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GRADED TRANSCENDENTAL EXTENSIONS OF GRADED FIELDS

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We study transcendency properties for graded field extension and give an application to valued field extensions.

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1. Introduction. An important tool to study rings with valuation is the so-called associated graded ring construction: to a valuation ring R, we can associate a ring $\text{gr}(R)$ graded by the valuation group. This ring is often easier to study, and one tries to lift properties back from $\text{gr}(R)$ to R. This principle has been recently applied to rings of differential operators (see [9]), the Brauer group (see, e.g., [8]), and to valuations on division algebras (see [1, 11, 12]). This has been one of the motivations to study graded rings, see [10] for a detailed discussion. In a sense, the easiest example of a graded ring is a graded field, this is a commutative graded ring in which every homogeneous element is invertible, and the terminology has been introduced in [13].

This note is a continuation of earlier work of the author (see [3, 4, 5, 6]), in which graded fields and graded division rings are studied with special emphasis on applications to valuation theory. The aim of this note is to introduce and study the notion of gr-transcendental graded field extension, at least in the case where the grading group is torsion-free abelian; application to valued field extensions leads to three different notions of transcendental extensions of valued fields.

In Section 2, we recall some basic results on graded ring theory and on gradings on polynomial rings. We introduce the notions of gr-algebraically freeness and gr-transcendental extension in Section 3 and prove some elementary properties (see, e.g., Proposition 3.4). In Section 4, we look at two special cases: unramified graded field extensions, where the grading groups of both graded fields are the same, and totally ramified extensions, where the parts of degree zero of both extensions coincide. The transcendency can be described explicitly in both cases; combination of the two situations leads to the existence of a gr-transcendency basis in general (Proposition 4.5) and to the notion of gr-transcendency degree. In Section 5, we give a structure theorem for purely gr-transcendental graded field extensions of divisible type (Proposition 5.1);

a graded field extension is of divisible type if the quotient of the two grading groups is torsion. In Section 6, we apply our results to valued fields and introduce the notions of gradually, residually, and valuatively transcendental valued field extensions.

2. Preliminaries

2.1. Graded rings. Let Γ be a torsion-free abelian group and R a commutative ring. All rings considered in this note will be commutative. Recall that R is called a Γ-graded ring if $R = \oplus_{\delta\in\Gamma} R_\delta$, where R_δ is an additive subgroup of R, such that $R_\delta R_\gamma \subset R_{\delta+\gamma}$, for all $\gamma, \delta \in \Gamma$. We say that $a \neq 0 \in R_\delta$ is homogeneous of degree δ, and we then write $\deg(a) = \delta$. Let $H(R) = (\cup_{\delta\in\Gamma} R_\delta) \setminus \{0\}$ be the set of all homogeneous elements of R. If $a = \sum_{\delta\in G} a_\delta$ with $a_\delta \in R_\delta$, then a_δ is called the homogeneous component of a of degree δ; $\Gamma_R = \{\lambda \in \Gamma \mid R_\lambda \neq \{0\}\}$ is called the *support* of the graded ring R and R is a domain if and only if $H(R)$ has no zero divisors; in this case Γ_R is a submonoid of Γ. We will say that R is a graded ring of type Γ if R is Γ-graded with $\Gamma_R = \Gamma$. A Γ-graded commutative ring R is called a *graded field* if every nonzero homogeneous element of R is invertible. If R is a graded field, then Γ_R is a subgroup of Γ and is called the grading group of R. The rational closure of Γ_R is then denoted by $\Delta_R = \Gamma_R \otimes_{\mathbb{Z}} \mathbb{Q}$. In this case, $H(R)$ is a group, called the group of homogeneous elements of R.

If R and S are Γ-graded rings, then $f : R \to S$ is called a homomorphism of graded rings if f is grade preserving, that is, $f(R_\tau) \subset S_\tau$ for all $\tau \in \Gamma$.

2.2. Gradings on polynomial rings. Throughout, Γ will be a torsion-free abelian group. Let R be a Γ-graded commutative ring and assume that $H(R)$ contains no zero divisors. Localizing R at the multiplicatively closed set $H(R)$, we obtain a Γ-graded field $\mathrm{Fr}_{\mathrm{gr}}(R)$, called the graded field of fractions of R. The support of $\mathrm{Fr}_{\mathrm{gr}}(R)$ is the subgroup $\{\alpha - \beta \mid \alpha, \beta \in \Gamma_R\}$ of Γ.

