

## ASYMPTOTIC EQUIVALENCE AND SUMMABILITY

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ABSTRACT. This paper is a study of the relationships between the asymptotic equivalence of two sequences ( $\lim_n x_n/y_n = 1$ ) and three variations of this equivalence. For a sequence-to-sequence transformation  $A$ , the three variations are given by the ratios  $R_m Ax/R_m Ay$ ,  $S_m Ax/S_m Ay$ , and  $\mu_m Ax/\mu_m Ay$ , where  $R_m Az := \sum_{n \geq m} |(Az)_n|$ ,  $S_m Az := \sum_{n \leq m} |(Az)_n|$ , and  $\mu_m Az := \sup_{n \geq m} |(Az)_n|$ .

KEY WORDS AND PHRASES. Asymptotically regular, Asymptotic equivalence, Nörlund-type matrix  $\tilde{N}_p$ .

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### 1. INTRODUCTION.

In [2] Pobyvanets introduced the concept of *asymptotically regular* matrices, which preserve the asymptotic equivalence of two nonnegative number sequences; that is, if the quotient  $x_n/y_n$  has limit 1 then the quotient  $(Ax)_n/(Ay)_n$  of the transformed sequences also has a limit 1. Stated symbolically this is  $x \sim y$  implies  $Ax \sim Ay$ . The frequent occurrence of terms having zero value makes a term-by-term ratio  $x_n/y_n$  inapplicable in many cases. Therefore in [1] Fridy introduced new ways of comparing rates of convergence. For  $x$  in  $\ell^1$  he used the remainder sum, whose  $m$ -th term is  $R_mx := \sum_{n=m}^{\infty} |x_n|$ , and examined the ratio  $R_mx/R_my$  as  $m \rightarrow \infty$ . For the case where  $x$  is not in  $\ell^1$  he used the sequence of partial sums of moduli which is given by  $S_mx := \sum_{n \leq m} |x_n|$ . If  $x$  is a bounded sequence he used the supremum of the remaining terms which is given by  $\mu_mx := \sup_{n \geq m} |x_n|$ . In each case the ratio  $x_n/y_n$  can be replaced by the appropriate new ratio:  $R_mx/R_my$ ,  $S_mx/S_my$ , or  $\mu_mx/\mu_my$ . The goal of the work in [1] was to study the relationship of these ratios when they tend to zero. In the present study we investigate the relationships of these ratios when they have limit one.

### 2. BASIC THEOREMS.

The following results are stated here for the convenience of the reader and to illustrate our notation.

THEOREM 1. (Pobyvanets [2]). The nonnegative matrix  $A$  is asymptotically regular if and only if for each fixed integer  $m$ ,  $\lim_{n \rightarrow \infty} a_{nm} / \sum_{k=1}^{\infty} a_{nk} = 0$ .

THEOREM 2. The Nörlund type matrix  $\tilde{N}_p$  is regular if and only if  $\lim_n P_n = \infty$ .

THEOREM 3. The matrix  $A$  is a  $c_0 - c_0$  matrix (i.e.,  $A$  preserves zero limits) if and only if

(a)  $\lim_n a_{nk} = 0$  for  $k = 1, 2, \dots$ , and

(b) There exists a number  $M > 0$  such that for each  $n$ ,  $\sum_{k=1}^{\infty} |a_{nk}| < M$ .

### 3. ASYMPTOTIC EQUIVALENCE.

In order to give the main results we first introduce some notation.

NOTATIONS:

- $P_\delta = \{ \text{the set of all real number sequences such that } x_k \geq \delta > 0 \text{ for all } k \}$
- $P_0 = \{ \text{the set of all nonnegative sequences which have at most a finite number of zero terms} \}$
- $P = \{ \text{the set of all sequences } x \text{ such that } x_k > 0 \text{ for all } k \}$

$$\bar{N}_p \text{ is defined by } \bar{N}_p[n, k] := \begin{cases} \frac{p_k}{P_n} & , \text{ if } k \leq n, \\ 0 & , \text{ if } k > n, \end{cases}$$

where  $\{p_n\}_{n=0}^\infty$  is the sequence of nonnegative real numbers with  $p_0 > 0$  and  $P_n := \sum_{k=0}^n p_k$ .

