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ON DEFINING THE GENERALIZED FUNCTIONS $\delta^\alpha(z)$ AND $\delta^n(x)$

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ABSTRACT. In a previous paper (see [5]), we applied a fixed δ-sequence and neutrix limit due to Van der Corput to give meaning to distributions δ^k and $(\delta')^k$ for $k \in (0,1)$ and $k = 2, 3, \cdots$. In this paper, we choose a fixed analytic branch such that $z^\alpha(-\pi < \arg z \leq \pi)$ is an analytic single-valued function and define $\delta^\alpha(z)$ on a suitable function space I_a. We show that $\delta^\alpha(z) \in I'_a$. Similar results on $(\delta^{(m)}(z))^\alpha$ are obtained. Finally, we use the Hilbert integral $\varphi(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\varphi(t)}{t-z}dt$ where $\varphi(t) \in D(R)$, to redefine $\delta^n(x)$ as a boundary value of $\delta^n(z - i\epsilon)$. The definition of $\delta^n(x)$ is independent of the choice of δ-sequence.

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1. INTRODUCTION

The difficulties inherent in defining products, powers or nonlinear functions of generalized functions have not prevented their appearance in the literature (see e.g. [1], [2] and [3]). In [4], a definition for a product of distributions is given using δ-sequences. However, they show that δ^2 does not exist. Similarly, such objects as $\sqrt{\delta}$ and sin δ appear to be meaningless. In this paper, we attempt to show a way out in some sense.

In a previous paper [5], we applied a fixed δ-sequence and neutrix-limit due to Van der Corput to give a meaning to distributions δ^k and $(\delta')^k$ for $k \in (0,1)$ and $k = 2, 3, \cdots$. In fact, our definition is a regularization of distributions which is similar to the method used to define some pseudofunctions by Hadamard's finite part. A method is introduced here with the help of the Cauchy's integral formula and the Hilbert integral transform. For all $\alpha \in R^+$, we define $\delta^\alpha(z)$ on a suitable function space I_a, and redefine $\delta^n(x)$ as the boundary value of $\delta^n(z - i\epsilon)$ as $\epsilon \to 0^+$. The definition of $\delta^n(x)$ is independent of the choice of the δ-sequence. No neutrix limit or Hadamard's finite part are needed.

2. THE GENERALIZED FUNCTIONS $\delta^\alpha(\alpha \in R^+)$

Let a be a fixed positive number greater than 1. Let $I_a = \{\varphi(z) : \varphi(z)$ is analytic on $|I_m z| < a\}$. We supply I_a with the following convergence concept.

Let $\varphi_n(z)$ be in I_a and $\varphi_n(z) \to 0$ in I_a iff it converges uniformly to zero over every compact domain contained in $|I_m z| < a$. I_a is a vector space with addition and multiplication by a scalar given by

$$(\varphi_1 + \varphi_2)(z) \triangleq \varphi_1(z) + \varphi_2(z) \quad \text{for} \quad \varphi_1, \varphi_2 \in I_a$$

$$(\alpha\varphi_1)(z) \triangleq \alpha\varphi_1(z) \quad \text{for} \quad \varphi_1 \in I_a \quad \text{and} \quad \alpha \in C.$$

We denote the dual of I_a by I_a'.

Suppose that C is an arbitrary closed curve in counter-clockwise direction containing zero in $|I_m z| < a$. Let C_1 and C_2 be the parts of C in the upper and lower plane respectively. Thus $C = C_1 \cup C_2$.

