# UNIFORM LIMIT POWER-TYPE FUNCTION SPACES

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To answer a question proposed by Mari in 1996, we propose $\mathcal{ULP}_\alpha(\mathbb{R}^+)$, the space of uniform limit power functions. We show that $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ has properties similar to that of $\mathcal{AP}(\mathbb{R}^+)$. We also proposed three other limit power function spaces.



## 1. Introduction

In literature of Fourier transforms and Wavelet transforms, the basic space is $L_2(\mathbb{R})$. From the point of view of signal analysis, a signal $f \in L_2(\mathbb{R})$ can only be transient (or "wavelets"). During recent years in some application areas, it has become more common to motivate a theory via persistent rather than transient signals (e.g., [16, 28, 30, 42]). To work on persistent signals, people have to seek a space different from $L_2(\mathbb{R})$. One important example of such spaces is $\mathcal{AP}(\mathbb{R})$, the space of almost periodic functions. People have developed a profound theory and applications for $\mathcal{AP}(\mathbb{R})$ (e.g., see [4, 5, 7–15, 18, 19, 24, 26, 30, 32, 37, 38]).

As in [30], a function $f$ is called limit power if the limit

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2 dt \tag{1.1}$$

exists. Denote by $H_2$ the set of all such functions.

It is well known that $\mathcal{AP}(\mathbb{R}) \subset H_2$ and so is the Besicovitch space $B_2$ [5], the completion of $\mathcal{AP}(\mathbb{R})$ in $H_2$. In fact, many useful persistent signals are in $H_2$, for example, the bounded power signals studied in Wiener's generalized harmonic analysis [36]. However, $H_2$ is not a linear set. An example in [29] shows that $H_2$ is not closed under addition. The lack of closedness under addition caused some difficulties in Robust control (e.g., see [27]).

As [29] pointed out that except for some subsets of $H_2$ which are already known to be vector spaces (e.g., $L_2(\mathbb{R})$, $\{f \in L_\infty(\mathbb{R}) : \lim_{|t|\to\infty} f(t) \text{ exists}\}$, $\mathcal{AP}(\mathbb{R})$), it is not clear whether a "nice" (e.g., Hilbert) large vector space could be defined.

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Let us recall when the spaces mentioned above were invented. The latest one is $B_2$ invented by Besicovitch in 1926 (see [5]); a year earlier is $\mathscr{AP}(\mathbb{R}^+)$ invented by Bohr [8–10]; $L_2(\mathbb{R})$ was invented even earlier. We remark that the function set studied by Wiener mentioned above is not closed under addition either. Some generalizations of $\mathscr{AP}(\mathbb{R})$, for example, the functions studied in [1, 2, 17, 21, 32–34, 38], are vector spaces. They are larger than $\mathscr{AP}(\mathbb{R})$ in $\mathscr{C}(\mathbb{R})$. However, they are the same with $\mathscr{AP}(\mathbb{R})$ in $H_2$. We remark that though $H_2$ is not linear, there have been Banach spaces containing $H_2$, for example, the space $B^2$ proposed in [11] (in [28] for the discrete setting). However, [11, 28] use $\overline{\lim}$ instead of lim in (1.1) to construct the spaces. In many cases, lim is needed too. The background of [29] and related problems being pointed out by some authors (e.g., [27, 28, 30] and references therein) show real needs for new, larger, nice spaces in $H_2$.

The purpose of the paper is to propose such spaces. One will see that the new spaces are so natural that they come from what we call generalized trigonometric polynomials in the same way as $\mathscr{AP}(\mathbb{R})$ and $B_2$ come from trigonometric polynomials. One will also see that they are so huge that to compare $\mathscr{AP}(\mathbb{R})$ and $B_2$ with them is the same as to compare one point with $\mathbb{R}^+$.

The layout of the paper is as follows. In the next section, we show the existence of a larger orthonormal basis. In Section 3, we develop a theory of uniform limit power functions in a way parallel to that of $\mathscr{AP}(\mathbb{R})$ (e.g., [12, 13]). In Section 4, we discuss the limit power type functions.

## 2. Orthonormal basis

It is well known that $\{e^{i\lambda t}\}$ is a complete orthonormal basis in $B_2$ [5]. In this section, we consider the set

$$\{e^{i\lambda t^\alpha}\}, \tag{2.1}$$

where $\lambda \in \mathbb{R}$ and $0 < \alpha < \infty$.

When $\alpha > 1$, in radar and sonar terminology, the function $e^{i\lambda t^\alpha}$ represents a chirp signal because it is reasonably well defined but steadily rising frequency. By analyzing $f(t) = \sin(\pi t^2)$, [25, Chapter 2] points out the fact that a chirp has a well-defined instantaneous frequency and ordinary Fourier analysis hides the fact. By using Windowed Fourier transform, the signal is reasonably well localized both in time and in frequency. In particular, when $\alpha = 2$, the function $e^{it^2}$, being an underlying kernel (e.g., in oscillatory integrals, optics, etc.), has important applications; we refer the reader to [3, 6, 20, 22, 23, 31, 35] for details.

When $\alpha < 1$, the function $e^{i\lambda t^\alpha}$ behaves conversely.

