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EIGHT-DIMENSIONAL REAL ABSOLUTE-VALUED ALGEBRAS WITH LEFT UNIT WHOSE AUTOMORPHISM GROUP IS TRIVIAL

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We classify, by means of the orthogonal group $\mathbb{O}_7(\mathbb{R})$, all eight-dimensional real absolute-valued algebras with left unit, and we solve the isomorphism problem. We give an example of those algebras which contain no four-dimensional subalgebras and characterise with the use of the automorphism group those algebras which contain one.

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1. Introduction. One of the fundamental results about finite-dimensional real division algebras is due to Kervaire [7] and Bott and Milnor [3], and states that the n-dimensional real vector space $\mathbb{R}^n$ possesses a bilinear product without zero divisors only in the case where the dimension $n = 1, 2, 4$, or 8. All eight-dimensional real division algebras that occur in the literature contain a four-dimensional subalgebra (see [1, 2, 4, 5, 6]). However, it is still an open problem whether a four-dimensional subalgebra always exists in an eight-dimensional real division algebra, even for quadratic algebras [4]. In [9], Ramírez Álvarez gave an example of a four-dimensional absolute-valued real algebra containing no two-dimensional subalgebras. On the other hand, any four-dimensional absolute-valued real algebra with left unit contains a two-dimensional subalgebra. Therefore, a natural question to ask is whether an eight-dimensional real absolute-valued algebra with left unit contains a four-dimensional subalgebra. In this note, we give a negative answer and we characterise the eight-dimensional absolute-valued real algebras with left unit containing a four-dimensional subalgebra in terms of the automorphism group.

2. Notation and preliminary results. For simplicity, we only consider vector spaces over the field $\mathbb{R}$ of real numbers.

Definition 2.1. Let A be an algebra; A is not assumed to be associative or unital.

(1) An element $x \in A$ is called invertible if the linear operators

$$L_x : y \longmapsto xy, \qquad R_x : y \longmapsto yx \tag{2.1}$$

are invertible in the associative unital algebra $\mathrm{End}(A)$. The algebra A is called a division algebra if all nonzero elements in A are invertible.

(2) A unital algebra A is called a quadratic algebra if $\{1, x, x^2\}$ is linearly dependent for all $x \in A$. If $(\cdot/\cdot)$ is a symmetric bilinear form over A, then a linear operator f on A is called an isometry with respect to $(\cdot/\cdot)$ if $(f(x)/f(y)) = (x/y)$ for all $x, y \in A$. If, moreover, $(xy/z) = (x/yz)$, for all $x, y, z \in A$, then $(\cdot/\cdot)$ is called a trace form over A.

(3) The algebra A is termed normed (resp., absolute-valued) if it is endowed with a space norm $\|\cdot\|$ such that $\|xy\| \leq \|x\|\|y\|$(resp., $\|xy\| = \|x\|\|y\|$) for all $x, y \in A$. A finite-dimensional absolute-valued algebra is obviously a division algebra and has a subjacent Euclidean structure (see [11]).

(4) An automorphism $f \in \mathrm{Aut}(A)$ is called a reflexion of A if $f \neq I_A$ and $f^2 = I_A$.

Write $\mathrm{Aut}(\mathbb{O}) = G_2$. We denote by $S(E)$ and $\mathrm{vect}\{x_1, \dots, x_n\}$, respectively, the unit sphere of a normed space E and the vector subspace spanned by $x_1, \dots, x_n \in E$.

It is known that a quadratic algebra A is obtained from an anticommutative algebra $(V, \wedge)$ and a bilinear form $(\cdot,\cdot)$ over V as follows: $A = \mathbb{R} \oplus V$ as a vector space, with product

$$(\alpha + x)(\beta + y) = \big(\alpha\beta + (x, y)\big) + (\alpha y + \beta x + x \wedge y). \tag{2.2}$$

We have a bilinear form associated to A, namely,

$$A \times A \longrightarrow \mathbb{R}, \qquad (\alpha + x, \beta + y) \longmapsto \alpha\beta + (x, y), \tag{2.3}$$

$(V, \wedge)$ is called the anticommutative algebra associated to A. The elements of V are called vectors, while the elements of $\mathbb{R}$ are called scalars. We write $A = (V, (\cdot,\cdot), \wedge)$ (see [8]).