**THEOREM 1.** Let  $A$  be a nonnegative  $c_0 - c_0$  matrix, and let  $x$  and  $y$  be bounded sequences such that  $x \sim y$  and  $x, y \in P_\delta$  for some  $\delta > 0$ , then  $\mu Ax \sim \mu Ay$ .

**PROOF.** Since  $x \sim y$  we can write  $x_n = y_n(1 + z_n)$  for some null sequence  $z$ . For each  $n$  we have

$$\begin{aligned} \frac{(\mu Ax)_n}{(\mu Ay)_n} &= \frac{\sup_{k \geq n} (Ax)_k}{\sup_{k \geq n} (Ay)_k} = \frac{\sup_{k \geq n} \sum_{i=0}^\infty a_{ki} x_i}{\sup_{k \geq n} \sum_{i=0}^\infty a_{ki} y_i} \\ &= \frac{\sup_{k \geq n} \sum_{i=0}^\infty a_{ki} (y_i + y_i z_i)}{\sup_{k \geq n} \sum_{i=0}^\infty a_{ki} y_i} \leq 1 + \frac{\sup_{k \geq n} \sum_{i=0}^\infty a_{ki} y_i |z_i|}{\sup_{k \geq n} \sum_{i=0}^\infty a_{ki} y_i} \\ &\leq 1 + \frac{\sup_{k \geq n} \sum_{i=0}^\infty a_{ki} y_i |z_i|}{\delta \sup_{k \geq n} \sum_{i=0}^\infty a_{ki}}. \end{aligned}$$

Since  $y$  is a positive bounded sequence,  $z$  is a null sequence, and  $A$  is a  $c_0 - c_0$  matrix, it follows that  $\lim_n \sum_{i=0}^\infty a_{ki} y_i |z_i| = 0$ , and therefore  $\lim_n ((\mu Ax)_n / (\mu Ay)_n) \leq 1$ . In a similar manner we can get  $\lim_n ((\mu Ax)_n / (\mu Ay)_n) \geq 1$ , hence  $\lim_n ((\mu Ax)_n / (\mu Ay)_n) = 1$ , i.e.  $\mu Ax \sim \mu Ay$ .

**THEOREM 2.** If  $A$  is a nonnegative  $\ell^\infty - \ell^1$  summability matrix then the following are equivalent:

- (i) if  $x$  and  $y$  are bounded sequences such that  $x \sim y$  and  $y \in P_\delta$  for some  $\delta > 0$ , then  $RAx \sim R Ay$ ;
- (ii) for each  $m$ ,

$$\lim_{n \rightarrow \infty} \left( \frac{\sum_{k=n}^\infty a_{km}}{\sum_{k=n}^\infty \sum_{j=0}^\infty a_{kj}} \right) = 0.$$

**PROOF.** [(ii)  $\Rightarrow$  (i)] The hypothesis  $x \sim y$  implies that for  $\epsilon > 0$  there exists a number  $J$  such that if  $k \geq J$  then  $|(x_k/y_k) - 1| < \epsilon$ . Hence, for all  $k \geq J$  we have

$$(1 - \epsilon)y_k \leq x_k \leq (1 + \epsilon)y_k. \tag{3.1}$$

First,

$$R_n Ax = \sum_{k=n}^\infty (Ax)_k \leq \sum_{k=n}^\infty \sum_{j=0}^{J-1} a_{kj} x_j + \sum_{k=n}^\infty \sum_{j=J}^\infty a_{kj} x_j.$$

By using (3.1) we have

$$R_n Ax \leq \left( \sum_{j=0}^{J-1} x_j \right) \sum_{k=n}^\infty \left( \max_{0 \leq j \leq J-1} a_{kj} \right) + (1 + \epsilon) \sum_{k=n}^\infty \sum_{j=0}^\infty a_{kj} y_j$$

Hence,

$$\frac{R_n Ax}{R_n Ay} \leq \frac{\left(\sum_{j=0}^{J-1} x_j\right) \sum_{k=n}^{\infty} \left(\max_{0 \leq j \leq J-1} a_{kj}\right)}{\delta \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}} + 1 + \epsilon$$

Now we claim that

$$\lim_{n \rightarrow \infty} \left[ \frac{\left(\sum_{k=n}^{\infty} \left(\max_{0 \leq j \leq J-1} a_{kj}\right)\right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}} \right] = 0.$$

