APPROXIMATION OF SIGNALS (FUNCTIONS) BELONGING TO THE WEIGHTED $W(L_p,\xi(t))$-CLASS BY LINEAR OPERATORS

M. L. MITTAL, B. E. RHOADES, AND VISHNU NARAYAN MISHRA

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In memory of Professor Brian Kuttner (1908–1992)

Mittal and Rhoades (1999–2001) and Mittal et al. (2006) have initiated the studies of error estimates $E_n(f)$ through trigonometric Fourier approximations (TFA) for the situations in which the summability matrix T does not have monotone rows. In this paper, we determine the degree of approximation of a function $\widetilde{f}$, conjugate to a periodic function f belonging to the weighted $W(L_p,\xi(t))$-class ($p \geq 1$), where $\xi(t)$ is nonnegative and increasing function of t by matrix operators T (without monotone rows) on a conjugate series of Fourier series associated with f. Our theorem extends a recent result of Mittal et al. (2005) and a theorem of Lal and Nigam (2001) on general matrix summability. Our theorem also generalizes the results of Mittal, Singh, and Mishra (2005) and Qureshi (1981-1982) for Nörlund (N_p)-matrices.

1. Introduction

Let f be 2π-periodic function (signal) in $L_1[-\pi,\pi]$. The Fourier series associated with f at a point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.1}$$

with partial sums $s_n(f;x)$. The conjugate series of Fourier series (1.1) of f is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x) \tag{1.2}$$

with partial sums $\widetilde{s}_n(f;x)$. Throughout this paper, we will call (1.2) as conjugate Fourier series of function f.

Define for all $n \geq 0$,

$$t_n(f;x) = \sum_{k=0}^{n} a_{n,k} s_k(f;x), \tag{1.3}$$

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where $T \equiv (a_{n,k})$ is a linear operator represented by the infinite lower triangular matrix. The series (1.1) is said to be T-summable to s if $t_n(f;x) \to s$ as $n \to \infty$. The T-operator reduces to the Nörlund N_p-operator if

$$a_{n,k} = \frac{p_{n-k}}{P_n}, \quad 0 \le k \le n, \\ = 0, \quad k > n, \tag{1.4}$$

where $P_n = \sum_{k=0}^{n} p_k \neq 0$ and $p_{-1} = 0 = P_{-1}$. In this case, the transform $t_n(f;x)$ reduces to the Nörlund transform $N_n(f;x)$.

A linear operator T is said to be regular if it is limit-preserving over c, the space of convergent sequences. Each matrix T in this paper has nonnegative entries with row sums one. If $\lim_{n\to\infty} a_{n,k} = 0$, for each k, then T is regular.

The L_p-norm is defined by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}, \quad p \ge 1, \\ \|f\|_\infty = \sup_{x\in[0,2\pi]} |f(x)|, \tag{1.5}$$

and the degree of approximation $E_n(f)$ is given by

$$E_n(f) = \mathrm{Min}_n \|f(x) - T_n(x)\|_p, \tag{1.6}$$

in terms of n, where $T_n(x)$ is a trigonometric polynomial of degree n. This method of approximation is called trigonometric Fourier approximation (TFA).

A function $f \in \mathrm{Lip}\,\alpha$ if

$$|f(x+t) - f(x)| = O(|t|^\alpha) \quad \text{for } 0 < \alpha \le 1, \tag{1.7}$$

and $f(x) \in \mathrm{Lip}(\alpha, p)$, for $0 \le x \le 2\pi$, if

$$\omega_p(t;f) = \left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(|t|^\alpha), \quad 0 < \alpha \le 1,\ p \ge 1. \tag{1.8}$$

Given a positive increasing function $\xi(t)$ and $p \ge 1$, $f(x) \in \mathrm{Lip}(\xi(t), p)$ if

$$\omega_p(t;f) = \left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(\xi(t)), \tag{1.9}$$

and $f(x) \in W(L_p,\xi(t))$ if

$$\left(\int_0^{2\pi} |[f(x+t) - f(x)] \sin^\beta x|^p dx\right)^{1/p} = O(\xi(t)), \quad (\beta \ge 0). \tag{1.10}$$

