

A NEW HILBERT-TYPE INTEGRAL INEQUALITY AND THE EQUIVALENT FORM

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Received 30 April 2006; Revised 10 August 2006; Accepted 21 August 2006

We give a new Hilbert-type integral inequality with the best constant factor by estimating the weight function. And the equivalent form is considered.

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1. Introduction

If f, g are real functions such that $0 < \int_0^\infty f^2(x) dx < \infty$ and $0 < \int_0^\infty g^2(x) dx < \infty$, then we have (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left\{ \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right\}^{1/2}, \quad (1.1)$$

where the constant factor π is the best possible. Inequality (1.1) is the well-known Hilbert's inequality. And inequality (1.1) had been generalized by Hardy in 1925 as follows.

If $f, g \geq 0$, $p > 1$, $1/p + 1/q = 1$, $0 < \int_0^\infty f^p(x) dx < \infty$, and $0 < \int_0^\infty g^q(x) dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x) dx \right\}^{1/q}, \quad (1.2)$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \int_0^\infty f^p(x) dx, \quad (1.3)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. When $p = q = 2$, (1.2) reduces to (1.1), inequality (1.2) is named of Hardy-Hilbert integral inequality, which is important in analysis and its applications. It has been studied and generalized in many directions by a number of mathematicians.

2 A new Hilbert-type integral inequality

In this paper, we give a new type of Hilbert's integral inequality as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy < c \left\{ \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right\}^{1/2}, \quad (1.4)$$

where $c = \sqrt{2}(\pi - 2 \arctan \sqrt{2}) = 1.7408\dots$

2. Main results

LEMMA 2.1. *Suppose $\varepsilon > 0$, then*

$$\int_1^\infty x^{-\varepsilon-1} \int_0^{x^{-1}} \frac{1}{1+t+\max\{1,t\}} t^{(-1-\varepsilon)/2} dt dx = O(1)(\varepsilon \rightarrow 0^+). \quad (2.1)$$

Proof. There exists $n \in \mathbb{N}$ which is large enough, such that $1 + (-1 - \varepsilon)/2 > 0$ for $\varepsilon \in (0, 1/n]$, we have

$$\int_0^{x^{-1}} \frac{1}{1+t+\max\{1,t\}} t^{(-1-\varepsilon)/2} dt < \int_0^{x^{-1}} t^{(-1-\varepsilon)/2} dt = \frac{1}{1+(-1-\varepsilon)/2} \left(\frac{1}{x}\right)^{1+(-1-\varepsilon)/2}. \quad (2.2)$$

Since for $a \geq 1$ the function $g(y) = (1/ya^y)$ ($y \in (0, \infty)$) is decreasing, we find

$$\frac{1}{1+(-1-\varepsilon)/2} \left(\frac{1}{x}\right)^{1+(-1-\varepsilon)/2} \leq \frac{1}{1+((-1-1)/n)/2} \left(\frac{1}{x}\right)^{1+((-1-1)/n)/2}, \quad (2.3)$$

so

$$\begin{aligned} 0 &< \int_1^\infty x^{-\varepsilon-1} \int_0^{x^{-1}} \frac{1}{1+t+\max\{1,t\}} t^{(-1-\varepsilon)/2} dt dx \\ &< \int_1^\infty x^{-1} \frac{1}{1+((-1-1)/n)/2} \left(\frac{1}{x}\right)^{1+((-1-1)/n)/2} dx \\ &= \left(\frac{1}{1+((-1-1)/n)/2} \right)^2. \end{aligned} \quad (2.4)$$

Hence the relation (2.1) is valid. The lemma is proved. \square

Now we study the following inequality.