We will write $(W, (\cdot/\cdot), \times)$ for the (quadratic) Cayley-Dickson octonions algebra $\mathbb{O}$ with its trace form $(\cdot/\cdot)$ and the anticommutative algebra $(W, \times)$. For $u \neq 0 \in W$, $W(u)$ will be the orthogonal subspace of $\mathbb{R} \cdot u$ in W. It is well known that $\mathbb{O}$ is an alternative algebra, that is, it satisfies the identities $x^2 y = x(xy)$ and $yx^2 = (yx)x$.

Remark 2.2. Let A be an eight-dimensional absolute-valued algebra with left unit e, and f is an isometry of the Euclidian space A such that $f(e) = e$. Let A_f be equal to A as a vector space, with a new product given by the formula $x * y = f(x)y$, for all $x, y \in A$. Then A_f is also an absolute-valued algebra with left unit e. It is clear that an f-invariant subalgebra of A is a subalgebra of A_f. In particular, if we consider the isometry R_e^{-1}, then we obtain an absolute-valued algebra $A_{R_e^{-1}}$ with unit e, which is isomorphic to $\mathbb{O}$ (see [12]).

3. Isometries of $\mathbb{O}$ with no invariant four-dimensional subalgebras. Let φ be an isometry of the Euclidian space $\mathbb{O} = \mathbb{R} \oplus W$, fixing the element 1. Then there exists an orthonormal basis $\mathcal{B} = \{1, x_1, \dots, x_7\}$ of $\mathbb{O}$ such that x_1 is an eigenvector of φ and $W_k = \text{vect}\{x_{2k}, x_{2k+1}\}$ is a φ-invariant subspace of $\mathbb{O}$, for $k = 1, 2, 3$. If B is a four-dimensional φ-invariant subspace of $\mathbb{O}$ containing 1, then the basis $\mathcal{B}$ can be chosen as an extension of an orthonormal basis $\{1, u, y, z\}$ of B, with $u \in W$ an eigenvector of φ, and $E = \text{vect}\{y, z\}$ is a φ-invariant subspace of B. Thus, B can be written as a direct orthogonal φ-invariant sum $\mathbb{R} \oplus \mathbb{R} \cdot u \oplus E$.

In the following important example, we use the notation introduced above.

Example 3.1. If φ fixes x_1 and its restriction to every W_k is the rotation with angle $k\pi/4$, then $\text{vect}\{1, x_1\}$ is the eigenspace $E_1(\varphi)$ of φ associated to the eigenvalue 1. The characteristic polynomial $P_\varphi(X)$ of φ is then

$$(X-1)^2\left(X^2-2X\cos\left(\frac{\pi}{4}\right)+1\right)\left(X^2-2X\cos\left(\frac{2\pi}{4}\right)+1\right)\left(X^2-2X\cos\left(\frac{3\pi}{4}\right)+1\right) = \prod_{0\le k\le 3} P_k(X) \tag{3.1}$$

with

$$P_k(X) = X^2 - 2X\cos\left(\frac{k\pi}{4}\right) + 1. \tag{3.2}$$

The characteristic polynomial $P_{\varphi_{//B}}(X)$ of the restriction of φ to B is a polynomial of degree 4, a multiple of $X-1$, and a divisor of $P_\varphi(X)$. Actually, $P_{\varphi_{//B}}(X) = (X-1)^2 P_k(X)$ for $k \in \{1,2,3\}$, and this "forces" B to be of the form $E_1(\varphi) \oplus W_k$ for a certain $k \in \{1,2,3\}$. In particular, if $\mathcal{B}$ is obtained from the canonical basis $\{1, e_1, \dots, e_7\}$ of $\mathbb{O}$ by taking

$$\begin{gathered} x_1 = e_5, \quad x_2 = \frac{e_1+e_2}{\sqrt{2}}, \quad x_3 = \frac{e_1-e_2}{\sqrt{2}}, \quad x_4 = \frac{e_3+e_4}{\sqrt{2}}, \\ x_5 = \frac{e_3-e_4}{\sqrt{2}}, \quad x_6 = \frac{e_6+e_7}{\sqrt{2}}, \quad x_7 = \frac{e_6-e_7}{\sqrt{2}}, \end{gathered} \tag{3.3}$$

then for each $i \neq j$ and l, $x_i \times x_j$ and x_l are not colinear. This shows that $E_1(\varphi) \oplus W_k$ is not a subalgebra of $\mathbb{O}$, for $k = 1, 2, 3$. It follows that $\mathbb{O}$ has no four-dimensional φ-invariant subalgebras.

