THE COMPACTIFICABILITY CLASSES OF CERTAIN SPACES

MARTIN MARIA KOVÁR

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We apply the theory of the mutual compactificability to some spaces, mostly derived from the real line. For example, any noncompact locally connected metrizable generalized continuum, the Tichonov cube without its zero point $\mathbb{I}^{\aleph_0} \setminus \{\mathbf{0}\}$, as well as the Cantor discontinuum without its zero point $\mathbb{D}^{\aleph_0} \setminus \{\mathbf{0}\}$ are of the same class of mutual compactificability as $\mathbb{R}$.

1. The notation and terminology

By a space, we always mean topological space. Throughout the paper, we mostly use the standard topological notions as in [1] or [3] with the exception that all spaces are assumed without any separation axioms. Especially, compactness is understood without the Hausdorff separation axiom. Some definitions (with broad references and explanations) of less standard notions (related especially to non-Hausdorff topology) may be found in the recent book [4]. The reader may find some topological notions (usually also non-Hausdorff) related to computer science and logic in [13] as well as in [4]. We take the terminology related to θ-regularity from [5, 7], but a relevant source is also [4]. An ordinal number is taken to be the set of smaller ordinals, and a cardinal number is the smallest ordinal equipotent with some fixed set. Let S be a set. We denote the cardinality of S by $|S|$. Let (X,τ) be a space. For our convenience and simplicity, sometimes we will speak just about the space X, while meaning, more precisely, the pair (X,τ). Similarly, if we first start speaking about the space X without specifying its topology explicitly, later we will usually denote the topology of X by τ or τ_X (in the case that we will work simultaneously with more topological spaces or more topologies on the same set). The *weight* of a space (X,τ) is defined as the least infinite cardinal number $w(X)$ such that (X,τ) has an open base $\tau_0 \subseteq \tau$ with $|\tau_0| \le w(X)$. The spaces with $w(X) = \aleph_0$ are called *second countable*. In a space X, a point $x \in X$ is in the θ-*closure* of a set $A \subseteq X$ ($x \in \mathrm{cl}_\theta A$) if every

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closed neighborhood of x intersects A. A filter base Φ in X has a *θ-cluster point* $x \in X$ if $x \in \bigcap\{\mathrm{cl}_\theta F \mid F \in \Phi\}$. We say that a space X is *θ-regular* if every filter base in X with a θ-cluster point has a cluster point. For more detailed characterization of θ-regularity, we refer the reader to [5, 7, 8]. The points x, y in a space X are *T_0-separable* if there is an open set containing only one of the points x, y. The points x, y are *T_2-separable* if they have open disjoint neighborhoods. Let X be a space. Two disjoint sets $A, B \subseteq X$ are said to be *pointwise separated in* X if every $x \in A$, $y \in B$ are T_2-separable in X. Recall that the preorder of specialization is the reflexive and transitive binary relation on X defined by $x \leqslant y$ if and only if $x \in \mathrm{cl}\{y\}$. This relation is antisymmetric, and hence a partial order if and only if X is a T_0 space. There are several modifications of local compactness established in the literature. In this paper, we say that a space is *(strongly) locally compact* if its every point has a compact (closed) neighborhood. It can be easily proved that a space is strongly locally compact if and only if it is θ-regular and locally compact. Let X be a strongly locally compact space which is dense in a compact space K and let the sets X, $K \smallsetminus X$ be pointwise separated in K. Then one can easily prove that X is an open subspace of K. A filter in a space X is said to be *ultra-closed* if it is maximal among all filters in X having a base consisting of closed sets [1]. By the *Wallman compactification* of X, we mean the set $\omega X = X \cup \{y \mid y \text{ is a nonconvergent ultra-closed filter in } X\}$. The sets $\mathcal{S}(U) = U \cup \{y \mid y \in \omega X \smallsetminus X, U \in y\}$, where U is open in X, constitute an open base of ωX (see [1]). The space X is called *homogeneous* provided that for all $x, y \in X$, there is a homeomorphism $h : X \to X$ with $h(x) = y$, see [11]. We say that the space X is *zero-dimensional* if X is T_1 and has a base consisting of open-and-closed sets. Clearly, every zero-dimensional space is $T_{3.5}$.