Let $X = \{X_i \mid i \in I\}$ be a (finite or infinite) set of variables and $\Delta_\Gamma = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ the divisible closure of Γ. Consider a map $\omega : X \to \Delta_\Gamma$ and write $\omega(X_i) = \delta_i$. Then ω defines a Δ_Γ-grading on $R[X]$ by taking $\mathrm{gr}(X_i) = \delta_i$. This graded ring is denoted by $R[X]^\omega$. Clearly, the support of $R[X]^\omega$ is contained in $\Gamma_R[\omega(X)]$, the submonoid of Δ_Γ generated by Γ_R and $\omega(X)$.

We call $P \in R[X]$ homogenizable if there exists $\omega : X \to \Delta_\Gamma$ such that $P \in H(R[X]^\omega)$.

Now assume that R is a graded field. Then $R[X]^\omega$ is a domain, and we can consider $\mathrm{Fr}_{\mathrm{gr}}(R[X]^\omega) = R(X)^\omega$, called the ω-graded field of fractions of $R[X]$.

Let R be a graded subring of a Γ-graded field S, and $A \subset H(S)$. Then the subring $R[A]$ of S generated by R and A is graded. Its graded field of fractions $R(A)$ is the smallest graded subfield of S containing R and A. If $B \subset H(S)$ is another subset, then $R[A \cup B] = R[A][B]$ and $R(A \cup B) = R(A)(B)$.

3. Graded transcendental extensions of graded fields. Let Γ be a torsion-free abelian group and $R \subset S$ an extension of Γ-graded fields. For $T = \{t_1, \dots, t_s\} \subset H(S)$, we consider the map $\omega_T : \{X_1, \dots, X_s\} \to \Gamma$, $\omega_T(X_i) = \deg(t_i)$.

(1) T is called gr-algebraically free over R if there are no nonzero $P \in H(R[X_1, \dots, X_s]^{\omega_T})$ such that $P(t_1, \dots, t_s) = 0$;
(2) $L \subset H(S)$ is called gr-algebraically free over R if every finite subset of L is gr-algebraically free;
(3) T is called a gr-transcendency basis of S over R if T is a maximal gr-algebraically free subset of $H(S)$ (for the inclusion);
(4) $a \in H(S)$ is called gr-transcendental over R if $\{a\}$ is gr-algebraically free; otherwise a is called gr-algebraic;
(5) $R \subset S$ is called a gr-*algebraic graded field extension* if every $a \in H(S)$ is gr-algebraic; otherwise $R \subset S$ is called a *transcendental graded field extension.*

Every graded field extension of finite degree is gr-algebraic (see [6] or [7]); consequently, $[S : R] = \infty$ if $R \subset S$ is transcendental.

If $T = \{t_1, \dots, t_s\} \subset H(S)$ is gr-algebraically free, then every t_i is gr-transcendental over R and $T \cap H(R) = \varnothing$.

Proposition 3.1. *Let S/R be a graded field extension and let $T \subset H(S)$. The set T is a* gr-*transcendency basis of S over R if and only if $S/R(T)$ is* gr-*algebraic and T is* gr-*algebraically free over R.*

Proof. First, Assume that T is a gr-transcendency basis and take $a \in H(S) \setminus T$. Then $T \cup \{a\}$ is not gr-algebraically free, so there exists $T' = \{t_1, \dots, t_s, a\} \subset T \cup \{a\}$ and $0 \neq f(X_1, \dots, X_s, Y) \in R[X_1, \dots, X_s, Y]^{\omega_{T'}}$ such that $f(t_1, \dots, t_s, a) = 0$. We can write

$$f(X_1, \dots, X_s, Y) = \sum_{i=1}^{r} f_i(X_1, \dots, X_s) Y^i \tag{3.1}$$

with at least one $f_i \neq 0$. It follows that

$$\sum_{i=1}^{r} f_i(t_1, \dots, t_s) a^i = 0, \tag{3.2}$$

and a is gr-algebraic over $R(T)$.

Conversely, let $T \subset H(S)$ be gr-algebraically free and assume that $S/R(T)$ is gr-algebraic. For $x \in H(S)$, there exists $f(X) = \sum_{i=1}^{r} a_i X^i \in H(R(T)[X]^{(\mathrm{gr}(x))})$ such that $f(x) = 0$. We can find $T' = \{t_1, \dots, t_s\} \subset T$ such that each a_i can be written in the form

$$a_i = \frac{N_i(t_1, \dots, t_s)}{D(t_1, \dots, t_s)} \tag{3.3}$$

with $N_i, D \in H(R[X_1,\dots,X_s]^{\omega_{T'}})$ and $d = D(t_1,\dots,t_s) \neq 0$. The polynomial

$$P(X_1,\dots,X_s,X) = \sum_{i=1}^{r} N_i(T_1,\dots,T_s)X^i \tag{3.4}$$

is homogenizable and $P(t_1,\dots,t_s,x) = 0$. Hence, $T \cup \{x\}$ is not gr-algebraically free. □

We call S a *pure* gr-*transcendental graded field extension* of R if there exists a (possibly empty) gr-transcendency basis T of R such that $S = R(T)$. Such a basis is called a *generating* gr-*transcendency base*.