As $\{e^{i\lambda t}\}$, the set $\{e^{i\lambda t^\alpha}\}$ is also orthonormal. We show this in the next two theorems.

Theorem 2.1. *For $\alpha \geq \beta \geq 0$ with $\alpha \geq 1$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq 0$, the following limit*

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{T+a} e^{i(\lambda t^\alpha + \mu t^\beta)}\,dt = \begin{cases} 1, & \alpha = \beta,\ \mu = -\lambda, \\ 0, & \textit{otherwise} \end{cases} \tag{2.2}$$

*exists uniformly with respect to $a \in \mathbb{R}^+$.*

*Proof.* The conclusion for the case of $\alpha = \beta$, $\mu = -\lambda$ is obvious, so we only consider the other cases. In these cases, if we can find $a_0 > 0$ such that

$$\frac{1}{T}\int_a^{T+a} e^{i(\lambda t^\alpha + \mu t^\beta)}dt \longrightarrow 0 \quad (T \longrightarrow \infty) \tag{2.3}$$

uniformly with respect to $a \in [a_0, \infty)$, then we have

$$\frac{1}{T}\int_a^{T+a} e^{i(\lambda t^\alpha + \mu t^\beta)}dt \longrightarrow 0 \quad (T \longrightarrow \infty) \tag{2.4}$$

uniformly with respect to $a \in \mathbb{R}^+$. In fact, for $a \in [0, a_0]$, one has

$$\begin{aligned} \left|\frac{1}{T}\int_a^{T+a} e^{i(\lambda t^\alpha + \mu t^\beta)}dt\right| &\leq \frac{1}{T}\left|\left(\int_a^{a_0} + \int_{a_0}^{T+a_0} - \int_{T+a}^{T+a_0}\right)e^{i(\lambda t^\alpha + \mu t^\beta)}dt\right| \\ &\leq \frac{2a_0}{T} + \frac{1}{T}\left|\int_{a_0}^{T+a_0} e^{i(\lambda t^\alpha + \mu t^\beta)}dt\right| \longrightarrow 0 \quad (T \longrightarrow \infty). \end{aligned} \tag{2.5}$$

Note that $\alpha$, $\beta$, $\lambda$, and $\mu$ are fixed, we can chose $a_0 > 1$ such that $|\lambda\alpha + \mu\beta a^{\beta-\alpha}| \geq \varepsilon_0 > 0$ for some $\epsilon_0 > 0$ and all $a \in [a_0, \infty)$. In fact, if $\beta = \alpha$ then $\lambda + \mu \neq 0$ and for all $a \in [1, \infty)$ one has $|\lambda\alpha + \mu\beta a^{\beta-\alpha}| = |\alpha(\lambda + \mu)| = \varepsilon_0 > 0$; if $\beta < \alpha$, then $t^{\beta-\alpha} \to 0$ as $t \to \infty$, and therefore there exists $a_0 > 1$ such that for all $a \in [a_0, \infty)$ one has $|\lambda\alpha + \mu\beta a^{\beta-\alpha}| \geq |\lambda\alpha/2| = \varepsilon_0 > 0$. For such $a$ and $t \geq a$, $\lambda\alpha t^{\alpha-1} + \mu\beta t^{\beta-1} = t^{\alpha-1}[\lambda\alpha + \mu\beta a^{\beta-\alpha}] \neq 0$ and

$$\begin{aligned} &\int_a^{T+a} e^{i(\lambda t^\alpha + \mu t^\beta)}dt \\ &\quad = \int_a^{T+a} \frac{e^{i(\lambda t^\alpha + \mu t^\beta)} i(\lambda\alpha t^{\alpha-1} + \mu\beta t^{\beta-1})}{i(\lambda\alpha t^{\alpha-1} + \mu\beta t^{\beta-1})}dt = \int_a^{T+a} \frac{e^{i(\lambda t^\alpha + \mu t^\beta)} di(\lambda t^\alpha + \mu t^\beta)}{i(\lambda\alpha t^{\alpha-1} + \mu\beta t^{\beta-1})} \\ &\quad = \frac{e^{i(\lambda t^\alpha + \mu t^\beta)}}{i(\lambda\alpha t^{\alpha-1} + \mu\beta t^{\beta-1})}\Bigg|_a^{T+a} - \frac{1}{i}\int_a^{T+a} e^{i(\lambda t^\alpha + \mu t^\beta)} d\frac{1}{(\lambda\alpha t^{\alpha-1} + \mu\beta t^{\beta-1})} = I_1 + I_2. \end{aligned} \tag{2.6}$$

So

$$\begin{aligned} |I_1| &\leq \frac{1}{|\lambda\alpha(T+a)^{\alpha-1} + \mu\beta(T+a)^{\beta-1}|} + \frac{1}{|\lambda\alpha a^{\alpha-1} + \mu\beta a^{\beta-1}|} \\ &= \frac{1}{(T+a)^{\alpha-1}|\lambda\alpha + \mu\beta(T+a)^{\beta-\alpha}|} + \frac{1}{a^{\alpha-1}|\lambda\alpha + \mu\beta a^{\beta-\alpha}|} \leq M_1, \end{aligned} \tag{2.7}$$

where $M_1$ is a constant which is independent of $T$ and $a \in [a_0, \infty)$.