2. Preliminaries and introduction

We will recall some notions and results from the previous papers [9, 10]. Let (X, τ_X), (Y, τ_Y) be spaces with $X \cap Y = \varnothing$. The space X is said to be *compactificable* by the space Y or, in other words, X, Y are called *mutually compactificable* if there exists a compact topology τ_K on $K = X \cup Y$ such that the topologies on X, Y induced by τ_K coincide with τ_X, τ_Y, respectively, and the sets X, Y are pointwise separated in (K, τ_K). Then we say that the topology τ_K is $\mathscr{C}$-acceptable. Recall that mutually compactificable spaces are always θ-regular, and any two disjoint strongly locally compact spaces are always mutually compactificable [9].

Let Top be the class of all topological spaces. For any pair of two spaces X, Z, we define $X \sim Z$ if for every nonempty space Y disjoint from X, Z, the space X is compactificable by Y if and only if Z is compactificable by Y. It can be easily seen that $\sim$ is reflexive, symmetric, transitive, and hence it is an equivalence relation. Let us denote by $\mathscr{C}(X)$ the equivalence subclass of Top with respect to $\sim$ containing X and call it the *compactificability class* of X. We proved in [10] that each compactificability class contains a T_1 representative, but there exist compactificability classes without any Hausdorff representatives. Because of completeness, we briefly repeat the main arguments witnessing that it really holds.

Let X be a topological space. We may assume that X is θ-regular, because otherwise, the desired T_1 representative of $\mathscr{C}(X)$ is any non-θ-regular T_1 space. Consider the net

$\mathrm{id}_M(M,\geqslant)$, where $\leqslant$ is the preorder of specialization on X and $M \subseteq X$ is a nonempty chain in the preordered set $(X,\leqslant)$. Then, $\mathrm{id}_M(M,\geqslant)$ θ-converges to any point $x \in M$. By θ-regularity of X, it also has a cluster point, say $z \in X$. It is an easy exercise to show that z is a lower bound of M. By Zorn's lemma, the set $X_1 \subseteq X$ of minimal points of $(X,\leqslant)$ is nonempty and each element of X is comparable with some element of X_1. If X is a T_0-space, then X_1 already is the desired T_1 representative of $\mathscr{C}(X)$. Otherwise, one may take the quotient space of X_1 with respect to the equivalence relation given by setting

$$x \sim y \Longleftrightarrow x \leqslant y, \quad y \leqslant x \tag{2.1}$$

for every x, $y \in X_1$, as the T_1 representative.

On the other hand, let X be a regular space on which every real-valued function is constant. Then X is a connected non-$T_{3.5}$space. Suppose that there exist a Hausdorff space Y such that $X \cap Y = \varnothing$ and a $\mathscr{C}$-acceptable topology on $K = X \cup Y$. Denote $H = \mathrm{cl}_K X \cap \mathrm{cl}_K Y$, $S = K \setminus \mathrm{cl}_K Y$ and $F = X \setminus S$. It is easy to show that both of F, S are $T_{3.5}$ and nonempty because of the normality of H and the local compactness of S. Take $x \in S$. Since X is regular, there exist an open set $U \in \tau_X$ with $x \in U$ and $\mathrm{cl}_X U \subseteq S$. Since X is connected, it follows that $\mathrm{cl}_X U \neq S$ which implies that $A = S \setminus U$ is a closed nonempty subset of S. Let $f : S \to I$ be a continuous function with $f(x) = 0$ and $f(A) = \{1\}$. Assigning the value 1 also to all the points of F, we get a continuous and nonconstant extension of f over X, which is a contradiction. Thus, X is compactificable by no Hausdorff space. Since X is compactificable by $\omega X \setminus X$, the class $\mathscr{C}(\omega X \setminus X)$ contains no Hausdorff representative. For more detail, we refer the reader to the previous papers [9, 10]. Note that it is unknown whether every compactificability class contains a sober (or sober T_1) representative.