If $\beta = 0$, our newly defined class $W(L_p,\xi(t))$ coincides with the class $\mathrm{Lip}(\xi(t), p)$. We observe that

$$\mathrm{Lip}\,\alpha \subseteq \mathrm{Lip}(\alpha, p) \subseteq \mathrm{Lip}(\xi(t), p) \subseteq W(L_p,\xi(t)) \quad \text{for } 0 < \alpha \le 1,\ p \ge 1. \tag{1.11}$$

We write

$$\psi(t)=\psi_x(t)=2^{-1}[f(x+t)-f(x-t)], \qquad W_n=|P_n^{-1}|\sum_{k=1}^{n}k|p_k-p_{k-1}|,$$
$$W_n(r)=\sum_{k=0}^{r}(k+1)|\Delta_k a_{n,n-k}|, \quad 0\le r\le n, \qquad J(n,t)=\sum_{k=0}^{n}a_{n,n-k}\cos\left(n-k+\frac{1}{2}\right)t,$$
$$\pi\tilde{f}(x)=\int_0^{\pi}\psi(t)\cot\left(\frac{t}{2}\right)dt=\lim_{h\to 0}\int_h^{\pi}\psi(t)\cot\left(\frac{t}{2}\right)dt, \qquad A_{n,k}=\sum_{r=k}^{n}a_{n,r},$$
$$V_{n,k}=\frac{(n-k+1)a_{n,k}}{A_{n,k}}, \qquad \overline{K}(n,t)=\frac{J(n,t)}{\sin(t/2)},$$
$$\tau=\left[\frac{1}{t}\right], \text{ the integral part of } \frac{1}{t}, \qquad \Delta_k a_{n,k}=a_{n,k}-a_{n,k+1}. \tag{1.12}$$

Furthermore, C denotes an absolute positive constant, not necessarily the same at each occurrence.

2. Previous results

Qureshi [12] has proved a theorem on the degree of approximation of a function $\tilde{f}(x)$, conjugate to a periodic function $f(x)$ with period 2π and belonging to the class Lipα, for $0<\alpha<1$, by N_p-means of its conjugate Fourier series. He has proved the following theorem.

Theorem 2.1 [12]. *If the sequence $\{p_n\}$ satisfies the conditions*

$$\begin{aligned}&\text{(i) } n|p_n|<C|P_n|,\\&\text{(ii) } W_n<C,\end{aligned} \tag{2.1}$$

then the degree of approximation of a function $\tilde{f}(x)$, conjugate to a periodic function $f(x)$ with period 2π and belonging to the class Lipα, $0<\alpha<1$, *by N_p-means of its conjugate Fourier series, is given by*

$$|\tilde{f}(x)-\tilde{t}_n(x)|=O\left(P_n^{-1}\sum_{k=1}^{n}\frac{P_k}{k^{\alpha+1}}\right), \tag{2.2}$$

where $\tilde{t}_n(x)$ is the N_p-mean of conjugate Fourier series (1.2).

Remark 2.2. Qureshi [12] has taken $p_n\ge 0$, so conditions (2.1) can be stated without modulus sign.

Generalizing Theorem 2.1 of Qureshi [12], many interesting results have been proved by various investigators such as Qureshi [13, 14], Lal and Nigam [2], Mittal et al. [8, 9] for functions of various classes (defined above) using N_p-matrices and general summability matrices.

Qureshi [13, 14] has extended his Theorem 2.1 for the functions of classes $\mathrm{Lip}(\alpha,p)$ and Weighted, that is, $W(L_p,\xi(t))(p \geq 1)$ respectively, using monotonicity on the elements of N_p-matrix. For the $W(L_p,\xi(t))$-class, he proved the following theorem.

Theorem 2.3 [14]. *If a 2π-periodic function f belongs to the class $W(L_p,\xi(t))$, then its degree of approximation by Nörlund means of a conjugate Fourier series is given by*

$$\|\tilde{f}(x) - \tilde{t}_n(x)\|_p = O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right) \tag{2.3}$$

provided that $\xi(t)$ satisfies the conditions

$$\left\{\int_0^{\pi/n}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^p \sin^{\beta p} t\,dt\right\}^{1/p} = O\left(\frac{1}{n}\right), \tag{2.4}$$

$$\left\{\int_{\pi/n}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O(n^{\delta}), \tag{2.5}$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$. Conditions (2.4) and (2.5) hold uniformly in x and

$$\left\{\int_0^{\pi/n}\left(\frac{\xi(t)}{t^{2+\beta}}\right)^q dt\right\}^{1/q} = O\left(n^{\beta+1+1/p}\xi\left(\frac{1}{n}\right)\right), \tag{2.6}$$

where $p^{-1}+q^{-1}=1$ and $1 \leq p \leq \infty$.