THEOREM 2.2. *Suppose $f(x), g(x) \geq 0$, $0 < \int_0^\infty f^2(x) dx < \infty$, $0 < \int_0^\infty g^2(x) dx < \infty$. Then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy < c \left\{ \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right\}^{1/2}, \quad (2.5)$$

where the constant factor $c = \sqrt{2}(\pi - 2 \arctan \sqrt{2}) = 1.7408\dots$ is the best possible.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy \\
&= \int_0^\infty \int_0^\infty \left[\frac{f(x)}{(x+y+\max\{x,y\})^{1/2}} \left(\frac{x}{y}\right)^{1/4} \right] \\
&\quad \times \left[\frac{g(y)}{(x+y+\max\{x,y\})^{1/2}} \left(\frac{y}{x}\right)^{1/4} \right] dx dy \tag{2.6} \\
&\leq \int_0^\infty \int_0^\infty \frac{f^2(x)}{x+y+\max\{x,y\}} \left(\frac{x}{y}\right)^{1/2} dx dy \\
&\quad \times \int_0^\infty \int_0^\infty \frac{g^2(y)}{x+y+\max\{x,y\}} \left(\frac{y}{x}\right)^{1/2} dx dy.
\end{aligned}$$

Define the weight function $\varpi(u)$ as

$$\varpi(u) := \int_0^\infty \frac{1}{u+v+\max\{u,v\}} \left(\frac{u}{v}\right)^{1/2} dv, \tag{2.7}$$

then the above inequality yields

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy \\
&\leq \left[\int_0^\infty \varpi(x) f^2(x) dx \right]^{1/2} \left[\int_0^\infty \varpi(y) g^2(y) dy \right]^{1/2}. \tag{2.8}
\end{aligned}$$

For fixed u , let $v = ut$, we have

$$\begin{aligned}
\varpi(u) &:= \int_0^\infty \frac{1}{1+t+\max\{1,t\}} \left(\frac{1}{t}\right)^{1/2} dt \\
&= \int_0^1 \frac{1}{2+t} \left(\frac{1}{t}\right)^{1/2} dt + \int_1^\infty \frac{1}{1+2t} \left(\frac{1}{t}\right)^{1/2} dt \tag{2.9} \\
&= \sqrt{2}(\pi - 2 \arctan \sqrt{2}).
\end{aligned}$$

Thus

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy \\
&\leq \sqrt{2}(\pi - 2 \arctan \sqrt{2}) \left\{ \int_0^\infty f^2(x) dx \right\}^{1/2} \left\{ \int_0^\infty g^2(x) dx \right\}^{1/2}. \tag{2.10}
\end{aligned}$$

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If (2.10) takes the form of the equality, then there exist constants a and b , such that they are not all zero and

$$a \frac{f^2(x)}{x+y+\max\{x,y\}} \left(\frac{x}{y}\right)^{1/2} = b \frac{g^2(y)}{x+y+\max\{x,y\}} \left(\frac{y}{x}\right)^{1/2} \quad (2.11)$$

a.e. on $(0, \infty) \times (0, \infty)$.

Then we have

$$axf^2(x) = byg^2(y) \quad \text{a.e. on } (0, \infty) \times (0, \infty). \quad (2.12)$$

Hence we have

$$axf^2(x) = byg^2(y) = \text{constant} = d \quad \text{a.e. on } (0, \infty) \times (0, \infty). \quad (2.13)$$

Without losing the generality, suppose $a \neq 0$, then we obtain $f^2(x) = d/ax$, a.e. on $(0, \infty)$, which contradicts the fact that $0 < \int_0^\infty f^2(x)dx < \infty$. Hence (2.10) takes the form of strict inequality; we get (2.5).

For $0 < \varepsilon < 1$, set $f_\varepsilon(x) = x^{(-\varepsilon-1)/2}$, for $x \in [1, \infty)$; $f_\varepsilon(x) = 0$, for $x \in (0, 1)$. $g_\varepsilon(y) = y^{(-\varepsilon-1)/2}$, for $y \in [1, \infty)$; $g_\varepsilon(y) = 0$, for $y \in (0, 1)$. Assume that the constant factor $c = \sqrt{2}(\pi - 2 \arctan \sqrt{2})$ in (2.2) is not the best possible, then there exists a positive number K with $K < c$, such that (2.5) is valid by changing c to K . We have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy < K \left\{ \int_0^\infty f^2(x)dx \right\}^{1/2} \left\{ \int_0^\infty g^2(x)dx \right\}^{1/2} = \frac{K}{\varepsilon}, \quad (2.14)$$

since

$$\int_0^\infty \frac{1}{1+t+\max\{1,t\}} t^{(-1-\varepsilon)/2} dt = \sqrt{2}(\pi - 2 \arctan \sqrt{2}) + o(1) \quad (\varepsilon \rightarrow 0^+). \quad (2.15)$$