To estimate $I_2$, we have

$$
\begin{aligned}
|I_2| &\le \int_a^{T+a} \left| \frac{\lambda\alpha(\alpha-1)t^{\alpha-2}+\mu\beta(\beta-1)t^{\beta-2}}{\left(\lambda\alpha t^{\alpha-1}+\mu\beta t^{\beta-1}\right)^2} \right| dt = \int_a^{T+a} \left| \frac{\lambda\alpha(\alpha-1)+\mu\beta(\beta-1)t^{\beta-\alpha}}{t^{2-\alpha}\left(\lambda\alpha t^{\alpha-1}+\mu\beta t^{\beta-1}\right)^2} \right| dt \\
&\le \int_a^{T+a} \frac{|\lambda\alpha(\alpha-1)|+|\mu\beta(\beta-1)|}{t^{\alpha}\left(\lambda\alpha+\mu\beta t^{\beta-\alpha}\right)^2} dt \le M_2 \int_a^{T+a} \frac{1}{t^{\alpha}} dt \\
&= \begin{cases} M_2\left(\ln\left(\dfrac{T}{a}+1\right)\right) \le M_2 \ln(T+1), & \alpha = 1, \\ M_2\left[\left(\dfrac{1}{a}\right)^{\alpha-1} - \left(\dfrac{1}{(T+a)}\right)^{\alpha-1}\right], & \alpha > 1, \end{cases}
\end{aligned}
\tag{2.8}
$$

where $M_2$ is a constant which is independent of $T$ and $a \in [a_0, \infty)$.

It follows from (2.6)–(2.8) that

$$
\lim_{T\to\infty} \frac{1}{T} \int_a^{T+a} e^{i(\lambda t^{\alpha}+\mu t^{\beta})} = 0 \tag{2.9}
$$

uniformly with respect to $a \in [a_0, \infty)$, and therefore with respect to $a \in \mathbb{R}^+$, the proof is complete. □

Corollary 2.2. *For $\alpha \ge 1$ and $\lambda \ne 0$, the limit*

$$
\lim_{T\to\infty} \frac{1}{T} \int_a^{T+a} e^{i\lambda t^{\alpha}} dt = 0 \tag{2.10}
$$

*exists uniformly with respect to $a \in \mathbb{R}^+$.*

*Proof.* Put $\mu = 0$ in Theorem 2.1 to get the conclusion. □

Theorem 2.3. *Let $1 > \alpha \ge \beta \ge 0$ with $\alpha > 0$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda \ne 0$. Then*

$$
\lim_{T\to\infty} \frac{1}{T} \int_0^{T} e^{i(\lambda t^{\alpha}+\mu t^{\beta})} dt = \begin{cases} 1, & \alpha = \beta,\ \mu = -\lambda, \\ 0, & \text{otherwise}. \end{cases} \tag{2.11}
$$

*Proof.* First, we consider the case $\alpha = \beta$ and $w = \lambda + \mu \ne 0$,

$$
\int_1^T e^{iwt^{\alpha}} dt = \int_1^T \frac{e^{iwt^{\alpha}} iw\alpha t^{\alpha-1}}{iw\alpha t^{\alpha-1}} dt = \frac{e^{iwt^{\alpha}}}{iw\alpha t^{\alpha-1}} \bigg|_1^T - \int_1^T \frac{e^{iwt^{\alpha}}}{iw\alpha t^{\alpha}} (1-\alpha) dt. \tag{2.12}
$$

So

$$\frac{1}{T}\int_0^T e^{iwt^\alpha}dt = \frac{1}{T}\left(\int_0^1 + \int_1^T e^{iwt^\alpha}dt\right) \longrightarrow 0 \tag{2.13}$$

as $T \to \infty$.

If $\alpha > \beta$, let $T > a > 1$ be so large that $|\lambda\alpha| > |\mu\beta a^{\beta-\alpha}|$. As in the proof of Theorem 2.1, one has

$$\begin{aligned}
&\int_a^T e^{i(\lambda t^\alpha + \mu t^\beta)}dt = I_3 + I_4,\\
|I_3| &\le \frac{1}{(\lambda\alpha T^{\alpha-1} + \mu\beta T^{\beta-1})} + \frac{1}{(\lambda\alpha a^{\alpha-1} + \mu\beta a^{\beta-1})}\\
&\le \frac{T^{1-\alpha}}{(\lambda\alpha + \mu\beta T^{\beta-\alpha})} + \frac{a^{1-\alpha}}{(\lambda\alpha + \mu\beta a^{\beta-\alpha})}\\
&\le T^{1-\alpha}\left[\frac{1}{(\lambda\alpha + \mu\beta T^{\beta-\alpha})} + \frac{1}{(\lambda\alpha + \mu\beta a^{\beta-\alpha})}\right]\\
&\le T^{1-\alpha}\left[\frac{1}{|\lambda\alpha| - |\mu\beta a^{\beta-\alpha}|} + \frac{1}{|\lambda\alpha| - |\mu\beta a^{\beta-\alpha}|}\right]\\
&\le M_3 T^{1-\alpha}.
\end{aligned} \tag{2.14}$$