Using (ii), we get

$$\sum_{j=0}^{J-1} \left( \frac{\left(\sum_{k=n}^{\infty} a_{kj}\right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}} \right) = 0,$$

where  $J$  is finite; hence,

$$0 \geq \lim_{n \rightarrow \infty} \left[ \frac{\left(\sum_{k=n}^{\infty} \left(\max_{0 \leq j \leq J-1} a_{kj}\right)\right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}} \right].$$

Since  $A$  is a nonnegative matrix we have proved the claim, and it follows that

$$\limsup_n \left( \frac{R_n Ax}{R_n Ay} \right) \leq 1 + \epsilon. \tag{3.2}$$

Second, we seek a lower bound for  $R_n Ax/R_n Ay$ . Consider

$$R_n Ax = \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj} x_j \geq \sum_{k=n}^{\infty} \left(\min_{0 \leq j \leq J-1} a_{kj}\right) \sum_{j=0}^{J-1} x_j + \sum_{k=n}^{\infty} \sum_{j=J}^{\infty} a_{kj} x_j.$$

Using Inequality (3.1) again we get

$$R_n Ax \geq \left(\sum_{j=0}^{J-1} x_j\right) \sum_{k=n}^{\infty} \left(\min_{0 \leq j \leq J-1} a_{kj}\right) + (1 - \epsilon) \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj} y_j - (1 - \epsilon) \sum_{k=n}^{\infty} \sum_{j=n}^{J-1} a_{kj} y_j.$$

Therefore

$$\begin{aligned} \frac{R_n Ax}{R_n Ay} &\geq \left(\frac{\sum_{j=0}^{J-1} x_j}{\sup_j y_j}\right) \left(\frac{\sum_{k=n}^{\infty} \min_{0 \leq j \leq J-1} a_{kj}}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}}\right) + (1 - \epsilon) \\ &\quad - \left(\frac{(1 - \epsilon) \sum_{j=0}^{J-1} y_j}{\delta}\right) \left(\frac{\sum_{k=n}^{\infty} \max_{0 \leq j \leq J-1} a_{kj}}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}}\right). \end{aligned}$$

Using the previous claim we have

$$\liminf_n \left( \frac{R_n Ax}{R_n Ay} \right) \geq 1 - \epsilon \tag{3.3}$$

From (3.2) and (3.3) we conclude that  $\lim_n (R_n Ax/R_n Ay) = 1$ , whence  $R_n Ax \sim R_n Ay$ .

To show that (i) implies (ii) we choose a fixed positive integer  $m$  and define  $x$  and  $y$  as follows:  $y_k := 1$  for all  $k$  and

$$x_k := \begin{cases} 0, & \text{if } k \leq m, \\ 1, & \text{if } k > m. \end{cases}$$

It is obvious that the conditions in (i) are satisfied by the sequences  $x$  and  $y$ . Consider

$$R_n Ax = \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj} x_j = \sum_{k=n}^{\infty} \sum_{j=m+1}^{\infty} a_{kj} = \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj} - \sum_{k=n}^{\infty} \sum_{j=0}^m a_{kj}.$$

Thus

$$\left( \frac{R_n Ax}{R_n Ay} \right) = 1 - \left[ \frac{\left( \sum_{k=n}^{\infty} \sum_{j=0}^m a_{kj} \right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj} y_i} \right].$$

The hypothesis that  $A$  is a nonnegative matrix implies

$$\left( \frac{R_n Ax}{R_n Ay} \right) \leq 1 - \left[ \frac{\left( \sum_{k=n}^{\infty} a_{km} \right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}} \right].$$

Hence,

$$\liminf_n \left( \frac{R_n Ax}{R_n Ay} \right) \leq 1 - \limsup_{n \rightarrow \infty} \left[ \frac{\left( \sum_{k=n}^{\infty} a_{km} \right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}} \right].$$

From (i) we get  $\limsup_n \left[ \left( \sum_{k=n}^{\infty} a_{km} \right) / \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj} \right] = 0$ , and since  $A$  is nonnegative matrix we have

$$\lim_{n \rightarrow \infty} \left[ \frac{\left( \sum_{k>n} a_{km} \right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}} \right] = 0.$$

Hence, the proof of the theorem is complete.