Remark 3.2. Let $A = \{a_i \mid i \in I\} \subset H(S)$ and consider $\omega_A : X = \{X_i \mid i \in I\} \to \Gamma$, $\omega(X_i) = \deg(a_i)$. Then we have a canonical surjection of graded rings $\phi_A : R[X]^{\omega_A} \to R[A]$, $\phi_A(X_i) = a_i$. The set A is gr-algebraically free if and only if ϕ_A is an isomorphism of graded rings. The map ϕ_A induces a morphism of graded fields

$$\psi_A : \mathrm{Fr}_{\mathrm{gr}}\left(R[X]^{\omega_A}\right) = R(X)^{\omega_A} \longrightarrow S \tag{3.5}$$

and A is a generating gr-transcendency base of S over R if and only if ψ_A is an isomorphism.

Proposition 3.3. *Let S/R be a Γ-graded field extension and $T \subset H(S)$. Assume that T is the disjoint union of two subsets L and C.*

(1) *The subset C is* gr-*algebraically free over $R(L)$.*
(2) *Every $h \in H(R(T)) \setminus R$ is* gr-*transcendental over R.*

Proof. (1) Assume that C is not gr-algebraically free over $R(L)$. Then there exist finite subsets $L_1 = \{t_1,\dots,t_s\} \subset L$ and $C_1 = \{t_{s+1},\dots,t_r\} \subset C$ and a homogeneous polynomial $P \in R(t_1,\dots,t_s)[X_{s+1},\dots,X_r]^{\omega_{C_1}}$ such that $P(t_{s+1},\dots,t_r) = 0$. The coefficients of P are quotients of homogeneous polynomials in $t_1,\dots,t_s$. Let Q be a common multiple of the denominators. Then we can write

$$P = \frac{F(t_1,\dots,t_s,X_{s+1},\dots,X_r)}{Q(t_1,\dots,t_s)} \tag{3.6}$$

with $F \in H(R[X_1,\dots,X_s,X_{s+1},\dots,X_r]^{\omega_{L_1} \cup \omega_{C_1}})$. Now,

$$F(t_1,\dots,t_s,t_{s+1},\dots,t_r) = 0, \tag{3.7}$$

contradicting the fact that T is gr-algebraically free over R.

(2) For $h \in H(R(T))$, we have a finite subset $T_1 = \{t_1,\dots,t_r\}$ of T such that $h \in H(R(T_1))$, and we can write

$$h = \frac{P(t_1,\dots,t_r)}{Q(t_1,\dots,t_r)} \tag{3.8}$$

with $P, Q \in H(R[X_1,\dots,X_r]^{\omega_{T_1}})$ and $Q(t_1,\dots,t_r) \neq 0$.

Assume that h is gr-algebraic over R. Then there exists $F \in H(R[X]^{\omega_{\{h\}}})$ such that $F(h) = 0$. Write $F(X) = \sum_{i=0}^{n} a_i X^i$. Then,

$$Q(t_1,\dots,t_r)^n F(h) = \sum_{i=0}^{n} a_i Q(t_1,\dots,t_r)^{n-i} P(t_1,\dots,t_r)^i = 0. \tag{3.9}$$

Now,

$$F(X_1,\dots,X_r) = \sum_{i=0}^{n} a_i Q(X_1,\dots,X_r)^{n-i} P(X_1,\dots,X_r)^i \tag{3.10}$$

is a homogeneous polynomial in $R[X_1,\dots,X_r]^{\omega_{T_1}}$ and $F(t_1,\dots,t_r) = 0$. This contradicts the fact that T is gr-algebraically free. □

Proposition 3.4. *Let Q/S and S/R be two Γ-graded field extensions and let T_Q be a* gr-*algebraically free subset of Q over S and T_S a* gr-*algebraically free subset of S over R. Then the following properties hold:*

(1) *$T_Q \cap T_S = \varnothing$;*
(2) *$T_Q \cup T_S$ is a* gr-*algebraically free subset of Q over R;*
(3) *$T_Q \cup T_S$ is a* gr-*transcendency base of Q over R if and only if T_Q and T_S are* gr-*transcendency bases, respectively of Q over S and of S over R;*
(4) *Q/R is a pure* gr-*transcendental extension if and only if Q/S and S/R are pure* gr-*transcendental extensions.*

Proof. (1) The proof follows from the fact that $T_Q \cap H(S) = \varnothing$ and $T_S \subset H(S)$.