Remark 2.4. Qureshi [12–14] has used monotonicity on the generating sequence $\{p_n\}$ in the proof of his theorems but has not mentioned it explicitly in the statement of these theorems.

Remark 2.5. Condition (2.1)(i) can be dropped in Theorem 2.1, as condition (2.1)(ii) implies condition (2.1)(i) [1, page 16].

Recently Mittal et al. [8] have generalized Theorem 2.3 by taking semimonotonicity on the generating sequence $\{p_n\}$ and also by dropping the condition (2.6). They proved the following theorem.

Theorem 2.6 [8]. *The degree of approximation of a function $\tilde{f}$, conjugate to a2π-periodic function f belonging to weighted class $W(L_p,\xi(t))(p \geq 1)$ by N_p-means of its conjugate Fourier series (1.2), is given by*

$$\|\tilde{f}(x) - \tilde{t}_n(x)\|_p = O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right), \tag{2.7}$$

provided that $\{p_n\}$ satisfies (2.1)(ii) and $\xi(t)$ satisfies conditions (2.4) and (2.5) uniformly in x and

$$\frac{\xi(t)}{t} \text{ is nonincreasing in } t, \tag{2.8}$$

where δ is an arbitrary number such that $0 \neq \delta q + 1 < q$, such that $p^{-1}+q^{-1}=1$, $1 \leq p \leq \infty$, and $\tilde{t}_n(x)$ is the same as in Theorem 2.1.

Recently, Mittal et al. [9] extended Theorem 2.1 [12] for functions of $\text{Lip}(\xi(t),p)(p \geq 1)$-class to matrix summability using semimonotonicity on the sequence $\{a_{n,k}\}$, which in turn generalizes a result of Lal and Nigam [2]. They proved the following theorem.

Theorem 2.7 [9]. *Let $T \equiv (a_{n,k})$ be an infinite regular triangular matrix with nonnegative entries such that*

$$W_n(r) = O(A_{n,n-r}), \quad 0 \leq r \leq n, \tag{2.9}$$

then the degree of approximation of a function $\tilde{f}(x)$, conjugate to a 2π-periodic function $f(x)$ belonging to $\text{Lip}(\xi(t),p)$-class, by using a matrix operator on its conjugate Fourier series, is given by

$$\|\tilde{f}(x) - \tilde{t}_n(x)\|_p = O\left(n^{1/p}\xi\left(\frac{1}{n}\right)\right), \tag{2.10}$$

provided that $\xi(t)$ is nonnegative, increasing, and satisfies conditions (2.4), (2.5) uniformly in x and (2.8), where δ is an arbitrary positive number such that $q(1-\delta)-1>0$ and $\tilde{t}_n(x)$ are the matrix means of the conjugate Fourier series (1.2).

3. Main result

Mittal [3], Mittal and Rhoades [4–6], and Mittal et al. [7] have obtained many interesting results on TFA (these approximations have assumed important new dimensions due to their wide application in signal analysis [10] in general, and in digital signal processing [11] in particular, in view of the classical Shannon sampling theorem), using summability methods without monotonicity on the rows of the matrix T. In this paper, we extend Theorem 2.6 to matrix (linear) operators and generalize Theorem 2.7 for functions of the weighted class $W(L_p,\xi(t))$. We prove the following theorem.

Theorem 3.1. *Let $T \equiv (a_{n,k})$ be an infinite regular triangular matrix with nonnegative entries satisfying (2.9), then the degree of approximation of function $\tilde{f}(x)$, conjugate to a 2π-periodic function $f(x)$ belonging to class $W(L_p,\xi(t))$, $p \geq 1$, by using a matrix operator on its conjugate Fourier series, is given by*

$$\|\tilde{f}(x) - \tilde{t}_n(x)\| = O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right) \tag{3.1}$$

provided that $\xi(t)$ satisfies (2.4) and (2.5) uniformly in x, in which δ is an arbitrary positive number with $q(1-\delta)-1>0$, where $p^{-1}+q^{-1}=1$, $1 \leq p \leq \infty$ and condition (2.8) holds.