Setting $y = tx$, by (2.1), we find

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy \\ &= \int_1^\infty \int_1^\infty \frac{x^{(-\varepsilon-1)/2} y^{(-\varepsilon-1)/2}}{x+y+\max\{x,y\}} dx dy \\ &= \int_1^\infty \int_{x^{-1}}^\infty \frac{x^{(-\varepsilon-1)/2} (tx)^{(-\varepsilon-1)/2}}{1+t+\max\{1,t\}} dx dt \\ &= \int_1^\infty x^{-\varepsilon-1} \left(\int_0^\infty \frac{1}{1+t+\max\{1,t\}} t^{(-1-\varepsilon)/2} dt - \int_0^{x^{-1}} \frac{1}{1+t+\max\{1,t\}} t^{(-1-\varepsilon)/2} dt \right) dx \\ &= \frac{1}{\varepsilon} [\sqrt{2}(\pi - 2 \arctan \sqrt{2}) + o(1)]. \end{aligned} \quad (2.16)$$

Since, for $\varepsilon > 0$ small enough, we have

$$\sqrt{2}(\pi - 2 \arctan \sqrt{2}) + o(1) < K, \quad (2.17)$$

thus we get $\sqrt{2}(\pi - 2 \arctan \sqrt{2}) \leq K$, then $c \leq K$, which contradicts the hypothesis. Hence the constant factor c in (2.5) is the best possible. \square

THEOREM 2.3. *Suppose $f \geq 0$ and $0 < \int_0^\infty f^2(x)dx < \infty$. Then*

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y+\max\{x,y\}} dx \right]^2 dy < c^2 \int_0^\infty f^2(x)dx, \quad (2.18)$$

where the constant factor $c^2 = 2(\pi - 2 \arctan \sqrt{2})^2 = 3.0305\dots$ is the best possible. Inequality (2.18) is equivalent to (2.5).

Proof. Setting $g(y)$ as

$$\int_0^\infty \frac{f(x)}{x+y+\max\{x,y\}} dx, \quad y \in (0, \infty), \quad (2.19)$$

then by (2.5), we find

$$\begin{aligned} 0 &< \int_0^\infty g^2(y)dy \\ &= \int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y+\max\{x,y\}} dx \right]^2 dy \\ &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy \\ &\leq \sqrt{2}(\pi - 2 \arctan \sqrt{2}) \left\{ \int_0^\infty f^2(x)dx \right\}^{1/2} \left\{ \int_0^\infty g^2(y)dy \right\}^{1/2}. \end{aligned} \quad (2.20)$$

Hence we obtain

$$0 < \int_0^\infty g^2(y)dy \leq 2(\pi - 2 \arctan \sqrt{2})^2 \int_0^\infty f^2(x)dx < \infty. \quad (2.21)$$

By (2.5), both (2.20) and (2.21) take the form of strict inequality, so we have (2.18).

On the other hand, suppose that (2.18) is valid. By Hölder's inequality, we find

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy \\ &= \int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y+\max\{x,y\}} dx \right] g(y) dy \\ &\leq \left\{ \int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y+\max\{x,y\}} dx \right]^2 dy \right\}^{1/2} \left\{ \int_0^\infty g^2(y)dy \right\}^{1/2}. \end{aligned} \quad (2.22)$$

Then by (2.18), we have (2.5). Thus (2.5) and (2.18) are equivalent.

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If the constant $c^2 = 2(\pi - 2 \arctan \sqrt{2})^2$ in (2.18) is not the best possible, by (2.22), we may get a contradiction that the constant factor c in (2.5) is not the best possible. Thus we complete the proof of the theorem. \square

Acknowledgment

The authors would like to thank the anonymous referees for their suggestions and corrections.

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