We choose a fixed analytic branch such that $Lnz = \ell n|z| + i \arg z \ (-\pi < \arg z \leq \pi)$ is an analytic single-valued function. So is z^α. By Cauchy's formula,

$$< \delta(z), \varphi(z) >= \varphi(0) = \frac{1}{2\pi i} \oint_C \frac{\varphi(z)}{z}\, dz. \tag{1}$$

Similarly,

$$< \delta'(z), \varphi(z) >= -\varphi'(0) = \frac{-1}{2\pi i} \oint_C \frac{\varphi(z)}{z^2} dz. \tag{2}$$

In (1), we may think of $\frac{1}{2\pi i z}$ as the Cauchy representation of $\delta(z)$ (cf. [3]). By the Cauchy Theorem on analytic functions, we know that this representation is unique up to an analytic function on the strip. We shall take this as the essential Cauchy representation. This suggests the possibility of operating on the Cauchy representation to effect the same operation on $\delta(z)$. Thus we have

DEFINITION 1: The generalized function $\delta^\alpha(z)$ on I_a is given by

$$< \delta^\alpha(z), \varphi(z) > \triangleq \frac{1}{(2\pi i)^\alpha} \oint_C \frac{\varphi(z)}{z^\alpha} dz, \ \alpha \in R^+, \varphi \in I_a \tag{3}$$

where C is defined as above.

DEFINITION 2: If δ^α and δ^β are as given in Definition 1, then we take the product $\delta^\alpha \cdot \delta^\beta$ as

$$< \delta^\alpha \cdot \delta^\beta, \varphi(z) >= \frac{1}{(2\pi i)^{\alpha+\beta}} \oint_C \frac{\varphi(z)}{z^{\alpha+\beta}} dz. \tag{4}$$

It is easy to see that the index law and the power rule of differentiation below are satisfied:

$$\delta^\alpha(z) \cdot \delta^\beta(z) = \delta^{\alpha+\beta}(z) \tag{5}$$

$$(\delta^\alpha(z))' = \alpha\delta^{\alpha-1}(z) \cdot \delta'(z). \tag{6}$$

These properties suggest that the definitions above are reasonable.

Our first main result is

THEOREM 1. $\delta^\alpha(z)$ belongs to the dual I'_a for $\alpha \in R^+$, and

$$\delta^\alpha(z) = 2i \sin \alpha\pi \cdot C_\alpha \sum_{n=0}^{+\infty} \frac{\delta^{(n)}(z)}{(n-\alpha+1)n!} \quad \text{for} \quad \alpha \neq 1,2,3,\cdots, \tag{7}$$

$$\delta^n(z) = \frac{2\pi i C_n}{(n-1)!}\, \delta^{(n-1)}(z) \quad \text{for} \quad n = 1,2,3,\cdots. \tag{8}$$

PROOF. Let $\varphi(z) \in I_a$ and let $\sum_{n=0}^{+\infty} a_n z^n$ be the Taylor series which converges to $\varphi(z)$ uniformly on C. For $\alpha \neq 1,2,3,\cdots$, equation (3) yields

$$\begin{aligned} < \delta^\alpha(z), \varphi(z) > &\triangleq C_\alpha \oint_C \frac{\varphi(z)}{z^\alpha}\, dz = C_\alpha \sum_{n=0}^{+\infty} a_n \oint_C z^{n-\alpha} dz \\ &= C_\alpha \sum_{n=0}^{+\infty} a_n \left[\int_{C_1} z^{n-\alpha} dz + \int_{C_2} z^{n-\alpha} dz \right]. \end{aligned} \tag{9}$$

Since $\frac{\varphi(z)}{z^\alpha}$ is analytic on $\{z : |I_m z| < a - \{0\}\}$, by Cauchy's Theorem, we can assume that C passes the x-axis at $(-1,0)$ and $(1,0)$.

Integrating and combining the right hand side of (9), we obtain

$$\begin{aligned} < \delta^\alpha(z), \varphi(z) > &= C_\alpha \sum_{n=0}^{+\infty} \frac{a_n}{n-\alpha+1}\, 2i \sin(n-\alpha+1)\pi \\ &= 2iC_\alpha \sin\, \alpha\pi \sum_{n=0}^{+\infty} \frac{(-1)^n a_n}{n-\alpha+1}. \end{aligned} \tag{10}$$

Since the Taylor series coefficient $a_n = \frac{\varphi^{(n)}(0)}{n!} = \frac{1}{n!}(-1)^n < \delta^{(n)}(z), \varphi(z) >$, we infer that

$$\delta^\alpha(z) = 2iC_\alpha \sin\, \alpha\pi \sum_{n=0}^{+\infty} \frac{\delta^{(n)}(z)}{(n-\alpha+1)n!}, \quad \alpha \neq 1,2,3,\cdots.$$

