2.6}$$

where M_i depends on T, b and function $a(x,t)$, $i = 1,\dots,8$.

The proof of (2.4) follows as the proof of part (a). For (2.5) we let $c(x,t) = \frac{a(x,t)-a(b,t)}{a(b,t)}$, then

$$L_7 = -\int_0^t K_x(b,\tau)(c(0,s-\tau)w_x(0,s-\tau) - c(0,t-\tau)w_x(0,t-\tau))d\tau$$

$$+\int_0^t\int_0^1 K_{xx}(b-\xi,\tau)c(\xi,s-\tau)(w_x(\xi,s-\tau)-w_x(\xi,t-\tau))d\xi\,d\tau$$

$$+\int_0^t\int_0^1 K_{xx}(b-\xi,\tau)(w_x(\xi,t-\tau)(c(\xi,s-\tau)-c(\xi,t-\tau))d\xi\,d\tau$$

$$= J_1+J_2+J_3.$$

Since $c(\xi,t)=\mathcal{O}(|\xi-b|)$, we obtain

$$|J_1| \leq K_1T^* \,\|h\|_{T^*}^{(\frac{1+\gamma}{2})}|t-s|^{\frac{1+\gamma}{2}} \tag{2.7}$$

$$|J_2| \leq K_2T^{*1/2}\,\|w\|_{T^*}^{2+\gamma,1+\gamma/2}|t-s|^{1+\gamma}, \tag{2.8}$$

and by (R2),

$$J_3=\int_0^t\int_0^1|\xi-b|^\delta K_{xx}(b-\xi,\tau)w_x(\xi,t-\tau)\int_t^s\frac{\partial}{\partial r}\frac{c(\xi,r-\tau)}{|\xi-b|^\delta}d\xi\,d\tau.$$

Hence

$$|J_3|\leq K_3T^{*\delta/2}\|w\|_{T^*}^{2+\gamma,1+\gamma/2}|t-s|. \tag{2.9}$$

We obtain (2.5) from (2.7), (2.8), (2.9) and the fact that $\|w\|_{T^*}^{2+\gamma,1+\gamma/2}\leq M\|h\|_{T^*}^{(\frac{1+\gamma}{2})}$, where M does not depend on T^* (see [11] Theorem 5.4, p. 322). With a similar argument we obtain (2.6), and the proof of the Lemma follows from (2.4) (2.5) and (2.6).

REMARK. Notice that Lemma 2.1(a) holds for any two functions φ,ψ for which u^φ, u^ψ are well defined, u_x^φ, u_x^ψ are continuous in $\overline{Q}_{T^*}$ and u_{xx}^φ, u_{xx}^ψ are bounded in $\overline{Q}_{T^*}$.

THEOREM 2.2. Assume that H_1, H_2, R_1, R_2 hold. Then there exists a unique solution $u=u(x,t)$ of Problem (B). This solution belongs to $H^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$ and satisfies

$$\|u\|_T^{2+\gamma,1+\gamma/2}\leq C(T)\left\{\|\tilde{E}\|^{1+(\frac{1+\gamma}{2})}+\|\tilde{g}\|_T^{(\frac{1+\gamma}{2})}+\|f\|_1^{2+\gamma}\right\}.$$

PROOF. Let $\epsilon>0$ as in Lemma 2.1 and $T^*<\epsilon$, then if we define the sequence $\varphi_1(x)=f'(0),\varphi_{i+1}=\mathcal{F}\varphi_i$, $i=1,2\ldots$, then Lemma 2.1 implies that the sequence of restrictions $\left\{\varphi_i\,|_{[0,T^*]}\right\}_{i\in\mathbf{N}}$ converges in $C[0,T^*]$ and in $H_{T^*}^{(\frac{1+\gamma}{2})}$ to a function φ_0.

Furthermore

$$\begin{aligned}\|\varphi_n\|_{T^*}^{(\frac{1+\gamma}{2})} &\leq \sum_{i=1}^{\infty}\|\varphi_{i+1}-\varphi_i\|_{T^*}^{(\frac{1+\gamma}{2})}+\|\varphi_1\|_{T^*}^{(\frac{1+\gamma}{2})}\\ &\leq 2\|\varphi_2-\varphi_1\|_{T^*}^{(\frac{1+\gamma}{2})}+\|\varphi_1\|_{T^*}^{(\frac{1+\gamma}{2})}\\ &\leq C_1\left\{\|\tilde{E}\|_{T^*}^{1+(\frac{1+\gamma}{2})}+\|\tilde{g}\|_{T^*}^{(\frac{1+\gamma}{2})}+\|f\|_1^{2+\gamma}\right\}.\end{aligned}$$

Then for $u:\overline{Q}_{T^*}\to\mathbf{R}$ defined by $u=u^{\varphi_0}$, we have

$$\|u\|_{T^*}^{2+\gamma,1+\gamma/2}\leq C_2\left\{\|\tilde{E}\|_{T^*}^{1+(\frac{1+\gamma}{2})}+\|\tilde{g}\|_{T^*}^{(\frac{1+\gamma}{2})}+\|f\|_1^{(2+\gamma)}\right\}.$$

Hence u is solution to the local problem. Since C_1 and C_2 depend on T^* only, a global solution u can be obtained by a standard step by step construction, and u satisfies

$$\|u\|_T^{2+\gamma,1+\gamma/2}\leq C\left\{\|\tilde{E}\|_T^{1+(\frac{1+\gamma}{2})}+\|\tilde{g}\|_T^{(\frac{1+\gamma}{2})}+\|f\|_1^{(2+\gamma)}\right\}.$$

Finally, the remark after Lemma 2.1 implies that any solution of (B) in $\overline{Q}_T$ has to be u.

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