4. Eight-dimensional real absolute-valued algebras with left unit. First recall the following result from [11].

Lemma 4.1. *Every homomorphism from a normed complete algebra into an absolute-valued algebra is contractive. In particular, every isomorphism of absolute-valued algebras is an isometry.*

As a consequence we have the following lemma.

Lemma 4.2. *Let $\psi : A \to B$ be an isomorphism of absolute-valued $\mathbb{R}$-algebras and $f : A \to A$ an isometry. Then $\psi \circ f \circ \psi^{-1} : B \to B$ is an isometry and $\psi : A_f \to B_{\psi \circ f \circ \psi^{-1}}$ is an isomorphism. In particular, $\psi : A_f \to \mathbb{O}$ is an isomorphism if and only if $\psi : A \to \mathbb{O}_{\psi \circ f^{-1} \circ \psi^{-1}}$ is an isomorphism.*

Proof. The first statement is a consequence of Lemma 4.1. For $x, y \in A$, we have

$$\psi(f(x)y) = \psi(f(x))\psi(y) = (\psi \circ f \circ \psi^{-1})(\psi(x))\psi(y), \tag{4.1}$$

hence $\psi : A_f \to B_{\psi \circ f \circ \psi^{-1}}$ is an isomorphism. □

Theorem 4.3. *Every eight-dimensional absolute-valued left unital algebra is isomorphic to $\mathbb{O}_f$ where f is an isometry of the Euclidian space $\mathbb{O}$ which fixes 1. Moreover, the following two properties are equivalent:*

(1) *$\mathbb{O}_f$ and $\mathbb{O}_g$ are isomorphic (f, g being two isometries of $\mathbb{O}$ fixing 1);*
(2) *there exists $\psi \in G_2$ such that $g = \psi \circ f \circ \psi^{-1}$, that is, f and g are in the same orbit of conjugations by isometries of $\mathbb{O}$ fixing 1.*

Proof. The first statement is a consequence of a Remark 2.2 and Lemma 4.2. The second statement can be proved as follows: $\psi : \mathbb{O}_f \to \mathbb{O}_g$ is an isomorphism if and only if $\psi : \mathbb{O} \to (\mathbb{O}_g)_{\psi \circ f^{-1} \circ \psi^{-1}} = \mathbb{O}_{\psi \circ f^{-1} \circ \psi^{-1} \circ g}$ is an isomorphism. This is equivalent to

$$\psi \circ f^{-1} \circ \psi^{-1} \circ g = I_{\mathbb{O}}, \quad \psi \in G_2. \tag{4.2}$$

□

5. Subalgebras and automorphisms of $\mathbb{O}_\varphi$.

The following preliminary result allows us to characterise the subalgebras of $\mathbb{O}_\varphi$.

Lemma 5.1. *If A is an algebra with left unit and without zero divisors, then every nontrivial finite-dimensional subalgebra of A contains the left unit element of A.*

Proof. Such a subalgebra B is a division algebra and for every $x \neq 0 \in B$, there exists $y \in B$ such that $yx = x$. On the other hand, if e is the left unit of A, then $ex = x$. Then the absence of zero divisors in A shows that $y = e \in B$. □

What are the subalgebras of $\mathbb{O}_\varphi$?

Proposition 5.2. *Let φ be an isometry of the Euclidian space $\mathbb{O}$ that fixes 1 and B is a subspace of $\mathbb{O}$. Then the following two properties are equivalent:*

(1) *B is a subalgebra of $\mathbb{O}_\varphi$;*
(2) *B is a φ-invariant subalgebra of $\mathbb{O}$.*

Proof. (1)⇒(2). The subalgebra B contains the left unit element 1 of $\mathbb{O}_\varphi$ and is φ-invariant. Indeed,

$$1 \in B, \quad \forall x \in B : \varphi(x) = \underbrace{\varphi(x)1}_{\text{product in } \mathbb{O}} = \overbrace{x * 1}^{\text{product in } \mathbb{O}_\varphi} \in B. \tag{5.1}$$

(2)⇒(1). See Remark 2.2. □

Remark 5.3. (1) The algebra $\mathbb{O}_\varphi$ has a two-dimensional subalgebra because φ has an eigenvector $x \in W$ and the subalgebra vect$\{1, x\}$ of $\mathbb{O}$ is φ-invariant. This argument shows that $\mathbb{H}_\varphi$ has a two-dimensional subalgebra.