Note 3.2. In the case of the N_p-transform, condition (2.9), for $r = n$, reduces to (2.1)(ii) and thus Theorem 3.1 extends Theorem 2.6 to matrix summability for the weighted class functions.

Note 3.3. Also for $\beta = 0$, Theorem 3.1 reduces to Theorem 2.7, and thus generalizes the theorem of Lal and Nigam [2].

4. Lemmas

Lemma 4.1 [3]. *Let $T \equiv (a_{n,k})$ be an infinite triangular matrix satisfying (2.9). Then*

$$V_{n,k} = O(1), \quad 0 \le k \le n. \tag{4.1}$$

For $k = 0$,

$$V_{n,0} = O(1), \tag{4.2}$$

that is,

$$(n+1)a_{n,0} = O(1). \tag{4.3}$$

Lemma 4.2 [9]. *Let $T \equiv (a_{n,k})$ be an infinite triangular matrix satisfying (2.9). Then*

$$|J(n,t)| = O(A_{n,n-\tau}) + O(t^{-1})\left(\sum_{k=\tau}^{n-1} |\Delta_k a_{n,n-k}| + a_{n,0}\right). \tag{4.4}$$

5. Proof of Theorem 3.1

It is well known that

$$\begin{aligned}
\widetilde{s}_{n-k}(x) - \widetilde{f}(x) &= -\frac{1}{2\pi}\int_0^{\pi} \psi(t)\left(\frac{\cos(n-k+1/2)t}{\sin t/2}\right)dt,\\
\widetilde{f}(x) - \widetilde{t}_n(x) &= \sum_{k=0}^{n} a_{n,n-k}\{\widetilde{f}(x) - \widetilde{s}_{n-k}(x)\}\\
&= \frac{1}{2\pi}\int_0^{\pi} \psi(t)\left(\sum_{k=0}^{n} a_{n,n-k}\frac{\cos(n-k+1/2)t}{\sin t/2}\right)dt.
\end{aligned} \tag{5.1}$$

Therefore,

$$\begin{aligned}
|\widetilde{f}(x) - \widetilde{t}_n(x)| &\le \frac{1}{2\pi}\int_0^{\pi} |\psi(t)|\,|\overline{K}(n,t)|\,dt\\
&= \frac{1}{2\pi}\left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right)|\psi(t)|\,|\overline{K}(n,t)|\,dt = \frac{1}{2\pi}[I_1 + I_2], \text{ say.}
\end{aligned} \tag{5.2}$$

Since $\overline{K}(n,t) = O(1/t)$, using Hölder's inequality, condition (2.4), and the fact that $(\sin t)^{-1} \le \pi/2t$, for $0 < t \le \pi/2$, and the second mean value theorem for integrals,

$$\begin{aligned}
I_1 &= \int_0^{\pi/n} |\psi(t)|\,|\overline{K}(n,t)|\,dt \\
&= O\left[\int_0^{\pi/n}\left\{\frac{t|\psi(t)|\sin^\beta t}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_0^{\pi/n}\left\{\frac{\xi(t)|\overline{K}(n,t)|}{t\sin^\beta t}\right\}^q dt\right]^{1/q} \\
&= O\left(\frac{1}{n}\right)\left[\int_h^{\pi/n} O\left\{\frac{\xi(t)}{t^2\sin^\beta t}\right\}^q dt\right]^{1/q} \\
&= O\left(\frac{1}{n}\right)\left[\left(\frac{\pi/n}{\sin\pi/n}\right)^{\beta q}\int_h^{\pi/n} O\left\{\frac{\xi(t)}{t^{2+\beta}}\right\}^q dt\right]^{1/q} \\
&= O\left(\frac{1}{n}\right)\left[\int_h^{\pi/n}\left\{\frac{\xi(t)}{t^{2+\beta}}\right\}^q dt\right]^{1/q}.
\end{aligned} \tag{5.3}$$