To estimate $I_4$ we have the following:

$$\begin{aligned}
|I_4| &\le \int_a^T \left|\frac{\lambda\alpha(\alpha-1)t^{\alpha-2} + \mu\beta(\beta-1)t^{\beta-2}}{(\lambda\alpha t^{\alpha-1} + \mu\beta t^{\beta-1})^2}\right| dt\\
&= \int_a^T \left|\frac{\lambda\alpha(\alpha-1)t^{\alpha-2} + \mu\beta(\beta-1)t^{\beta-2}}{t^{2(\alpha-1)}(\lambda\alpha + \mu\beta t^{\beta-\alpha})^2}\right| dt\\
&= \int_a^T \left|\frac{\lambda\alpha(\alpha-1)t^{-\alpha} + \mu\beta(\beta-1)t^{\beta-2\alpha}}{(\lambda\alpha + \mu\beta t^{\beta-\alpha})^2}\right| dt\\
&\le \int_a^T \frac{|\lambda\alpha(\alpha-1)|t^{-\alpha} + |\mu\beta(\beta-1)|t^{\beta-2\alpha}}{[|\lambda\alpha| - |\mu a^{\beta-\alpha}|]^2} dt\\
&\le M_4 t^{1-\alpha}\Big|_a^T + M_5 t^{1+\beta-2\alpha}\Big|_a^T,
\end{aligned} \tag{2.15}$$

where $M_4$ and $M_5$ are constants which are independent of $T$.

It follows that

$$\begin{aligned}
\left|\frac{1}{T}\int_0^T e^{i(\lambda t^\alpha + \mu t^\beta)}dt\right| &= \frac{1}{T}\left|\int_0^a + \int_a^T e^{i(\lambda t^\alpha + \mu t^\beta)}dt\right|\\
&\le \left|\frac{1}{T}\int_0^a e^{i(\lambda t^\alpha + \mu t^\beta)}dt\right| + \left|\frac{1}{T}\int_a^T e^{i(\lambda t^\alpha + \mu t^\beta)}dt\right| \longrightarrow 0
\end{aligned} \tag{2.16}$$

as $T \to \infty$. The proof is complete. □

It follows from Theorems 2.1 and 2.3 that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(e^{i\lambda t^{\alpha}}\cdot e^{-i\mu t^{\beta}}\right)dt=\begin{cases}1, & \alpha=\beta,\ \mu=\lambda,\\ 0, & \text{otherwise.}\end{cases} \tag{2.17}$$

That is, the set $\{e^{i\lambda t^{\alpha}}\}$ constitutes an orthonormal basis.

*Remark 2.4.* Since the domain of the function $e^{i\lambda t^{\alpha}}$ in general is $\mathbb{R}^+$, we consider $\mathbb{R}^+$ only in the paper. For special numbers of $\alpha$, for example, $\alpha's$ are positive integers, the domain will be $\mathbb{R}$. In this case all the results will hold for $\mathbb{R}$.

## 3. Uniform limit power functions

We call the functions

$$\sum_{k=1}^{n} a_k e^{i\lambda_k t^{\alpha}} \tag{3.1}$$

$\alpha$-trigonometric polynomial, where $a_k \in C$ and $\lambda_k \in \mathbb{R}$. As $\mathscr{AP}(\mathbb{R})$, we have the following definition.

*Definition 3.1.* Let $\alpha > 0$ be fixed. A function $f$ on $\mathbb{R}^+$ is called uniform limit power if for each $\epsilon > 0$ there exists an $\alpha$-trigonometric polynomial $P_\epsilon$ such that

$$\|f-P_\epsilon\| = \sup\left\{\left|f(t)-P_\epsilon(t)\right| : t\in\mathbb{R}^+\right\} < \epsilon. \tag{3.2}$$

Denote by $\mathscr{ULP}_\alpha(\mathbb{R}^+)$ the set of all such functions.

One sees that when $\alpha = 1$, $\mathscr{ULP}_\alpha(\mathbb{R}^+) = \mathscr{AP}(\mathbb{R}^+)$. Also one sees that $\mathscr{ULP}_\alpha(\mathbb{R}^+)$ is the completion of $\alpha$-trigonometric polynomial in $C(\mathbb{R}^+)$. Since the set of $\alpha$-trigonometric polynomials are closed under addition, multiplication, and conjugation, so is the completion $\mathscr{ULP}_\alpha(\mathbb{R}^+)$. Thus we have shown the following statement: $\mathscr{ULP}_\alpha(\mathbb{R}^+)$ is a $C^*$-subalgebra of $C(\mathbb{R}^+)$ containing the constant functions.

Next, we discuss the Fourier expansion of $f \in \mathscr{ULP}_\alpha(\mathbb{R}^+)$. First of all, we show that the mean exists.