It is a natural question what the compactificability classes look like if we assume something more for the $\mathscr{C}$-acceptable topology—for instance, Hausdorffness. We call such a modification of the original concept mutual T_2-compactificability in [9]. For example, Thomas in [12] constructed an (relatively elementary) example of a regular non-$T_{3.5}$ space, which is compactificable by the countably infinite discrete space, as shown in [9], but certainly, T_2-compactificable by no topological space. These initial considerations witness that the compactificability classes and T_2-compactificability classes are essentially different and even the corresponding decomposition of Top is not a simple refinement of the other. For spaces which are Hausdorff, it seems to be more natural and important to study the mutual T_2-compactificability, but this we will do in a separate paper, in which we also will attempt to give a deeper insight into the relationship between these two modifications of the concept. The aim of this paper is to continue in an initial study of the concept and we start with the version which seems to be less complicated. We also should note that the T_2 version of the theory cannot distinguish between the spaces which are not at least $T_{3.5}$, because the non-$T_{3.5}$ spaces form a T_2-compactificability class (similarly as the non-θ-regular spaces form a compactificability class).

Now, for all spaces X, Z, we put $\mathscr{C}(X) \succcurlyeq \mathscr{C}(Z)$ if for every nonempty space Y, the following hold. If the space X is compactificable by Y disjoint from X, Z, then Z is compactificable by Y. Obviously, the relation $\succcurlyeq$ is reflexive, antisymmetric, transitive, and hence it is an order relation between the compactificability classes. If for some spaces X, Z, it holds that $\mathscr{C}(X) \succcurlyeq \mathscr{C}(Z)$ but $\mathscr{C}(X) \neq \mathscr{C}(Z)$, we write $\mathscr{C}(X) \succ \mathscr{C}(Z)$. The previously mentioned fact that every compactificability class contains a T_1 representative will be very important for verifying the relation $\mathscr{C}(X) \succcurlyeq \mathscr{C}(Z)$ in the next section. Indeed, it is easy to show that $\mathscr{C}(X) \succcurlyeq \mathscr{C}(Z)$ if and only if the following hold. If X is compactificable by a T_1 space (or a T_0 space), Y disjoint from X, Z, then Z is compactificable by Y.

3. Main results

We will have two main theorems in this section. The first one will state some relationship between the compactificability classes of a strongly locally compact space and its closed subspace. The second theorem will compare the compactificability class of a given space with the compactificability class of some known space; more concretely, a certain space constructed from the Cantor cube. Thus we will be able to determine the compactificability classes of some familiarly known spaces derived and constructed from the real line $\mathbb{R}$. But before formulating and proving these results, we need some more preparation.

Proposition 3.1. *Let (X, τ_X) be a closed subspace of a strongly locally compact space (Z, τ_Z). Let (Y, τ_Y) be a nonempty T_0 space such that $Y \cap Z = \varnothing$ and on the set $K = X \cup Y$, there exists a compact topology τ_K which induces on X, Y their original topologies and the sets X, Y are in (K, τ_K) pointwise separated.*

Then there exists a topology τ_L on the set $L = Y \cup Z$ which induces on the sets Y, Z, K their original topologies such that the sets Y, Z are pointwise separated in (L, τ_L). Moreover, if X is dense in K, Z is dense in L.

Proof. We will define a topology on L by its open base. Let $m \in Y$ be a point such that $\{m\}$ is a closed set in (Y, τ_Y). Note that since (Y, τ_Y) is θ-regular and T_0, it follows from Zorn's lemma that each point in Y is comparable with some minimal point $m \in Y$ with respect to the preorder of specialization, and, certainly, then $\{m\} = \mathrm{cl}\{m\}$ is a closed set. For more detail, the reader is referred to [10, Lemma 2.14]. We will define two types of neighborhoods.