Since $\xi(t)$ is nondecreasing with t and also using condition (2.8),

$$\begin{aligned}
I_1 &= O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{\pi}{n}\right)\right)\left(\int_h^{\pi/n} t^{-(2+\beta)q}\,dt\right)^{1/q} \\
&= O\left(n^{-1}\xi\left(\frac{1}{n}\right)\right)O\left(n^{2+\beta-1/q}\right) = O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right).
\end{aligned} \tag{5.4}$$

Using Hölder's inequality, condition (2.5), Lemma 4.2, Minkowski's inequality, and condition (2.8),

$$\begin{aligned}
I_2 &= \int_{\pi/n}^{\pi} |\psi(t)|\,|\overline{K}(n,t)|\,dt \\
&\le \left\{\int_{\pi/n}^{\pi}\left|\frac{t^{-\delta}\psi(t)\sin^\beta t}{\xi(t)}\right|^p dt\right\}^{1/p}\left\{\int_{\pi/n}^{\pi}\left|\frac{\xi(t)\overline{K}(n,t)}{t^{-\delta}\sin^\beta t}\right|^q dt\right\}^{1/q} \\
&= \left\{\int_{\pi/n}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|\,|\sin^\beta t|}{\xi(t)}\right)^p dt\right\}^{1/p}\left\{\int_{\pi/n}^{\pi}\left|\frac{\xi(t)J(n,t)}{t^{-\delta}\sin^\beta t\sin(t/2)}\right|^q dt\right\}^{1/q} \\
&= O(n^\delta)\left[\int_{\pi/n}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\right)^q O\left\{A_{n,n-\tau} + t^{-1}\left(a_{n,0} + \sum_{k=\tau}^{n-1}|\Delta_k a_{n,n-k}|\right)\right\}^q dt\right]^{1/q} \\
&= O\left(n^{\delta+1}\xi\left(\frac{\pi}{n}\right)\right)O\left[\left\{\int_{\pi/n}^{\pi}(t^{\delta-\beta}A_{n,n-\tau})^q dt\right\}^{1/q} + \left\{\int_{\pi/n}^{\pi}(t^{\delta-1-\beta}a_{n,0})^q dt\right\}^{1/q}\right. \\
&\qquad\left. + \left\{\int_{\pi/n}^{\pi}\left(t^{\delta-1-\beta}\sum_{k=\tau}^{n-1}|\Delta_k a_{n,n-k}|\right)^q dt\right\}^{1/q}\right] \\
&= O\left(n^{\delta+1}\xi\left(\frac{1}{n}\right)\right)\left[I_{2,1} + I_{2,2} + I_{2,3}\right],\ \text{say}.
\end{aligned} \tag{5.5}$$

Since A has nonnegative entries and row sums one,

$$I_{2,1}=\left\{\int_{\pi/n}^{\pi}\left(t^{\delta-\beta}A_{n,n-\tau}\right)^q dt\right\}^{1/q}=O\left\{\int_{\pi/n}^{\pi}t^{(\delta-\beta)q}dt\right\}^{1/q}=O\left(n^{\beta-\delta-1/q}\right). \tag{5.6}$$

Using Lemma 4.1,

$$\begin{aligned}I_{2,2}&=O\left\{\int_{\pi/n}^{\pi}\left(t^{\delta-1-\beta}a_{n,0}\right)^q dt\right\}^{1/q}=O\left(a_{n,0}\right)\left\{\int_{\pi/n}^{\pi}t^{(\delta-1-\beta)q}dt\right\}^{1/q}\\&=O\left(n^{-1}\right)\left(n^{\beta-\delta+1-1/q}\right)=O\left(n^{\beta-\delta-1/q}\right).\end{aligned} \tag{5.7}$$