THEOREM 3.2. *If $f \in \mathscr{ULP}_\alpha(\mathbb{R}^+)$, then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\,dt \tag{3.3}$$

*exists. In the case of $\alpha \geq 1$,*

$$\lim_{T\to\infty}\frac{1}{T}\int_a^{T+a} f(t)\,dt \tag{3.4}$$

*exists uniformly with respect to $a \in \mathbb{R}^+$.*

*Proof.* We first show the theorem in the case that $f$ is an $\alpha$-trigonometric polynomial. Let

$$f(t) = P(t) = c_0 + \sum_{k=1}^{n} c_k e^{i\lambda_k t^\alpha}. \tag{3.5}$$

Then by Theorems 2.1 and 2.3,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P(t)dt = c_0. \tag{3.6}$$

If $f$ is an arbitrary function in $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ then for $\epsilon > 0$ there exists an $\alpha$-trigonometric polynomial $P_\epsilon$ such that (3.2) holds. Since $\lim_{T\to\infty}(1/T)\int_0^T P_\epsilon(t)dt$ exists, we can find a number $T_0$ such that when $T_1, T_2 > T_0$,

$$\left| \frac{1}{T_1}\int_0^{T_1} P_\epsilon(t)dt - \frac{1}{T_2}\int_0^{T_2} P_\epsilon(t)dt \right| < \epsilon. \tag{3.7}$$

It follows from (3.2) and the last inequality above that when $T_1, T_2 > T_0$,

$$\begin{aligned} \left| \frac{1}{T_1}\int_0^{T_1} f(t)dt - \frac{1}{T_2}\int_0^{T_2} f(t)dt \right| &\le \frac{1}{T_1}\int_0^{T_1} |f(t) - P_\epsilon(t)|\,dt \\ &\quad + \left| \frac{1}{T_1}\int_0^{T_1} P_\epsilon(t)dt - \frac{1}{T_2}\int_0^{T_2} P_\epsilon(t)dt \right| \\ &\quad + \frac{1}{T_2}\int_0^{T_2} |f(t) - P_\epsilon(t)|\,dt < 3\epsilon. \end{aligned} \tag{3.8}$$

Similarly, one shows the existence of the second limit. The proof is complete. □

We call the limit in Theorem 3.2 the mean of $f$ and denote it by $M(f)$.

For $\lambda \in \mathbb{R}$ and $f \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ since the function $fe^{-i\lambda t^\alpha}$ is in $\mathcal{ULP}_\alpha(\mathbb{R}^+)$, the mean exists for the function. We write

$$a(\lambda) = M(fe^{-i\lambda t^\alpha}). \tag{3.9}$$

As the proof for $\mathcal{AP}(\mathbb{R}^+)$ (see [12, 13, 18, 26, 37, 38]), for a function $f \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ the frequency set

$$\text{Freq}(f) = \{\lambda \in \mathbb{R} : a(\lambda) \neq 0\} \tag{3.10}$$

is countable (or finite). Let $\text{Freq}(f) = \{\lambda_k\}$ and $A_k = a(\lambda_k)$. Thus $f$ has an associated Fourier series

$$f(t) \sim \sum_{k=1}^{\infty} A_k e^{i\lambda_k t^\alpha}, \tag{3.11}$$

and Parseval's equality holds:

$$\sum_{k=1}^{\infty} |a(\lambda_k)|^2 = M(|f|^2). \tag{3.12}$$

The unique theorem for almost periodic function is well known. That is, distinct almost periodic functions have distinct Fourier series. We point out that this is also true for $\mathcal{ULP}_\alpha(\mathbb{R}^+)$. To show this we need to set up some correspondence between $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ and $\mathcal{AP}(\mathbb{R}^+)$. For the $\alpha$-trigonometric polynomial $P_\epsilon$ in Definition 3.1, let $s = t^\alpha$. Then $P_\epsilon$ becomes trigonometric polynomial of $s$. That is,

$$P_\epsilon(s) = \sum_{k=1}^{n} a_k e^{i\lambda_k s}. \tag{3.13}$$

For $f \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ define the function

$$\widetilde{f}(s) = f(s^{1/\alpha}). \tag{3.14}$$

Thus, (3.2) becomes

$$|\widetilde{f}(s) - P_\epsilon(s)| < \epsilon \quad (s \in \mathbb{R}^+). \tag{3.15}$$

So, $\widetilde{f} \in \mathcal{AP}(\mathbb{R}^+)$. Conversely, let $h \in \mathcal{AP}(\mathbb{R}^+)$. By the approximation theorem of $\mathcal{AP}(\mathbb{R}^+)$ for $\epsilon > 0$, there exists a trigonometric polynomial $\sum_{k=1}^{n} a_k e^{i\lambda_k s}$ such that

$$\left| h(s) - \sum_{k=1}^{n} a_k e^{i\lambda_k s} \right| < \epsilon \quad (s \in \mathbb{R}^+). \tag{3.16}$$

Let $s = t^\alpha (t \in \mathbb{R}^+)$ and let $\overline{h}(t) = h(t^\alpha)$. It follows that

$$\left| \overline{h}(t) - \sum_{k=1}^{n} a_k e^{i\lambda_k t^\alpha} \right| < \epsilon \quad (t \in \mathbb{R}^+). \tag{3.17}$$

Therefore, we have $\overline{h} \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$. Thus (3.14) is the correspondence between $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ and $\mathcal{AP}(\mathbb{R}^+)$.

Note the translate property of almost periodic function, that is, for $\epsilon > 0$ there exists $l > 0$ with the property that any interval $I \subset \mathbb{R}^+$ of length $l$ has a number $\tau \in I$ such that

$$|\widetilde{f}(s+\tau) - \widetilde{f}(s)| < \epsilon \quad (s \in \mathbb{R}^+). \tag{3.18}$$

By the correspondence (3.14), we have in fact already shown the following theorem.