(i) A neighborhood of type 1 is the set of the form

$$W(m, V, C) = V \cup [(Z \setminus X) \setminus C], \tag{3.1}$$

where $m \in V \in \tau_K$ and $C \subseteq Z$ is a set compact and closed in (Z, τ_Z), such that $V \cap C = \varnothing$.

(ii) A neighborhood of type 2 is any set $W \subseteq L$ such that $m \notin W$, $W \cap Z \in \tau_Z$, $W \cap K \in \tau_K$.

Let $W(m, V_1, C_1)$, $W(m, V_2, C_2)$ be two neighborhoods of type 1. Then $W(m, V_1, C_1) \cap W(m, V_2, C_2) = (V_1 \cap V_2) \cup [(Z \setminus X) \setminus (C_1 \cup C_2)]$. Moreover, $(V_1 \cap V_2) \cap (C_1 \cup C_2) = \varnothing$, which means that the intersection of two neighborhoods of type 1 is a neighborhood of type 1. It is clear that the intersection of two neighborhoods of type 2 is again a neighborhood of type 2. Let $W(m, V, C) = V \cup [(Z \setminus X) \setminus C]$ be a neighborhood of type 1 and W a neighborhood of type 2. Then, $W(m, V, C) \cap Z = (V \cap Z) \cup [(Z \setminus X) \setminus C]$.

Since $V \cap C = \varnothing$, we have $V \cap Z = (V \cap Z) \setminus C$, so

$$W(m,V,C) \cap Z = [(V \cap Z) \setminus C] \cup [(Z \setminus X) \setminus C] = [(V \cap Z) \cup (Z \setminus X)] \setminus C. \tag{3.2}$$

Since $V \in \tau_K$, it follows that $V \cap Z = V \cap X = S \cap X$ for some $S \in \tau_Z$. Then

$$\begin{aligned} W(m,V,C) \cap Z &= [(S \cap X) \cup (Z \setminus X)] \setminus C \\ &= [(S \cup (Z \setminus X)) \cap (X \cup (Z \setminus X))] \setminus C \\ &= [(S \cup (Z \setminus X)) \cap Z] \setminus C = [S \cup (Z \setminus X)] \setminus C \in \tau_Z. \end{aligned} \tag{3.3}$$

Let $U = W(m,V,C) \cap W$. Then $U \cap Z = [W(m,V,C) \cap Z] \cap (W \cap Z) \in \tau_Z$. Further,

$$W(m,V,C) \cap K = V \cup [((Z \setminus X) \setminus C) \cap K] = V \cup [((Z \setminus K) \setminus C) \cap K] = V \in \tau_K, \tag{3.4}$$

so $U \cap K = V \cap (W \cap K) \in \tau_K$. Since clearly $m \notin U$, U is a neighborhood of type 2. Since $W(m,K,\varnothing) = L$, the neighborhoods of type 1 together with the neighborhoods of type 2 form an open base of some topology, say τ_L, on the set $L = Y \cup Z$. Moreover, we have just checked that the topologies on the sets Z, K, respectively, induced from the space (L,τ_L) are equal or weaker than the original topologies τ_Z, τ_K, respectively.

Conversely, let $S \in \tau_Z$. Then $S \cap X \in \tau_X$, so there exists $V \in \tau_K$, such that $S \cap X = V \cap X = S \cap K$. We put $U = S \cup (V \setminus \{m\})$. Clearly, $m \notin U$. We have

$$\begin{aligned} U \cap Z &= S \cup [(V \setminus \{m\}) \cap Z] = S \cup [(V \setminus \{m\}) \cap X] \\ &= S \cup (V \cap X) = S \cup (S \cap X) = S \in \tau_Z, \\ U \cap K &= (S \cap K) \cup (V \setminus \{m\}) = (V \cap X) \cup (V \setminus \{m\}) \\ &= [(V \setminus \{m\}) \cap X] \cup (V \setminus \{m\}) = V \setminus \{m\} \in \tau_K. \end{aligned} \tag{3.5}$$

Hence, U is a neighborhood of type 2 which induces the given set $S \in \tau_Z$ on Z. Therefore, the topology on Z induced from (L,τ_L) equals to τ_Z.