This proves equation (7). To prove equation (8) we let $\alpha \to n$, and note that all terms in the series vanish except the $(n-1)$th term. Thus,

$$\delta^n(z) = \frac{2\pi i C_n}{(n-1)!}\, \delta^{(n-1)}(z).$$

This concides with the result from the definition $< \delta^n(z), \varphi(z) >= C_n \oint_C \frac{\varphi(z)}{z^n}\, dz$.

Let $\varphi_m(z) \to 0$ in I_a. There exists $M > 0$, $\max\limits_{|z|=a_1} |\varphi_m(z)| \leq M$ for all $m = 1,2,3,\cdots$, where $1 < a_1 < a$. It follows from the fact that $\varphi(z)$ is analytic on $|I_m z| < a$ that

$$\frac{|\varphi^{(n)}(0)|}{n!} \leq \frac{M(a_1)}{a_1^n}$$

Since $\sum\limits_{n=1}^{\infty} \frac{1}{a_1^n}$ converges, it follows from (10) that $\delta^\alpha(z) \in I'_a$. Clearly, $\delta^n(z) \in I'_a$.

From Cauchy's integral formula,

$$\varphi^{(m)}(0) = \frac{m!}{2\pi i} \oint_C \frac{\varphi(z)}{z^{m+1}}\, dz.$$

We define

$$< (\delta^{(m)}(z))^{\alpha}, \varphi(z) > = C_{\alpha,m} \oint_C \frac{\varphi(z)}{z^{(m+1)\alpha}}\, dz$$

where $C_{\alpha,m} = \left(\frac{(-1)^m m!}{2\pi i}\right)^{\alpha}$.

Replacing α by $(m+1)\alpha$ in Theorem 1.1 and quoting the result given in the theorem, we get

THEOREM 2. $(\delta^{(m)}(z))^{\alpha} \in I'_a$ and

$$(\delta^{(m)}(z))^{\alpha} = 2iC_{\alpha,m}\sin(m+1)\alpha\pi \sum_{n=0}^{+\infty} \frac{\delta^{(n)}(z)}{(n+1-m\alpha-\alpha)n!} \quad \alpha \neq 1,2,3,\cdots$$

$$(\delta^{(m)}(z))^{k} = \frac{2\pi i C_{k,m}}{(mk+k-1)!}\, \delta^{(mk+k-1)}(z)$$

for $k = 1,2,3,\cdots$.

We now use Theorem 1 to extend our definition to an analytic function of the Dirac distribution, $\delta(z)$.

Let $f(z)$ be an analytic function on $|z| < 1$ with $f(0) = 0$. We can assume that $f(z) = \sum_{n=1}^{+\infty} a_n z^n$. $f(\delta(z))$ is defined on I_a in the following way for $\varphi(z) \in I_a$

$$< f(\delta(z)), \varphi(z) > \triangleq \sum_{n=1}^{+\infty} a_n < \delta^n(z), \varphi(z) > .$$

By Theorem 1.1 we get

$$< f(\delta(z)), \varphi(z) > = \sum_{n=1}^{+\infty} a_n \frac{(-1)^{n-1}\varphi^{(n-1)}(0)}{(2\pi i)^{n-1}(n-1)!}.$$

Hence, for $1 < a_1 < a$,

$$| < f(\delta(z)), \varphi(z) > | \leq \sum_{n=1}^{+\infty} |a_n| \frac{M(a_1)}{a_1^{n-1}}.$$

The series on the right side converges and $f(\delta(z))$ is well defined.