(2) Clearly, finite subsets

$$L_1 = \{t_1,\dots,t_s\} \subset T_Q, \qquad L_2 = \{t_{s+1},\dots,t_{s+r}\} \subset T_S \tag{3.11}$$

are gr-algebraically free over R. Assume that $L = L_1 \cup L_2$ is not gr-algebraically free over R. Then $L_1 \neq \varnothing$ and $L_2 \neq \varnothing$, and there exists a nonzero homogenizable $f \in R[X_1,\dots,X_{s+r}]$ such that $f(t_1,\dots,t_{s+r}) = 0$. Then $g = f(X_1,\dots,X_s, t_{s+1},\dots,t_{s+r})$ is a homogeneous polynomial in $S[X_1,\dots,X_s]$, and $g(t_1,\dots,t_s) = 0$, hence Q is not gr-algebraically free over S, which is a contradiction.

(3) Let T_Q and T_S be gr-transcendency bases, respectively of Q over S and of S over R, and consider free $L \subset H(Q)$ gr-algebraically over R and strictly containing $T_Q \cup T_S$. Then

$$L = (L \cap (Q \setminus S)) \cup (L \cap S). \tag{3.12}$$

We have that at least one of the two inclusions $L \cap (Q \setminus S) \subset T_Q$ and $L \cap S \subset T_S$ is strict, and $L \cap (Q \setminus S) \subset H(Q)$ and $L \cap S \subset H(S)$ are gr-algebraically free over, respectively, S and R. This contradicts the hypothesis.

Conversely, assume that $T_Q \cup T_S$ is a gr-transcendency base of Q over R and that T_Q is not a maximal gr-algebraically free part of $H(Q)$ over S. Let L be a gr-algebraically free part of $H(Q)$ over S strictly containing T_Q. Then $L \cup T_S$ strictly contains $T_Q \cup T_S$ and is gr-algebraically free over R by part (2), which is a contradiction. We use the same argument if T_S is not maximal.

(4) If $S = R(T_S)$ and $Q = S(T_Q)$, then $Q = S(T_Q) = R(T_S)(T_Q) = R(S_T \cup T_Q)$, and it follows from part (2) that $S_T \cup T_Q \subset H(Q)$ is gr-algebraically free over R.

Conversely, let $Q = R(L)$ with $L \subset H(Q)$ gr-algebraically free over R. Let $L_S = L \cap H(S)$ and $L_Q = L \setminus L_S$. It follows from part (3) that L_S and L_Q are gr-transcendency bases of S over R and of Q over S. If there exists $x \in S \setminus R(L_S)$, then x is gr-algebraic over $R(L_S)$. But it follows from part (2) of Proposition 3.3 that every $x \in Q \setminus R(L_S)$ is gr-transcendental over $R(L_S)$, so we have a contradiction.

Finally, $Q = R(L) = R(L_Q \cup L_S) = R(L_S)(L_Q) = S(L_Q)$. □

4. Unramified and totally ramified graded field extensions. We call an extension S/R of Γ-graded fields *unramified* if $\Gamma_S = \Gamma_R$.

Proposition 4.1. *Let S/R be an unramified Γ-graded field extension.*

(1) *Every transcendency basis T of S_0/R_0 is a* gr*-transcendency basis of S/R.*

(2) *If T is a* gr*-transcendency basis of S/R, then for each $t \in T$, there exists $r_t \in H(R)$ such that $T_0 = \{t/r_t \mid t \in T\}$ is a transcendency basis of S_0/R_0.*

Proof. (1) If T is not gr-algebraically free over R, then there exist $\{t_1, \dots, t_s\} \subset T \subset R_0$ and $P \in H(R[X_1, \dots, X_s]^{\omega_T})$ such that

$$P(t_1, \dots, t_s) = 0. \tag{4.1}$$

In $R[X_1, \dots, X_s]^{\omega_T}$, we have that $\deg(X_i) = 0$, for all i, and we can conclude that all the coefficients of P are homogeneous of the same degree δ (which is also the degree of P). Take $x \neq 0 \in R_\delta$, then $x^{-1}P(t_1, \dots, t_s) = 0$, and $x^{-1}P \in R_0[X_1, \dots, X_s]$, hence T is not algebraically free over R_0, which is a contradiction. From the fact that $S_0/R(T)_0$ is algebraic and $\Gamma_S = \Gamma_R = \Gamma_{R(T)}$ is a torsion group over $\Gamma_{R(T)}$, we conclude that $S/R(T)$ is gr-algebraic (see [6, Proposition 1, page 24]).