(2) Let φ be the isometry considered in Example 3.1. Then $\mathbb{O}_\varphi$ has no four-dimensional subalgebras.

The following elementary result is useful for characterising the automorphisms of the algebra $\mathbb{O}_\varphi$.

Lemma 5.4. *Let A be an algebra with left unit e and without zero divisors. If $f \in \mathrm{Aut}(A)$, then $f(e) = e$.*

Proof. We have $(f(e) - e)f(e) = 0$. □

What are the automorphisms of the algebra $\mathbb{O}_\varphi$?

Proposition 5.5. *If φ is an isometry of the Euclidian space $\mathbb{O}$ that fixes 1, then $f \in \mathrm{Aut}(\mathbb{O}_\varphi)$ if and only if $f \in G_2$ and f commutes with φ.*

Proof. For all $x, y \in \mathbb{O}$, we have that $f(\varphi(x)y) = \varphi(f(x))f(y)$, hence $f(\varphi(x)) = f(\varphi(x)1) = \varphi(f(x))f(1) = \varphi(f(x))$, and $f \circ \varphi = \varphi \circ f$ and $f \in G_2$. □

Remark 5.6. If $f \in \mathrm{Aut}(\mathbb{O}_\varphi) \setminus \{I_{\mathbb{O}}\}$ is a reflexion, then $B = \mathrm{Ker}(f - I_{\mathbb{O}})$ is a four-dimensional subalgebra of $\mathbb{O}_\varphi$.

6. The relation in $\mathbb{O}_\varphi$ between four-dimensional subalgebras and nontrivial automorphisms. We begin with the following useful preliminary result taken from [10].

Lemma 6.1. *Every four-dimensional subalgebra B of $\mathbb{O} = (W, (\cdot/\cdot), \times)$ coincides with the square of its orthogonal $B^\perp$ and satisfies the equality $BB^\perp = B^\perp B = B^\perp$.*

Proof. Let $v \in S(B^\perp)$, then $B^\perp = vB$. Indeed, taking into account the trace property of $(\cdot/\cdot)$, we have for all $x, y \in B$ that $(vx/y) = (v/xy) = 0$, hence $vB \subset B^\perp$, and we have equality because the dimensions of both spaces are equal. Using the middle Moufang identity, we compute that

$$(vx)(vy) = -(vx)(yv) = v(xy)v = xy \tag{6.1}$$

for all $x, y \in B$. Taking into account the anticommutativity of the product $\times$, we find that $BB^\perp = B^\perp B$. Finally, the trace property of $(\cdot/\cdot)$ shows that $BB^\perp$ is orthogonal to B, hence $BB^\perp \subset B^\perp$. □

Proposition 6.2. *Let B be a φ-invariant four-dimensional subalgebra of $\mathbb{O}$. Then the map*

$$f:\mathbb{O}=B\oplus B^{\perp}\longrightarrow\mathbb{O},\quad f(a+b)=a-b, \tag{6.2}$$

is a reflexion which commutes with φ.

Proof. Take $a,x\in B$ and $b,y\in B^{\perp}$. Using Lemma 6.1, we compute

$$\begin{aligned} f\big((a+b)(x+y)\big) &= f(ax+by+ay+bx) \\ &= (ax+by)-(ay+bx) \\ &= (a-b)(x-y) \\ &= f(a+b)f(x+y), \end{aligned} \tag{6.3}$$

$B^{\perp}$ is φ-invariant since B is φ-invariant, and we have

$$\begin{aligned} (f\circ\varphi)(a+b) &= f\big(\varphi(a)+\varphi(b)\big) \\ &= \varphi(a)-\varphi(b) \\ &= \varphi(a-b) \\ &= (\varphi\circ f)(a+b). \end{aligned} \tag{6.4}$$

□

Theorem 6.3. *If φ is an isometry of the Euclidian space $\mathbb{O}$ which fixes 1, then the following four properties are equivalent:*

(1) *$\mathbb{O}_{\varphi}$ contains a four-dimensional subalgebra;*
(2) *$\mathbb{O}$ contains a φ-invariant four-dimensional subalgebra;*
(3) *$\mathrm{Aut}(\mathbb{O}_{\varphi})$ contains a reflexion;*
(4) *$\mathrm{Aut}(\mathbb{O}_{\varphi})$ is not trivial.*

Proof. The only thing that remains to be shown is that (4) implies (1). Let $g\in\mathrm{Aut}(\mathbb{O}_{\varphi})-\{I_{\mathbb{O}}\}$. If g is a reflexion, then the result follows from Remark 5.6. By assuming that g is not a reflexion, we distinguish two cases.