In the following theorem we use another notation that was introduced by Fridy in [1] to study sequences that converge at different rates. Now, we use the same notation to study sequences that converge at the same rate. For sequences  $x$  and  $y$  that are not in  $\ell^1$  Fridy used the partial sum  $S_N x := \sum_{k \leq N} |x_k|$  to say that  $y$  diverges faster than  $x$  provided that  $S_N x = o(S_N y)$ . Here we use  $S_N Ax \sim S_N Ay$  to indicate that  $\sum_{n \leq N} (Ax)_n$  and  $\sum_{n \leq N} (Ay)_n$  diverge at the same rate.

**THEOREM 3.** If  $A$  is a nonnegative matrix, then the following statements are equivalent:

- (a) if  $x$  and  $y$  are sequences such that  $x \sim y$ ,  $x \in P_0$ , and  $y \in P_\delta$  for some  $\delta > 0$ , then  $Ax$  and  $Ay$  are not in  $\ell^1$  and  $S_A x \sim S_A y$ ;
- (b)  $A$  satisfies the following conditions:
  - (i)  $\left\{ \sum_{j=0}^{\infty} a_{kj} \right\}_{k=0}^{\infty} \notin \ell^1$ ;
  - (ii) for each  $m$ ,

$$\lim_{N \rightarrow \infty} \left( \frac{\sum_{k=1}^N a_{km}}{\sum_{k=1}^N \sum_{j=0}^{\infty} a_{kj}} \right) = 0.$$

PROOF. Suppose that condition (b) is satisfied and assume that  $x$  and  $y$  are sequences such that  $x \sim y$ ,  $x \in P_0$ , and  $y \in P_\delta$  for some  $\delta > 0$ . Using condition (i), it is clear that  $Ay$  is not in  $\ell^1$ . Also, the fact that  $x \sim y$  implies that for each given  $\varepsilon > 0$  there exists an integer  $J$  such that  $|x_k/y_k - 1| < \varepsilon$  whenever  $k \geq J$ , or equivalently, if  $k \geq J$  then

$$(1 - \varepsilon)y_k \leq x_k \leq (1 + \varepsilon)y_k \quad (3.4)$$

From this it is easy to prove that  $Ax$  is not in  $\ell^1$ .

Now, we need to show that  $SAx \sim SAy$ . Consider

$$S_N Ax = \sum_{k=1}^N \sum_{j=1}^{\infty} a_{kj} x_j = \sum_{k=1}^N \sum_{j=1}^{J-1} a_{kj} x_j + \sum_{k=1}^N \sum_{j=J}^{\infty} a_{kj} x_j.$$

By using (3.4) we get

$$\begin{aligned} S_N Ax &\leq \sum_{k=1}^N \sum_{j=1}^{J-1} a_{kj} x_j + (1 + \varepsilon) \sum_{k=1}^N \sum_{j=J}^{\infty} a_{kj} y_j \\ &\leq \left( \sum_{j=1}^{J-1} x_j \right) \sum_{k=1}^N \left( \max_{0 \leq j \leq J-1} a_{kj} \right) + (1 + \varepsilon) \sum_{k=1}^N \sum_{j=0}^{\infty} a_{kj} y_j \\ &\leq \left( \sum_{j=1}^{J-1} x_j \right) \sum_{k=1}^N \left( \max_{0 \leq j \leq J-1} a_{kj} \right) + (1 + \varepsilon) S_N Ay. \end{aligned}$$

Therefore

$$\frac{S_N Ax}{S_N Ay} \leq \left( \frac{\sum_{j=1}^{J-1} x_j}{\delta} \right) \left( \frac{\sum_{k=1}^N \left( \max_{0 \leq j \leq J-1} a_{kj} \right)}{\sum_{k=1}^N \sum_{j=0}^{\infty} a_{kj}} \right) + (1 + \varepsilon).$$