A similar argument to the proof of Theorem 1 can be used to show that $f(\delta(z)) \in I'_a$. Here are two examples. Since $\sin z$ and $\ell n(1+z)$ are analytic on $|z| < 1$ and vanish at $z = 0$,

$$\sin\delta(z) = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(2n+1)!}\, \delta^{2n+1}(z),$$

and

$$\ell n(1+\delta(z)) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{\delta^n(z)}{n}.$$

3. THE POWER $\delta^n(x)$ IN THE SCHWARTZ SENSE ($n \in N$)

In Section 2, we define $\delta^{\alpha}(z)$ on the space I_a. We shall now give a definition on the Schwartz space of test functions $D(R)$. We note that I_a and $D(R)$ intersect at a single element $\{0\}$. Thus, we expect different results when applying the definition of $\delta^n(x)$.

Let $\varphi(t) \in D(R)$. Construct the Hilbert integral $\varphi(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\varphi(t)}{t-z}\, dt$ where $I_m z > 0$. Since $\varphi(t)$ has a compact support, $\varphi(z)$ is well defined. By a theorem about Dirichlet boundary problem of the upper plane (see Chapter 5 of [6]) $\varphi(z)$ is analytic on $I_m z > 0$ and $\lim_{I_m z \to 0^+} Re\varphi(z) = \varphi(x)$.

The following lemma will be needed.

LEMMA 1. Let $\varphi(t) \in D(R)$, then $\lim_{Im z \to 0^+} Re\varphi^{(n)}(z) = \varphi^{(n)}(x)$.

PROOF. Let $\varphi(t) \in D(R)$. We integrate the following integral by parts

$$\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\varphi^{(n)}(t)}{t-z}\,dt = \frac{1}{\pi i}\left[\frac{\varphi^{(n-1)}(t)}{t-z}\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{\varphi^{(n-1)}(t)}{(t-z)^2}\,dt\right]$$
$$= \cdots = \frac{n!}{\pi i}\int_{-\infty}^{+\infty} \frac{\varphi(t)}{(t-z)^{n+1}}\,dt = \varphi^{(n)}(z).$$

Since $\varphi^{(n)}(t) \in D(R)$, we obtain

$$\lim_{Im z \to 0^+} Re\varphi^{(n)}(z) = \varphi^{(n)}(x).$$

We may now use Lemma 1 to define $\delta^n(x)$ as the boundary value of $\delta^n(z - i\varepsilon)$ as $\varepsilon \to 0^+$.

$$< \delta^n(x), \varphi(x) > \triangleq \lim_{\epsilon \to 0^+} Re < \delta^n(z - i\epsilon), \varphi(z) > \triangleq \lim_{\epsilon \to 0^+} Re\frac{1}{(2\pi i)^n} \oint_{|z-i\epsilon| = \frac{\epsilon}{2}} \frac{\varphi(z)}{(z-i\epsilon)^n}\,dz.$$

By Cauchy's integral formula,

$$< \delta^n(x), \varphi(x) > = \lim_{\epsilon \to 0^+} Re\frac{1}{(2\pi i)^{n-1}}\,\frac{\varphi^{(n-1)}(i\epsilon)}{(n-1)!} = Re\frac{1}{(2\pi)^{n-1}}(-i)^{n-1}\,\frac{\varphi^{(n-1)}(0)}{(n-1)!}$$
$$= \begin{cases} 0 & n = 2k, \quad k = 1,2,3,\cdots \\ \frac{(-1)^k\varphi^{(2k)}(0)}{(2\pi)^{2k}(2k)!} & n = 2k+1, \quad k = 0,1,2,\cdots. \end{cases}$$

Thus, we obtain our main theorem

THEOREM 3.

$$\delta^{2k}(x) = 0 \quad \text{for} \quad k = 1,2,3,\cdots$$

and

$$\delta^{2k+1}(x) = \frac{(-1)^k\delta^{(2k)}(x)}{(2\pi)^{2k}(2k)!} \quad \text{for} \quad k = 0,1,2,\cdots.$$

This theorem is quite similar to Theorem 1 of [5]. In both cases, the even powers of $\delta(x)$ turn out to be zero while the odd powers are expressible as a constant multiple of a derivative of $\delta(x)$. This theorem, however, is obtained without recourse to a delta sequence nor to Van der Corput's neutrix limit, an idea that requires a bit more machinery.

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