Similarly, let $V \in \tau_K$. At first, suppose that $m \notin V$. We have $V \cap X \in \tau_X$, so there exists $S \in \tau_Z$ such that $V \cap X = S \cap X = S \cap K$. We put $U = S \cup V$. Of course, $m \notin U$. We have

$$\begin{aligned} U \cap Z &= S \cup (V \cap Z) = S \cup (V \cap X) = S \cup (S \cap X) = S, \\ U \cap K &= (S \cap K) \cup V = (V \cap X) \cup V = V \in \tau_K. \end{aligned} \tag{3.6}$$

Then U is a neighborhood of type 2 which induces the given set $V \in \tau_K$ on K. Finally, suppose that $m \in V$. We put $U = W(m,V,\varnothing) = V \cup (Z \setminus X)$. Then

$$U \cap K = V \cup [(Z \setminus X) \cap K] = V \cup [(Z \setminus K) \cap K] = V, \tag{3.7}$$

so U is a neighborhood of type 1 which induces the given set $V \in \tau_K$ on K. Hence, the topology on K induced from (L,τ_L) equals to τ_K. Then also the topology on $Y \subseteq K$ induced from (L,τ_L) equals to τ_Y.

Let $z \in Z$, $y \in Y$. Since Z is locally compact and θ-regular, it is strongly locally compact, so there exists $S \in \tau_Z$ such that $z \in S$ and $C = \mathrm{cl}_Z S$ is compact in (Z, τ_Z). Then $D = C \cap X$ is compact in (X, τ_X), and hence in (K, τ_K). Since X, Y are pointwise separated in K, there exist $V, P \in \tau_K$ such that $y \in V$, $D \subseteq P$, and $V \cap P = \varnothing$. We put $U = V \cup [(Z \setminus X) \setminus C]$. Then $y \in U$ and $U \cap S = (V \cap S) \cup [((Z \setminus X) \setminus C) \cap S]$. But $S \subseteq C$, so

$$\begin{gathered} V \cap S \subseteq V \cap C = V \cap K \cap Z \cap C = V \cap X \cap C = V \cap D \subseteq V \cap P = \varnothing, \\ \big[(Z \setminus X) \setminus C\big] \cap S \subseteq (L \setminus C) \cap C = \varnothing, \end{gathered} \tag{3.8}$$

so $U \cap S = \varnothing$. We also can see that $V \cap C = \varnothing$. If $z \in Z \setminus X$, we put $W = S \cap (Z \setminus X)$. Then $z \in W$, $W \cap Z = W \in \tau_Z$, $W \cap K = \varnothing \in \tau_K$, so W is a neighborhood of type 2, and $W \cap U = \varnothing$. Otherwise, that is, if $z \in X$, we construct the set W in a different way. The set $S \cap X \in \tau_X$ is nonempty and there exists $T \in \tau_K$ such that $S \cap X = T \cap X$. Without loss of generality, we may assume that $m \notin T$. On the other hand, $P \in \tau_K$, so $P \cap X \in \tau_X$ and there exists some $Q \in \tau_Z$ such that $P \cap X = Q \cap X$. We put $W = (P \cup Q) \cap (S \cup T)$. We have $m \notin W$ and $z \in S \cap X \subseteq C \cap X = D \subseteq P$, so $z \in W$. Further,