Hence, from condition (ii) we get

$$\lim_{N \rightarrow \infty} \frac{S_N Ax}{S_N Ay} \leq 1 + \varepsilon, \quad (3.5)$$

similarly, using the left-hand inequality in (3.4) we have

$$\begin{aligned} S_N Ax &\geq \sum_{k=1}^N \sum_{j=0}^{J-1} a_{kj} x_j + (1 - \varepsilon) \sum_{k=1}^N \sum_{j=J}^{\infty} a_{kj} y_j \\ &\geq \left( \sum_{k=1}^N \left( \min_{0 \leq j \leq J-1} a_{kj} \right) \right) \sum_{j=0}^{J-1} x_j + (1 - \varepsilon) \sum_{k=1}^N \sum_{j=0}^{\infty} a_{kj} y_j - (1 - \varepsilon) \sum_{k=1}^N \sum_{j=0}^{J-1} a_{kj} y_j. \end{aligned}$$

Thus

$$\begin{aligned} \frac{S_N Ax}{S_N Ay} &\geq \left( \frac{\sum_{j=0}^{J-1} x_j}{\sup_j y_j} \right) \frac{\sum_{k=1}^N \left( \max_{0 \leq j \leq J-1} a_{kj} \right)}{\sum_{k=1}^N \sum_{j=0}^{\infty} a_{kj}} + 1 - \varepsilon \\ &\quad - \left( \frac{(1 - \varepsilon) \sum_{j=0}^{J-1} y_j}{\delta} \right) \frac{\sum_{k=0}^N \left( \max_{0 \leq j \leq J-1} a_{kj} \right)}{\sum_{k=1}^N \sum_{j=0}^{\infty} a_{kj}}. \end{aligned}$$

As before, from condition (ii) we see that the first and the third terms in the previous inequality tend to zero as  $N \rightarrow \infty$ . Hence,

$$\lim_{N \rightarrow \infty} \frac{S_N A x}{S_N A y} \geq 1 - \varepsilon, \quad (3.6)$$

From (3.5) and (3.6) we conclude that  $\lim_{N \rightarrow \infty} S_N A x / S_N A y = 1$ , i.e.,  $S A x \sim S A y$ .

To prove that (a) implies (b), we first need to show the necessity of condition (ii). Thus, we form two sequences  $x$  and  $y$  such that  $x \sim y$  and both satisfy the hypothesis in (a), but  $S A x$  and  $S A y$  are not asymptotically equivalent if condition (ii) does not hold. Let  $m$  be an arbitrary fixed positive integer and define  $y_k := 1$  for all  $k$ , and

$$x_k := \begin{cases} 0, & \text{if } k \leq m, \\ 1, & \text{if } k > m. \end{cases}$$

It is clear that  $x \sim y$ ,  $x \in P_0$ , and  $y \in P_1$ . Assume condition (ii) does not hold. For any such  $m$  we have

$$\begin{aligned} S_N A x &= \sum_{k=1}^N \sum_{m+1}^{\infty} a_{kj} = \sum_{k=1}^N \sum_{j=0}^n a_{kj} - \sum_{k=1}^N \sum_{j=0}^m a_{kj} \\ &\leq \sum_{k=1}^N \sum_{j=0}^{\infty} a_{kj} - \sum_{k=1}^N a_{km}. \end{aligned}$$

Therefore,

$$\liminf_N \frac{S_N A x}{S_N A y} \leq 1 - \limsup_{N \rightarrow \infty} \frac{\sum_{k=1}^N a_{km}}{\sum_{k=1}^N \sum_{j=0}^{\infty} a_{kj}}.$$

Since  $A$  is a nonnegative matrix and condition (ii) does not hold, we conclude that

$$\lim_{N \rightarrow \infty} \frac{S_N A x}{S_N A y} < 1,$$

where  $S A x$  and  $S A y$  are not asymptotically equivalent.

It remains to show that the sequence  $\{\sum_{j=0}^{\infty} a_{kj}\}_{k=1}^{\infty}$  is not in  $\ell^1$ . Since  $y$  is identically 1, we get  $(A y)_k = \sum_{j=0}^{\infty} a_{kj}$ , and since the sequence  $A y$  is not in  $\ell^1$ , we see that  $\{\sum_{j=0}^{\infty} a_{kj}\}_{k=0}^{\infty}$  is not in  $\ell^1$ . Therefore the proof is complete.

In the following theorem we study the relationship between the original asymptotic equivalence that uses term by term comparison and the asymptotic equivalence that uses the remainder sum comparison.

**THEOREM 4.** If  $x$  and  $y$  are nonnegative sequences in  $\ell^1$  such that  $x \sim y$ , then  $R x \sim R y$ .

**PROOF.** The assumption that  $x \sim y$  implies that for a given  $\varepsilon > 0$  there exists an integer  $N$  such that  $|x_n/y_n - 1| \leq \varepsilon$  for all  $n \geq N$ .