Case 1. The automorphism g admits two linearly independent orthonormal eigenvectors $u,y\in W$. Then $g(uy)=g(u)g(y)=(\pm u)(\pm y)=\pm uy$ and $\mathrm{vect}\{1,u,y,uy\}=\mathrm{Ker}(g^2-I_{\mathbb{O}})$ is a φ-invariant four-dimensional subalgebra of $\mathbb{O}$.

Case 2. The automorphism g has only one eigenvector $u\in S(W)$ except the sign. Then u is an eigenvector of φ and g and φ induce isometries

$$g_u,\varphi_u:W(u)\longrightarrow W(u). \tag{6.5}$$

Using the minimal polynomials $P(X)$ and $Q(X)$ of g_u and φ_u, we will first show that $W(u)$ contains a two-dimensional g-invariant and φ-invariant subspace of E. The irreducible factors of $P(X)$ are polynomials of degree two with negative discriminant. However $Q(X)$ can have a factor of degree one, and then the existence of E is assured by the fact that the eigenspaces of φ_u are f-invariant, and their direct sum is of even dimension. So we can assume that $Q(X)$ is a product of polynomials of degree two with negative discriminant. Now, we have three different cases.

(a) $P(X) = X^2 - \alpha X - \beta$ and $Q(X) = X^2 - \lambda X - \mu$ are polynomials of degree two. Since $\alpha^2 + 4\beta < 0$ and $\lambda^2 + 4\mu < 0$, there exists $\omega \in \mathbb{R}^*$ such that $\alpha^2 + 4\beta = \omega^2(\lambda^2 + 4\mu)$, and we have

$$\begin{aligned}\left(g_u - \frac{\alpha}{2} I_{W(u)}\right)^2 &= \left(\frac{\alpha^2}{4} + \beta\right) I_{W(u)} \\ &= \omega^2 \left(\frac{\lambda^2}{4} + \mu\right) I_{W(u)} \\ &= \omega^2 \left(\varphi_u - \frac{\lambda}{2} I_{W(u)}\right)^2.\end{aligned} \tag{6.6}$$

Now g_u and φ_u commute, so

$$\left(g_u - \frac{\alpha}{2} I_{W(u)} - \omega\left(\varphi_u - \frac{\lambda}{2} I_{W(u)}\right)\right) \circ \left(g_u - \frac{\alpha}{2} I_{W(u)} + \omega\left(\varphi_u - \frac{\lambda}{2}\right)\right) \equiv 0. \tag{6.7}$$

(i) If $g_u - (\alpha/2) I_{W(u)} = \pm\omega(\varphi_u - (\lambda/2) I_W(u))$, then every g-invariant two-dimensional subspace of $W(u)$ is φ-invariant, as well as its orthogonal.

(ii) If $g_u - (\alpha/2) I_{W(u)} \neq \pm\omega(\varphi_u - (\lambda/2) I_{W(u)})$, then

$$H = \operatorname{Ker}\left(g_u - \frac{\alpha}{2} I_{W(u)} - \omega\left(\varphi_u - \frac{\lambda}{2} I_{W(u)}\right)\right) \tag{6.8}$$

and $H^\perp$ are g_u-invariant and φ_u-invariant proper subspaces of $W(u)$. One of them is necessarily two-dimensional and the other one is four-dimensional.

(b) If $\deg(P(X)) > 2$, then we consider an irreducible component $P_1(X)$ of $P(X)$. The kernel $\operatorname{Ker}(P_1(g_u))$ and its orthogonal are then g_u-invariant and φ_u-invariant proper subspaces of $W(u)$.

(c) The case $\deg(Q(X)) > 2$ is similar to the case $\deg(P(X)) > 2$.

The subspace $\operatorname{vect}\{1, u\} \oplus E$ is then a subalgebra of $\mathbb{O}$. Indeed, $E = \operatorname{vect}\{y, z\}$, with $y, z \in W(u)$ orthogonal, and there exist $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ such that the matrix of the restriction of g to E is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ b & -a \end{pmatrix}. \tag{6.9}$$

Thus, $g(yz) = g(y)g(z) = \pm(a^2 + b^2)yz = \pm yz$, and consequently $yz = \pm u$. Using alternativity and anticommutativity for vectors, we then obtain that $uy = \pm z$ and $uz = \pm y$. □

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