$$\begin{aligned} W \cap Z &= (P \cup Q) \cap (S \cup T) \cap Z = (P \cup Q) \cap \big[S \cup (T \cap Z)\big] \\ &= (P \cup Q) \cap \big[S \cup (T \cap X)\big] = (P \cup Q) \cap \big[S \cup (S \cap X)\big] \\ &= (P \cup Q) \cap S = (P \cap S) \cup (Q \cap S) = (P \cap K \cap Z \cap S) \cup (Q \cap S) \\ &= (P \cap X \cap S) \cup (Q \cap S) = (Q \cap X \cap S) \cup (Q \cap S) = Q \cap S \in \tau_Z, \\ W \cap K &= (P \cup Q) \cap (S \cup T) \cap K = \big[P \cup (Q \cap K)\big] \cap (S \cup T) \\ &= \big[P \cup (Q \cap X)\big] \cap (S \cup T) = \big[P \cup (P \cap X)\big] \cap (S \cup T) \\ &= P \cap (S \cup T) = (P \cap S) \cup (P \cap T) = (P \cap K \cap Z \cap S) \cup (P \cap T) \\ &= (P \cap X \cap S) \cup (P \cap T) = (P \cap X \cap T) \cup (P \cap T) = P \cap T \in \tau_K. \end{aligned} \tag{3.9}$$

Then

$$\begin{aligned} W &= W \cap L = W \cap (Z \cup K) = (W \cap Z) \cup (W \cap K) = (Q \cap S) \cup (P \cap T) \\ &\subseteq P \cap S = P \cap K \cap Z \cap S = P \cap X \cap S \subseteq P \cap X. \end{aligned} \tag{3.10}$$

Hence,

$$\begin{aligned} W \cap U &= (P \cap X) \cap \big[V \cup (Z \setminus X)\big] = (P \cap X \cap V) \cup \big[P \cap X \cap (Z \setminus X)\big] \\ &\subseteq P \cap V \cup \big[X \cap (Z \setminus X)\big] = P \cap V = \varnothing. \end{aligned} \tag{3.11}$$

Moreover, one can see that W is a neighborhood of z of type 2.

Now, if $y = m$, then $U = V \cup [(Z \setminus X) \setminus C]$ is a neighborhood of y of type 1, and hence, the points z, y are pointwise separated. If $y \neq m$, without loss of generality, we can choose $V \in \tau_K$ such that $m \notin V$. Then $m \notin U$ and $U \cap Z = (V \cap Z) \cup [(Z \setminus X) \setminus C]$.

Since $V \cap C = \varnothing$, we have $V \cap Z = (V \cap Z) \smallsetminus C$, so

$$U \cap Z = [(V \cap Z) \smallsetminus C] \cup [(Z \smallsetminus X) \smallsetminus C] = [(V \cap Z) \cup (Z \smallsetminus X)] \smallsetminus C$$
$$= [(V \cap X) \cup (Z \smallsetminus X)] \smallsetminus C. \tag{3.12}$$

But $V \cap X \in \tau_X$, so there is some $R \in \tau_Z$ such that $V \cap X = R \cap X$. Then

$$U \cap Z = [(R \cap X) \cup (Z \smallsetminus X)] = [R \cup (Z \smallsetminus X)] \cap [X \cup (Z \smallsetminus X)]$$
$$= [R \cup (Z \smallsetminus X)] \cap Z = R \cup (Z \smallsetminus X) \in \tau_Z. \tag{3.13}$$

Similarly,

$$U \cap K = V \cup [((Z \smallsetminus X) \smallsetminus C) \cap K] \subseteq V \cup [(Z \smallsetminus X) \cap K]$$
$$= V \cup [(Z \smallsetminus K) \cap K] = V \in \tau_K. \tag{3.14}$$

Hence, U is a neighborhood of y of type 2.