Theorem 3.3. *Let $f \in C(\mathbb{R}^+)$. Then the following statements are equivalent:*

(1) $f \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$;

(2) *for $\epsilon > 0$ there exists $l > 0$ with the property that any interval $I \subset \mathbb{R}^+$ of length $l$ has a number $\tau \in I$ such that*

$$|f[(t+\tau)^{1/\alpha}] - f(t^{1/\alpha})| < \epsilon. \tag{3.19}$$

Furthermore, if $f \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ then so is $|f(\cdot)|$.

Now, we make use of the unique theorem for $\widetilde{f}$ to get the same conclusion for $f$.

LEMMA 3.4. *Let $f \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ be nonnegative and $f(t_0) > 0$ for some $t_0 \in \mathbb{R}^+$. Then $M(f) > 0$.*

*Proof.* Let $\widetilde{f}$ be the function in (3.14). So $\widetilde{f}(s) \geq 0$ and $\widetilde{f}(s_0) > 0$, where $s_o = t_0^\alpha$. Since $\widetilde{f} \in \mathcal{AP}(\mathbb{R}^+)$, one has

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{T+a} \widetilde{f}(s)ds = M(\widetilde{f}) > 0 \tag{3.20}$$

uniformly with respect to $a \in \mathbb{R}^+$.

To show the theorem we need to discuss two cases.

(1) $\alpha \geq 1$:

$$\begin{aligned} \frac{1}{T}\int_1^T f(t)dt &= \frac{1}{T}\int_1^{T^\alpha} f\left(s^{1/\alpha}\right)\frac{s^{1/\alpha-1}}{\alpha}ds = \frac{1}{\alpha}\frac{1}{T}\int_1^{T^\alpha} \frac{\widetilde{f}(s)}{s^{1-1/\alpha}}ds \\ &\geq \frac{1}{\alpha}\frac{1}{T}\int_1^{T^\alpha} \frac{\widetilde{f}(s)}{\left(T^\alpha\right)^{1-1/\alpha}}ds = \frac{1}{\alpha}\frac{1}{T^\alpha}\int_1^{T^\alpha} \widetilde{f}(s)ds. \end{aligned} \tag{3.21}$$

It follows from (3.20) that

$$M(f(t)) \geq \frac{1}{\alpha}M(\widetilde{f}(s)) > 0. \tag{3.22}$$

(2) $0 < \alpha < 1$: in this case,

$$\begin{aligned} \frac{1}{T}\int_0^T f(t)dt &= \frac{1}{T}\int_0^{T^\alpha} f\left(s^{1/\alpha}\right)\frac{s^{1/\alpha-1}}{\alpha}ds = \frac{1}{\alpha}\frac{1}{T}\int_0^{T^\alpha} \widetilde{f}(s)s^{1/\alpha-1}ds \\ &\geq \frac{1}{\alpha}\frac{1}{T}\int_{T^\alpha/2}^{T^\alpha} \widetilde{f}(s)\left(\frac{T^\alpha}{2}\right)^{1/\alpha-1}ds = \frac{1}{\alpha}\frac{1}{T}\frac{T^{1-\alpha}}{2^{1/\alpha-1}}\int_{T^\alpha/2}^{T^\alpha} \widetilde{f}(s)ds \\ &= \frac{1}{\alpha 2^{1/\alpha}}\frac{1}{T^\alpha/2}\int_{T^\alpha/2}^{T^\alpha} \widetilde{f}(s)ds. \end{aligned} \tag{3.23}$$

So, in this case we also have

$$M(f(t)) \geq \frac{1}{\alpha 2^{1/\alpha}}M(\widetilde{f}(s)) > 0. \tag{3.24}$$

The proof is complete. □

By the lemma above we are able to show the following unique theorem.

THEOREM 3.5. *Distinct uniform limit power functions have distinct Fourier series.*

*Proof.* Suppose that the distinct functions $f_1, f_2 \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ have the same Fourier series. Then $f_1 - f_2$ will have Fourier series of all term zero. By Parseval's equality $M(|f_1 - f_2|^2) = 0$. However, by Lemma 3.4 we get $M(|f_1 - f_2|^2) > 0$. This is a contraction. The proof is complete. □

For $f(t) \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ since $\widetilde{f}(s) \in \mathcal{AP}(\mathbb{R}^+)$, the function $\widetilde{f}$ has an associated Fourier series

$$\widetilde{f}(s) \sim \sum_{k=1}^{\infty} a_k e^{i\mu_k s}, \tag{3.25}$$

where $a_k = M(\widetilde{f}(s)e^{-i\mu_k s})$. It is well known (e.g., see [12, 13, 38]) that $\widetilde{f}$ can be approximated uniformly on $\mathbb{R}$ by the Bocher-Fejér trigonometric polynomials

$$\sigma_m(s) = \sum_{k=1}^{n(m)} r_{m,k} a_k e^{i\mu_k s}, \tag{3.26}$$

where the rational numbers $0 \le r_{m,k} \le 1$ and $\lim_{m\to\infty} r_{m,k} = 1$. Thus, replacing $s$ in (3.26) by $t^\alpha$ we have

$$\left| \sum_{k=1}^{n(m)} r_{m,k} a_k e^{i\mu_k t^\alpha} - \widetilde{f}(t^\alpha) \right| = \left| \sum_{k=1}^{n(m)} r_{m,k} a_k e^{i\mu_k t^\alpha} - f(t) \right| \longrightarrow 0, \tag{3.27}$$

uniformly on $\mathbb{R}^+$.