(2) Let T be a gr-transcendency basis of S/R. For every $t \in T$, we choose $r_t \in R$ such that $\deg(r_t) = \deg(t)$ (using the fact that $\Gamma_R = \Gamma_S$). Then $L_T = \{t/r_t \mid t \in T\} \subset S_0$ is still a gr-transcendency basis of S/R. From the fact that L_T is gr-algebraically free, it follows immediately that L_T is algebraically free; also the fact that $S/R(L_T)$ is gr-algebraic entails that $S_/(R(L_T)_0)$ is algebraic, and the proof is finished after we remark that $R(L_T)_0 = R_0(L_T)$. □

Corollary 4.2. *Every unramified graded field extension S/R has a* gr*-transcendency basis and all the* gr*-transcendency bases have the same cardinality, equal to the transcendency degree of S_0/R_0.*

We call an extension S/R of Γ-graded fields *totally ramified* if $R_0 = S_0$.

Proposition 4.3. *If S/R is a totally ramified extension of Γ-graded fields, then $T \subset H(S)$ is* gr*-algebraically free over R if and only if $\Gamma_T = \{\deg(t) \mid t \in T\}$ is linearly free in Γ_S/Γ_R.*

Proof. Assume first that T is not algebraically free. Then there exist $T' = \{t_1,\dots,t_s\} \subset T$ and a nonzero $P \in H(R[X_1,\dots,X_s]^{\omega_T})$ such that $P(t_1,\dots,t_s) = 0$. The polynomial P can be written as a sum of monomials, and at least two of them are different from 0, say

$$aX_1^{n_1}\cdots X_s n_s, \qquad bX_1^{m_1}\cdots X_s m_s. \tag{4.2}$$

All these monomials have the same degree, hence

$$\begin{gathered}\deg(a) + \sum_{j=1}^{s} n_j \deg(t_j) = \deg(b) + \sum_{j=1}^{s} m_j \deg(t_j), \\ \sum_{j=1}^{s} (n_j - m_j)\deg(t_j) = \deg(b) - \deg(a) \in \Gamma_R,\end{gathered} \tag{4.3}$$

so Γ_T is not linearly free modulo Γ_R.

Conversely, assume that there exists $T' = \{t_1,\dots,t_s\} \subset T$ such that $\Gamma_{T'}$ is not linearly free modulo Γ_R. Then there exist $l_1,\dots,l_s \in \mathbb{Z}$ such that

$$\sum_{k=1}^{s} l_k \deg(t_k) = \lambda \in \Gamma_R. \tag{4.4}$$

Take $a \neq 0 \in H(R)$ such that $\deg(a) = \lambda$. Then,

$$b = a^{-1} t_1^{l_1} \cdots t_s^{l_s} \in R_0. \tag{4.5}$$

For every $m \in \{1,\dots,s\}$, we take

$$\begin{aligned} i_m &= l_m, \quad j_m = 0 \quad \text{if } l_m \geq 0, \\ i_m &= 0, \quad j_m = -l_m \quad \text{if } l_m < 0. \end{aligned} \tag{4.6}$$

The polynomial

$$P(X_1,\dots,X_s) = a^{-1}b^{-1}X_1^{i_1}\cdots X_s i_s - X_1^{j_1}\cdots X_s j_s \tag{4.7}$$

is homogeneous in $R[X_1,\dots,X_s]^{\omega_{T'}}$, and $P(t_1,\dots,t_s) = 0$, so it follows that T' is not gr-algebraically free. □

Corollary 4.4. *Every totally ramified graded field extension S/R has a* gr*-transcendency basis and all the* gr*-transcendency bases have the same cardinality, equal to the rank of the abelian group Γ_S/Γ_R.*

Proof. Take a maximal free subgroup F of Γ_S/Γ_R; then $\Gamma_S/\Gamma_R(F)$ is torsion. For every $f \in F$, choose $t_f \in H(S)$ such that $\deg(t_f)$ represents f in Γ_S/Γ_R.

It follows from Proposition 4.3 that $T = \{t_f \mid f \in F\}$ is gr-algebraically free. Finally, $\Gamma_{R(T)} = \Gamma_R(\Gamma_T)$ and $R(T)_0 = R_0$, so $S_0/R(T)_0 = R_0/R_0$ is algebraic. It follows that $S/R(T)$ is gr-algebraic if and only if $(\Gamma_S/\Gamma_R(\Gamma_T) = \Gamma_S)/\Gamma_R(F)$ is torsion, see [6, Proposition 1, page 24]. □

We now look at the general case: if S/R is an extension of Γ-graded fields, then $S/R(S_0)$ is a totally ramified extension and $R(S_0)/R$ is an unramified extension. The above results show that S/R has a gr-transcendency basis with cardinality equal to the sum of the transcendency degree of S_0/R_0 and the rank of Γ_S/Γ_R. Moreover, we have the following result.