In any case, the points z, y have disjoint neighborhoods in (L, τ_L). It remains to show that (L, τ_L) is compact. Let $\Omega \subseteq \tau_L$ be an open cover of L. Without loss of generality, we may assume that Ω consists of neighborhoods of type 1 or type 2. Since the topology on K induced from τ_L coincides with τ_K, K is a compact subspace of L. Hence, there exist $W_1, W_2, \ldots, W_k \in \Omega$ such that $K \subseteq \bigcup_{i=1}^{k} W_i$. There exist some $i \in \{1, 2, \ldots, k\}$, say $i = 1$, such that $m \in W_1$. Hence, W_1 is a neighborhood of type 1, so there exist $V \in \tau_K$ and $C \subseteq Z$ compact and closed in (Z, τ_Z) such that $W_1 = W(m, V, C) = V \cup [(Z \smallsetminus X) \smallsetminus C]$ and $V \cap C = \varnothing$. Since τ_Z agrees with the topology on Z induced by τ_L, it follows that C is a compact subspace of L. Hence, there exist $W_{k+1}, W_{k+2}, \ldots, W_n \in \Omega$ such that $C \subseteq \bigcup_{i=k+1}^{n} W_i$. Let $x \in L$. Suppose that $x \notin \bigcup_{i=1}^{k} W_i$. Then $x \notin K$, so $x \in L \smallsetminus K = Z \smallsetminus X$. But also $x \notin W_1$, so $x \in C$. But then, $x \in \bigcup_{i=k+1}^{n} W_i$. Therefore, $L = \bigcup_{i=1}^{n} W_i$. Hence, L is compact.

Finally, suppose that X is dense in K. Let $y \in Y$ and suppose that $y \in W \in \tau_L$. Then $W \cap K \in \tau_K$, so $W \cap Z \supseteq W \cap X = W \cap K \cap X \neq \varnothing$, which means that Z is dense in L. Now, we can see that τ_L satisfies all the conditions stated in the proposition, and the proof is complete. □

As an immediate corollary of the previous proposition, we can formulate our first theorem.

Theorem 3.2. *Let (X, τ_X) be a closed subspace of a strongly locally compact space (Z, τ_Z). Then $\mathscr{C}(X) \succcurlyeq \mathscr{C}(Z)$.*

Proof. Let Y be a space disjoint from Z and suppose that X is compactificable by Y. Take any T_0 representative of $\mathscr{C}(Y)$, say Y_0, such that $Y_0 \cap Z = \varnothing$. Then X, Y_0 are mutually compactificable. By Lemma 3.3, the spaces Z, Y_0 are mutually compactificable, which yields that Z is compactificable also by Y. Hence, we have $\mathscr{C}(X) \succcurlyeq \mathscr{C}(Z)$. □

Now we need to make some additional denotations. We denote by $\mathbb{D}_3$ the three-element set $\{0, 1, 2\}$, equipped by the discrete topology. Similarly, we also consider the set

$\mathbb{D} = \{0,1\}$ with the discrete topology. Further, we denote $\mathbb{I} = [0,1]$ and $\mathbb{A} = [1,\infty)$ with the Euclidean topology induced from $\mathbb{R}$. For a space (X,τ) and $\sigma \subseteq \tau$, we denote by $\mathbf{0}$ the constant mapping such that $\mathbf{0}(U) = 0$ for every $U \in \sigma$. We also call it the *"left corner"* of the cube $\mathbb{D}_3^\sigma$ or $\mathbb{D}^\sigma$. By αX, we denote the one-point Alexandroff compactification of a space (X,τ). The aim of the following considerations is to prove that for a locally compact Hausdorff space (X,τ) with $w(X) \geqslant \aleph_0$, it holds that $\mathscr{C}(X) \succcurlyeq \mathscr{C}(\mathbb{D}^{w(x)} \smallsetminus \{\mathbf{0}\})$. We present here a proof which is elementary and relatively independent of the literature, but, on the other hand, rather long and technical. The reader who will find it too boring may skip the next part starting from the following paragraph and continue after Corollary 3.12 with Remark 3.13.