The following remark tells us an important conclusion.

*Remark 3.6.* In the section all the results are achieved under the assumption of fixed $\alpha$. We may release the restriction on $\alpha$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be nonnegative sequence. A generalized trigonometric polynomial is a function of the form

$$\sum_{k=1}^{n} a_k e^{i\lambda_k t^{\alpha_k}}, \tag{3.28}$$

where $a_k \in C$ and $\lambda_k \in \mathbb{R}$, $1 \le k \le n$. If in Definition 3.1 $P_\epsilon$ is a generalized trigonometric polynomial, then the function $f$ is also called uniform limit power and $\mathcal{ULP}(\mathbb{R}^+)$ is denoted the set of all such functions. It is not difficult to show that $\mathcal{ULP}(\mathbb{R}^+)$ is a Banach space and (3.10)–(3.12) are valid. For the question if an $f \in \mathcal{ULP}(\mathbb{R}^+)$ can be approximated by the Bochner-Fejer polynomials, as well as how to construct the polynomials, we refer the reader to [40, 41] for details. Also

$$\mathcal{ULP}(\mathbb{R}^+) = \overline{\operatorname{span}}\{\mathcal{ULP}_\alpha(\mathbb{R}^+) : 0 < \alpha < \infty\} \subset C(\mathbb{R}^+) \cap H_2, \tag{3.29}$$

where the closure is taken in $C(\mathbb{R}^+)$. One can see how huge $\mathcal{ULP}(\mathbb{R}^+)$ is by comparing with $\mathcal{AP}(\mathbb{R}^+)$.

*Remark 3.7.* As chirps, some existing results enable us to analyze and reconstruct $f \in \mathcal{ULP}(\mathbb{R}^+)$. For example, by [30, Theorem 2.1] a windowed Fourier transform of $f$ exists, by Theorem 2.2 of the same paper the transform satisfies some Parseval's relation, and by Theorem 2.4 of that paper again a generalized frame exists.

One more remark is needed to end the section.

*Remark 3.8.* (1) The conclusion in the paragraph before Remark 2.4 is mentioned in [39, Section 2] without proof. Here we not only prove it in details, but we also distinguish the limits between the case $\alpha \geq 1$ and the case $\alpha < 1$ in Theorems 2.1 and 2.3, respectively. (2) Also, the results in Section 3 are presented in [39, Section 2] in an abstract-like form. To convince the reader the correctness of these results, we present and prove them in details here.

## 4. Limit power type functions

In this section, we will define and investigate three types of limit power function which are corresponding to the three types of well-known almost periodic functions (e.g., see [1, 2, 17, 21, 31, 33, 34, 38]).

Let

$$\begin{aligned} C_0(\mathbb{R}^+) &= \{\varphi \in C(\mathbb{R}^+) : \lim_{t\to\infty} \varphi(t) = 0\}, \\ \mathcal{PAP}_0(\mathbb{R}^+) &= \{\varphi \in C(\mathbb{R}^+) : M(|\varphi|) = 0\}. \end{aligned} \tag{4.1}$$

*Definition 4.1.* Let $f \in C(\mathbb{R}^+)$. A function $f$ is called asymptotic limit power if

$$f(t) = g(t) + \varphi(t) \quad (t \in \mathbb{R}^+), \tag{4.2}$$

where $g \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ and $\varphi \in C_0(\mathbb{R}^+)$. Denote by $\mathcal{ALP}_\alpha(\mathbb{R}^+)$ all such functions.

By (3.14), we have

$$\widetilde{f}(s) = \widetilde{g}(s) + \widetilde{\varphi}(s). \tag{4.3}$$

It is easy to check that $\varphi(t)$ is in $C_0(\mathbb{R}^+)$ if and only if $\widetilde{\varphi}(s)$ is in $C_0(\mathbb{R}^+)$. Since $\widetilde{g}(s) \in \mathcal{AP}(\mathbb{R}^+)$, one gets that $\widetilde{f}(s) \in \mathcal{AAP}(\mathbb{R}^+)$, the space of asymptotically almost periodic functions. Therefore, (3.14) is also a correspondence between $\mathcal{AAP}(\mathbb{R}^+)$ and $\mathcal{ALP}_\alpha(\mathbb{R}^+)$. Note the characterization of $\mathcal{AAP}(\mathbb{R}^+)$ (e.g., see [38, Theorem 1.2.11]), we have the following corresponding characterization.