Proposition 4.5. *Let S/R be a Γ-graded field extension. Then all* gr*-transcendency bases of S/R have the same cardinality, equal to the sum of the transcendency degree of S_0/R_0 and the rank of Γ_S/Γ_R.*

Proof. Let T be a gr-transcendency basis of S/R. Then $\Gamma_{R(T)} = \Gamma_R(\Gamma_T)$, and $\Gamma_S/\Gamma_{R(T)}$ is torsion since $S/R(T)$ is gr-algebraic. Applying Zorn's lemma to the set $\mathcal{T}$ consisting of $T^* \subset T$ such that Γ_{T*} is linearly free over Γ_R, and such that two different elements in T^* have different degrees, we obtain a maximal subset T_m satisfying these two properties. Then $\Gamma_R(\Gamma_T)/\Gamma_R(\Gamma_{T_m})$ and, a fortiori, $\Gamma_S/\Gamma_R(\Gamma_{T_m})$ are torsion, and T_m is a basis of Γ_S/Γ_R, proving that T_m is a gr-transcendency basis of $S/R(S_0)$.

On the other hand, the map $\deg : H(S) \to \Gamma_S$ is a group homomorphism and its kernel S_0^* is a multiplicative subgroup of S_0. The image of $H(R(T_m))$ under deg is $\Gamma_R(\Gamma_{T_m})$ and the inverse image of $\Gamma_R(\Gamma_{T_m})$ is $H(R(T_m))S_0^*$.

For every $t \in \overline{T} = T \setminus T_m$, there exists $n_t \in \mathbb{N}$ such that $n_t \deg(t) \in \Gamma_R(\Gamma_{T_m})$ or, equivalently, $t^{n_t} \in H(R(T_m))S_0^*$. Therefore,

$$t^{n_t} H(R(T_m)) \cap S_0^* \neq \varnothing. \tag{4.8}$$

Let G be the multiplicative subgroup of $H(R)(T)$ generated by $H(R)$ and T in $H(S)$. Then an element $x \in R(T) \cap S_0$ can be written as a quotient $\sum_i a_i$ by $\sum_j b_j$, where the a_i and b_j are elements of $G \cap S_0$. Hence,

$$R(T) \cap S_0 = R_0(G) \cap S_0 \tag{4.9}$$

and $S_0/R_0(G \cap S_0)$ is algebraic.

For every $t \in \overline{T}$, we choose $\tilde{t} \in t^{n_t} H(R(T_m)) \cap S_0$ and we put

$$\tilde{T} = \{\tilde{t} \mid t \in T\} \tag{4.10}$$

and $\tilde{G} = R_0^*(\tilde{T})$, the subgroup of S_0^* generated by R_0^* and $\tilde{T}$. Then $(G \cap S_0)/\tilde{G}$ is torsion, hence $R(T) \cap S_0$ is algebraic over $R_0(\tilde{G}) = R_0(\tilde{T})$. Now $S_0/R(T) \cap S_0$ is algebraic, so $S_0/R_0(\tilde{T})$ is also algebraic. We know that $\tilde{T}$ is gr-algebraically free over R_0, so $\tilde{T}$ is a gr-transcendency basis of $R(S_0)/R$ and $T_m \cup \tilde{T}$ is a gr-transcendency basis of S/R. To finish the proof, it suffices to remark that the map $T \to T_m \cup \tilde{T}$, mapping t to t if $t \in T_m$, and to $\tilde{t}$ otherwise, is a bijection. □

The cardinality of a gr-transcendency basis of S/R is called the gr-*transcendency degree* of S/R, and is denoted by $[S:R]_t$.

5. Extensions of divisible type. We say that an extension S/R of Γ-graded fields is of divisible type if Γ_S/Γ_R is a torsion group. In this situation, we have that $\Gamma_S \subset \Delta_R$.

Proposition 5.1. *Let R be a Γ-graded field $X = \{X_i \mid i \in I\}$ a set of variables, and $\omega : X \to \Delta_R$.*

(1) *The graded field extension $R(X)^\omega/R$ is pure* gr-*transcendental of divisible type; X is a generating* gr-*transcendency basis.*

(2) *Every pure* gr-*transcendental graded field extension of divisible type of R is* gr-*isomorphic to $R(X)^\omega$ for a suitable choice of X and ω.*

Proof. (1) It is clear that $R(X)^\omega/R$ is an extension of divisible type since $\Gamma_{R(X)^\omega} \subset \Delta_R$ and Δ_R/Γ_R is torsion. If X is not gr-algebraically free over R, then there exist $F = \{X_1,\dots,X_r\} \subset X$ finite and a polynomial $P \in H(R[Y_1,\dots,Y_r]^{\omega_{|F|}})$ such that $P(X_1,\dots,X_r) = 0$. But then P is the zero polynomial.