Finally from condition (2.9),

$$\begin{aligned}I_{2,3}&=\left\{\int_{\pi/n}^{\pi}\left(t^{\delta-1-\beta}\sum_{k=\tau}^{n-1}\left|\Delta_k a_{n,n-k}\right|\right)^q dt\right\}^{1/q}\\&=O\left\{\int_{\pi/n}^{\pi}\left(t^{\delta-\beta}(\tau+1)\sum_{k=\tau}^{n-1}\left|\Delta_k a_{n,n-k}\right|\right)^q dt\right\}^{1/q}\\&=O\left\{\int_{\pi/n}^{\pi}\left(t^{\delta-\beta}\sum_{k=\tau}^{n-1}(k+1)\left|\Delta_k a_{n,n-k}\right|\right)^q dt\right\}^{1/q}\\&=O\left\{\int_{\pi/n}^{\pi}\left(t^{\delta-\beta}\sum_{k=0}^{n}(k+1)\left|\Delta_k a_{n,n-k}\right|\right)^q dt\right\}^{1/q}\\&=O\left\{\int_{\pi/n}^{\pi}\left(t^{\delta-\beta}W_n(n)\right)^q dt\right\}^{1/q}=O\left\{\int_{\pi/n}^{\pi}\left(t^{\delta-\beta}A_{n,0}\right)^q dt\right\}^{1/q}=O\left(n^{\beta-\delta-1/q}\right).\end{aligned} \tag{5.8}$$

Combining (5.6), (5.7), and (5.8),

$$I_2=O\left(n^{\delta+1}\xi\left(\frac{1}{n}\right)\right)\left(n^{\beta-\delta-1/q}\right)=O\left(n^{\beta+1/p}\xi(1/n)\right). \tag{5.9}$$

Combining I_1 and I_2 yields

$$\left|\widetilde{f}(x)-\widetilde{t}_n(x)\right|=O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right). \tag{5.10}$$

Now, using the L_p-norm, we get

$$\begin{aligned}\left\|\widetilde{f}(x)-\widetilde{t}_n(x)\right\|_p&=\left\{\int_0^{2\pi}\left|\widetilde{f}(x)-\widetilde{t}_n(x)\right|^p dx\right\}^{1/p}=O\left\{\int_0^{2\pi}\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right)^p dx\right\}^{1/p}\\&=O\left[n^{\beta+1/p}\xi\frac{1}{n}\left(\int_0^{2\pi}dx\right)^{1/p}\right]=O\left[n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right].\end{aligned} \tag{5.11}$$

6. Applications

The following corollaries can be derived from Theorem 3.1.

Corollary 6.1 [13]. *If $\xi(t) = t^\alpha$, $0 < \alpha \leq 1$, then the weighted class $W(L_p,\xi(t))$, $p \geq 1$, reduces to the class* $\mathrm{Lip}(\alpha,p)$ *and the degree of approximation of a function $\tilde{f}(x)$, conjugate to a 2π-periodic function f belonging to the class* $\mathrm{Lip}(\alpha,p)$, *is given by*

$$\left|\tilde{t}_n(x) - \tilde{f}(x)\right| = O\left(\frac{1}{n^{\alpha-1/p}}\right). \tag{6.1}$$

Proof. The result follows by setting $\beta = 0$ in (3.1). □

Corollary 6.2 [12]. *If $\xi(t) = t^\alpha$ for $0 < \alpha < 1$ and $p = \infty$ in Corollary 6.1, then $f \in \mathrm{Lip}\,\alpha$. In this case, using (6.1), one has Theorem 2.1.*

Proof. For $p = \infty$, we get

$$\begin{aligned} \|\tilde{f}(x) - \tilde{t}_n(x)\|_\infty &= \sup_{0\leq x\leq 2\pi}\left|\tilde{f}(x) - \tilde{t}_n(x)\right| = O\left(\frac{1}{n^\alpha}\right) \\ &= O\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k}{k^{\alpha+1}}\right) \quad \text{for } p_n \geq p_{n+1}, \end{aligned} \tag{6.2}$$

that is,

$$\left|\tilde{f}(x) - \tilde{t}_n(x)\right| = O\left(P_n^{-1}\sum_{k=1}^{n}\frac{P_k}{k^{\alpha+1}}\right). \tag{6.3}$$

□

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M. L. Mittal: Department of Mathematics, IIT Roorkee, Roorkee 247667, Uttaranchal, India
E-mail address: mymitfma@iitr.ernet.in

B. E. Rhoades: Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
E-mail address: rhoades@indiana.edu

Vishnu Narayan Mishra: Department of Mathematics, IIT Roorkee, Roorkee 247667, Uttaranchal, India
E-mail address: vishnu_narayanmishra@yahoo.co.in