Theorem 4.2. *Let $f \in C(\mathbb{R}^+)$. Then the following statements are equivalent:*

(1) $\widetilde{f} \in \mathcal{ALP}_\alpha(\mathbb{R}^+)$;

(2) *the set* $\{f[(t+x)^{1/\alpha}] : x \in \mathbb{R}^+\}$ *is relatively compact in* $C(\mathbb{R}^+)$;

(3) *for any* $\epsilon > 0$ *there exists a bounded closed interval* $C = [0,a]$ *and* $l > 0$ *such that any interval* $I \subset \mathbb{R}^+$ *of length* $l$ *has a number* $\tau \in I$ *with the property*

$$|f[(t+\tau)^{1/\alpha}] - f(t)^{1/\alpha}| < \epsilon \quad (t, t+\tau \in \mathbb{R}^+ \setminus C). \tag{4.4}$$

If we only require the set in Theorem 4.2(2) to be weakly compact, then we get the following concept.

*Definition 4.3.* An $f \in C(\mathbb{R}^+)$ is called weak limit power if the set in Theorem 4.2(2) is weakly compact in $C(\mathbb{R}^+)$. Denote by $\mathcal{WLP}_\alpha(\mathbb{R}^+)$ of all such functions.

To get the decomposition of a function $f \in \mathcal{WLP}_\alpha(\mathbb{R}^+)$, we introduce the following set:

$$\mathcal{WLP}_0(b)\mathbb{R}^+ = \left\{\varphi \in \mathcal{WLP}_\alpha(\mathbb{R}^+) : 0 \in \overline{\{\varphi[(t+x)^{1/\alpha}] : x \in \mathbb{R}^+\}}\right\}, \tag{4.5}$$

where the closure is taken under weak topology in $C(\mathbb{R}^+)$. The following result is a correspondence in $\mathcal{WLP}_\alpha(\mathbb{R}^+)$ to that in $\mathcal{WAP}(\mathbb{R}^+)$.

Theorem 4.4. *Let $f \in C(\mathbb{R}^+)$. Then the following statements are equivalent:*
(1) $f \in \mathcal{WLP}_\alpha(\mathbb{R}^+)$;
(2) $f = g + \varphi$, *where* $g \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ *and* $\varphi \in \mathcal{WLP}_0(\mathbb{R}^+)$.

Finally we give the following concept.

*Definition 4.5.* An $f \in C(\mathbb{R}^+)$ is called pseudolimit power if $f$ has the form $f = g + \varphi$, where $g \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ and $\varphi \in \mathcal{PAP}_0(\mathbb{R}^+)$. Denote by $\mathcal{PLP}_\alpha(\mathbb{R}^+)$ all such functions.

Let $f \in \mathcal{PLP}_\alpha(\mathbb{R}^+)$. Since $f(t)e^{-i\lambda t^\alpha} = g(t)e^{-i\lambda t^\alpha} + \varphi(t)e^{-i\lambda t^\alpha}$ and

$$\lim_{T\to\infty} 1/T \int_0^T f(t)e^{-i\lambda t^\alpha}\,dt = M(ge^{-i\lambda t^\alpha}) \tag{4.6}$$

for all $\lambda \in \mathbb{R}$, Theorem 3.5 implies that $\mathcal{PLP}_\alpha(\mathbb{R}^+)$ is a direct sum of $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ and $\mathcal{PAP}_0(\mathbb{R}^+)$. Since the ranges $Rf$ and $R\tilde{f}$ are the same (so $Rg$ and $R\tilde{g}$, $R\varphi$ and $R\tilde{\varphi}$) and $\overline{R\tilde{f}} \supset R\tilde{g}$ [38, Lemma 1.5.2], we have $\overline{Rf} \supset Rg$. By this, we can show the following theorem.

Theorem 4.6. *$\mathcal{PLP}_\alpha(\mathbb{R}^+)$ is a Banach space.*

*Proof.* Let $\{f_n\} \subset \mathcal{PLP}_\alpha(\mathbb{R}^+)$ be Cauchy. Since $\overline{Rf_n} \supset Rg_n$, $\{g_n\}$ is Cauchy too. Note $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ is closed in $C(\mathbb{R}^+)$, there exists $g \in \mathcal{ULP}_\alpha(\mathbb{R}^+)$ such that $\|g_n - g\| \to 0$ as $n \to \infty$. Since $\{f_n - g_n\}$ is also Cauchy and $\mathcal{PAP}_0(\mathbb{R}^+)$ is closed in $C(\mathbb{R}^+)$, there exists $\varphi \in \mathcal{PAP}_0(\mathbb{R}^+)$ such that $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$. Let $f = g + \varphi$. Then $f \in \mathcal{PLP}_\alpha(\mathbb{R}^+)$ and $\|f_n - f\| \to 0$ as $n \to \infty$. The proof is complete. □

Since

$$\mathcal{AP}(\mathbb{R}^+) \subset \mathcal{AAP}(\mathbb{R}^+) \subset \mathcal{WAP}(\mathbb{R}^+) \subset \mathcal{PAP}(\mathbb{R}^+), \tag{4.7}$$

one has the following inclusion relationship:

$$\mathcal{ULP}_\alpha(\mathbb{R}^+) \subset \mathcal{ALP}_\alpha(\mathbb{R}^+) \subset \mathcal{WLP}_\alpha(\mathbb{R}^+) \subset \mathcal{PLP}_\alpha(\mathbb{R}^+). \tag{4.8}$$

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