(2) Let S/R be a pure gr-transcendental graded field extension of divisible type with gr-transcendency basis $T = \{t_i \mid i \in I\}$. Then $\deg(t_i) \in \Delta_R$. Let $X = \{X_i \mid i \in I\}$ be a set of indeterminates and consider $\omega : X \to \Delta_R$, $\omega(X_i) = \deg(t_i)$. By Remark 3.2, the map $\psi : R(X)\omega \to S$ defined by $\psi(X_i) = t_i$ is an isomorphism of graded fields. □

Proposition 5.2. *Let R be a Γ-graded field and $\omega : X = \{X_i \mid i \in I\} \to \Delta_R$. Then $(R(X)^\omega)_0/R_0$ is a pure transcendental field extension and $[(R(X)^\omega)_0 : R_0]_t = [R(X)^\omega : R]_t$.*

Proof. Let $\mathcal{L}$ be the set consisting of all couples (Y,B_Y) with $Y \subset X$ and $(R(Y)^{\omega_{|Y}})_0/R_0$ a pure transcendental field extension with generating transcendency basis B_Y. The set $\mathcal{L}$ is partially ordered: $(Y,B_Y) \le (Z,B_Z)$ if and only if $Y \subset Z$ and $B_Y \subset B_Z$.

Take $(Y,B_Y) \in \mathcal{L}$ and $Y \subset Z \subset X$. Remark that there exists $B_Z \subset R(Z)_0^{\omega_{|Z}}$ such that $B_Y \subset B_Z$ and $(Z,B_Z) \in \mathcal{L}$ if and only if $R(Z)_0^{\omega_{|Z}}/R(Y)_0^{\omega_{|Y}}$ is a purely transcendental field extension. Indeed, if B is a generating transcendency basis, then $B_Z = B \cup B_Y$ satisfies the required conditions.

Every totally ordered subset $\{(Y_i,B_{Y_i}) \mid i \in J\} \subset \mathcal{L}$ has an upper bound, namely $(X_J = \cup Y_i,\ B_J = \cup B_{Y_i})$. Indeed, if B_J is not algebraically free over R_0, then there exists a finite subset $P \subset B_J$ which is not algebraically free. P is contained in some B_j, which is algebraically free, so we have a contradiction. On the other hand, $B_J \subset R(X_J)_0^{(\omega(X_J))}$ and, for each Y_i, the generating transcendency basis B_i of $R(Y_i)_0^{\omega_{|Y_i}}/R_0$ is included in B_J, hence $R(Y_i)_0^{\omega_{|Y_i}} \subset R_0(B_J)$ and

$R(X_J)_0^{(\mathrm{gr}(X_J))} \subset R_0(B_J)$. So, $(X_J = \cup Y_i,\ B_J = \cup B_{Y_i}) \in \mathcal{L}$ and is bigger than all the (Y_i, B_{Y_i}). It then follows from Zorn's lemma that there exists a maximal element $(X_m, B_{X_m}) \in \mathcal{L}$. We show that $X_m = X$.

Assume that there exists $x \in X \setminus X_m$ and let $Y = X_m \cup \{x\}$. Then $R(Y)_0^{\omega_{|Y}}/R(X_m)_0^{\omega_{|Xm}}$ is purely transcendental of degree one with generating transcendency base $\{x\}$. Let α be the order of $\deg(x)$ over $\Gamma_R(\omega(X_m))$ and take $z \in R(X_m)_0^{\omega_{|Xm}}$ such that $\deg(z) = \alpha \deg(x)$. Then

$$R(Y)_0^{\omega_{|Y}} = R(X_m)_0^{\omega_{|Xm}}(z^{-1}x^{\alpha}), \tag{5.1}$$

where $z^{-1}x^{\alpha}$ is transcendental over $R(X_m)_0^{\omega_{|Xm}}$, and $R(Y)_0^{\omega_{|Y}}/R(X_m)_0$ is purely transcendental. This implies that (X_m, B_{X_m}) is not maximal in $\mathcal{L}$, a contradiction. We conclude that $R(X)_0^{(\omega(X))}/R$ is purely transcendental and we have a generating transcendency basis B_X indexed by X, so $[(R(X)^{\omega})_0 : R_0]_t = \#(X) = [R(X)^{\omega} : R]_t$ by Proposition 5.1. □

As an immediate consequence of Propositions 5.1 and 5.2, we have the following corollary.