Hence,

$$R_n x \leq (1 + \varepsilon) \sum_{k \geq n} y_k = (1 + \varepsilon) R_n y$$

and,

$$R_n x \geq (1 - \varepsilon) \sum_{k \geq n} y_k = (1 - \varepsilon) R_n y.$$

Therefore, for all  $n \geq N$ ,

$$(1 - \varepsilon) \leq \frac{R_n x}{R_n y} \leq (1 + \varepsilon),$$

and this implies that  $\lim_{n \rightarrow \infty} R_n x / R_n y = 1$ , i.e.,  $R x \sim R y$ .

The next theorem shows the relationship between the original asymptotic equivalence and the asymptotic equivalence that uses the partial sum comparison.

**THEOREM 5.** If  $x$  and  $y$  are nonnegative sequences not in  $\ell^1$  such that  $x \sim y$ , then  $S x \sim S y$ .

PROOF The assumption that  $x \sim y$  implies that  $x_n/y_n - 1 = \varepsilon_n$ , where  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . Since  $x$  is nonnegative we have

$$S_m x = \sum_{n \leq m} |x_n| = \sum_{n \leq m} (y_n + \varepsilon_n y_n) = \sum_{n \leq m} y_n + \sum_{n \leq m} \varepsilon_n y_n.$$

Therefore,

$$\begin{aligned} \frac{S_m x}{S_m y} &= 1 + \frac{\sum_{n \leq m} \varepsilon_n y_n}{\sum_{n \leq m} y_n} = 1 + \sum_{n \leq m} \left( \frac{y_n}{\sum_{j \leq m} y_j} \right) \varepsilon_n \\ &= 1 + (\bar{N}y\varepsilon)_m, \end{aligned}$$

where  $\bar{N}y$  is the Nörlund-type matrix [3, p. 45]. Since  $y$  is not in  $\ell^1$ ,  $\bar{N}y$  is regular; therefore  $\lim_{m \rightarrow \infty} (\bar{N}y\varepsilon)_m = 0$ . Thus we have  $\lim_{m \rightarrow \infty} S_m x/S_m y = 1$ , i.e.,  $Sx \sim Sy$ .

In the last result we study the relationship between the original asymptotic equivalence and the asymptotic equivalence that compares the "supremum of the remaining terms."

THEOREM 6. If  $x$  and  $y$  are nonvanishing null sequences such that  $x \sim y$ , then  $\mu x \sim \mu y$ .

PROOF. The hypothesis  $x \sim y$  implies that for  $\varepsilon > 0$  there exists an integer  $N$  such that if  $k \geq N$  then

$$(1 - \varepsilon)y_k \leq x_k \leq (1 + \varepsilon)y_k.$$

For  $n \geq N$  we have

$$\mu_n x = \sup_{k \geq n} |x_k| \leq (1 + \varepsilon) \sup_{k \geq n} |y_k|.$$

Hence,

$$\limsup_n \frac{\mu_n x}{\mu_n y} \leq 1 + \varepsilon.$$

Also,

$$\mu_n x = \sup_{k \geq n} |x_k| \geq (1 - \varepsilon) \sup_{k \geq n} |y_k|.$$

Hence,

$$\liminf_n \frac{\mu_n x}{\mu_n y} \geq 1 - \varepsilon.$$

Therefore we conclude that  $\lim_{n \rightarrow \infty} \mu_n x/\mu_n y = 1$ , i.e.,  $\mu x \sim \mu y$ .

REMARK. The converse of Theorems 5 and 6 is not true i.e., there exist sequences  $x$  and  $y$  such that  $Sx \sim Sy$  and  $\mu x \sim \mu y$  but  $x$  is not asymptotically equivalent to  $y$ . It is enough here to give the two sequences  $x$  and  $y$  and leave the proof to the reader.

$$x_n := \begin{cases} \frac{2}{n+1}, & \text{if } n \text{ is odd,} \\ \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

$$y_n := \begin{cases} \frac{1}{n+1}, & \text{if } n \text{ is odd,} \\ \frac{2}{n}, & \text{if } n \text{ is even.} \end{cases}$$

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