Corollary 5.3. *If S/R is a purely* gr*-transcendental extension of Γ-graded fields of divisible type, then S_0/R_0 is purely transcendental and $[S : R]_t = [S_0 : R_0]_t$.*

6. Application to valued extensions

6.1. The associated graded field. Let (F, v) be a valued field with valuation group Γ_F and let $\overline{F}$ be the residue field. For $\lambda \in \Gamma_F$, we have that $F_\lambda = \{x \in F \mid v(x) \geq \lambda\}$ is a subgroup of $(F, +)$ and $F_{\lambda^+} = \{x \in K \mid v(x) > \lambda\}$ is a subgroup of F_λ. We also write $\mathrm{gr}(F)_\lambda = F_\lambda/F_{\lambda^+}$. In particular, $\mathrm{gr}(F)_0 = \overline{F}$. On

$$\mathrm{gr}(F) = \bigoplus_{\Gamma_F} \mathrm{gr}(F)_\lambda, \tag{6.1}$$

we define a multiplication as follows:

$$(a + F_{\lambda^+})(b + F_{\delta^+}) = (ab + F_{(\lambda+\delta)^+}) \tag{6.2}$$

for $a \in \mathrm{gr}(F)_\lambda$ and $b \in \mathrm{gr}(F)_\delta$. This multiplication extends linearly to $\mathrm{gr}(F)$ and makes $\mathrm{gr}(F)$ into a Γ_F-graded field, called the *associated graded field.*

Let $\pi_\lambda : F_\lambda \to \mathrm{gr}(F)_\lambda$ be the canonical projection. For every $x \in F$, we put $\tilde{x} = \pi_{v(x)}(x)$. The group Γ_F is totally ordered, and therefore torsion-free. Notice that if E/F is an extension of valued fields, then $\mathrm{gr}(E)/\mathrm{gr}(F)$ is an extension of graded fields. More details on the associated graded field (or division ring) can be found in [2, 3, 6, 7, 8, 11].

For an extension of valued fields E/F, we now define the following notions:

(1) $[E:F]_{t\cdot g} = [\mathrm{gr}(E):\mathrm{gr}(F)]_t$, the *gradual transcendence degree* of E/F;

(2) $[E:F]_{t\cdot r} = [\overline{E}:\overline{F}]_t$, the *residual transcendence degree* of E/F;

(3) $[E:F]_{t\cdot v} = \mathrm{rank}(\Gamma_E/\Gamma_F)$, the *valuative transcendence degree* of E/F.

We call E a gradually (resp., residually, resp., valuatively) transcendental valued extension of F if $\mathrm{gr}(E)/\mathrm{gr}(F)$ is a gr-transcendental graded field extension (resp., if $\overline{E}/\overline{F}$ is a transcendental field extension, resp., if Γ_E/Γ_F is free).

If $T \subset E$ is such that $\tilde{T} = \{\tilde{t} \mid t \in T\}$ is gr-algebraically free over $\mathrm{gr}(E)$ (resp., a gr-transcendency basis of $\mathrm{gr}(E)/\mathrm{gr}(F)$), then we call T gradually algebraically free (resp., a gradually transcendental basis of E/F).

Observe that if T is a gradually transcendental basis of E/F, then there exists $T_1 \subset E$ such that $T \cup T_1$ is a transcendency basis of E/F. Moreover, if $[E:F]_t$ is finite, then the cardinality of T_1 is independent of the choice of T. Indeed, we can take for T_1 a transcendency basis of $E/F(T)$, and if $[E:F]_t$ is finite, then the cardinality of T_1 is nothing but the transcendency degree of $E/F(T)$.

Corollary 6.1. *Let E/F be an extension of valued fields. Then*

(1) $[E:F]_{t\cdot g} \leq [E:F]_t$ *and* $[E:F]_{t\cdot g} = [E:F]_{t\cdot r} + [E:F]_{t\cdot v}$*;*

(2) $\mathrm{rank}(\Gamma_E/\Gamma_F) \leq [E:F]_t$ *and* $[\overline{E}:\overline{F}]_t \leq [E:F]_t$*;*

(3) *if E/F is gradually purely transcendental (i.e.,* $\mathrm{gr}(E)/\mathrm{gr}(F)$ *is purely* gr-*transcendental) and* Γ_E/Γ_F *is torsion (*$[E:F]_{t\cdot v} = 0$*), then E/F is residually purely transcendental.*

We call a valued field extension E/F gr-defective if $[E:F]_{t\cdot g} < [E:F]_t$, and non-gr-defective if $[E:F]_{t\cdot g} = [E:F]_t$. The extension E/F is non-gr-defective if and only if there exists a transcendency basis T of E such that $\widetilde{T}$ is a gr-transcendency basis of $\mathrm{gr}(E)/\mathrm{gr}(F)$.

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