Recall that the *Khalimsky line* (see, e.g., [4]) is the set $\mathbb{Z}$ of integers equipped with the topology induced by the open subbase $\{\{2i, 2i+1, 2i+2\} \mid i \in \mathbb{Z}\}$ (in some papers, the roles of odd and even numbers may be exchanged). The standard reference for the Khalimsky space is [6]. This space is mostly used in digital topology and perhaps that could be a reason why, as far as the author knows, its universal properties yet have not been systematically studied. The *Sierpiński space* (see [4, page 20]) is the set $\mathbb{S} = \{0,1\}$, where the open sets are $\varnothing$, the whole space $\{0,1\}$, and one of the singletons, say $\{1\}$. Note that considered as a poset where $0 < 1$, in which the partial order $\leq$ may arise as the preorder of specialization from the Sierpiński topology, $(\mathbb{S}, \leq)$ is called *Sierpiński frame* (see, e.g., [13, page 22]). The power of the Sierpiński space is well known as the *Alexandroff cube* (see [3, page 116], or [4, page 103]) and it is easy to show that any T_0 space of weight $\mathfrak{m}$ can be embedded into the Alexandroff cube $\mathbb{S}^\mathfrak{m}$. These universal properties of the Sierpiński space are broadly well known, however, there exist finite T_0 spaces which can do things that the Sierpiński space can never do. A useful and interesting example of a universal T_0 space was given by R. Sikorski and later studied by Dow and Watson in [2]. It has three points $\{0,1,2\}$ and five open sets $\varnothing$, $\{2\}$, $\{1,2\}$, $\{0,2\}$, $\{0,1,2\}$.

Inspired by the paper of Dow and Watson, we equip the set $\{0,1,2\}$ by another four-element topology $\{\varnothing, \{0\}, \{2\}, \{0,1,2\}\}$, which can be induced from the Khalimsky line. Considered with this topology, we denote $\{0,1,2\}$ by $\mathbb{K}_3$. For a given T_0 space, we define a canonical mapping of X into the cube $\mathbb{K}_3^\sigma$. Let $x \in X$, $U \in \sigma$. We put

$$f_x(U) = \begin{cases} 0 & \text{for } x \notin \operatorname{cl} U, \\ 1 & \text{for } x \in \operatorname{fr} U, \\ 2 & \text{for } x \in U. \end{cases} \tag{3.15}$$

Then we put $h(x) = f_x$. This defines a (canonical) mapping $h : X \to \mathbb{K}_3^\sigma$. The following lemma establishes more precisely what we can (naturally) expect from the space $\mathbb{K}_3$ and the mapping h.

Lemma 3.3. *Let (X,τ) be a T_0 space, $\sigma \subseteq \tau$ open base of its topology. Then the canonical mapping $h : X \to h(x) = Y \subseteq \mathbb{K}_3^\sigma$ is a homeomorphic embedding of X onto a subspace $Y \subseteq \mathbb{K}_3^\sigma$.*

Proof. We will prove that h is injective. Let $x, y \in X$, $x \neq y$. Since X is a T_0 space, without loss of generality, we may assume that there exists $U \in \sigma$ such that $x \in U$, $y \notin U$. Then $h(x)(U) = 2$, while $h(y)(U) \in \{0,1\}$. Hence, $h(x)$, $h(y)$ are different mappings.

Let us show that h is continuous. Denote by $\pi_U : \mathbb{K}_3^\sigma \to \mathbb{K}_3$ the projection such that for every $f \in \mathbb{K}_3^\sigma$, it holds that $\pi_U(f) = f(U) \in \mathbb{K}_3$. According to the topology of $\mathbb{K}_3$, we have to verify that the inverse images of $Y \cap \pi_U^{-1}(0)$ and $Y \cap \pi_U^{-1}(2)$ in the mapping h are open in (X,τ). We leave to the reader to check that $h^{-1}(Y \cap \pi_U^{-1}(0)) = X \setminus \operatorname{cl} U$ and $h^{-1}(Y \cap \pi_U^{-1}(2)) = U$. Then h is continuous. From the second equality, it also follows that $h(U) = Y \cap \pi_U^{-1}(2)$, which means that h is also an open mapping. Then $h : X \to Y$ is a homeomorphism, which completes the proof. □

Proposition 3.4. *Let (X,τ) be a noncompact strongly locally compact T_0 space, $\sigma \subseteq \tau$ an open topology base such that for every $U \in \sigma$, the set $\operatorname{cl} U$ is compact. Then the mapping $h : X \to h(X) \subseteq \mathbb{K}_3^\sigma$ can be extended to the homeomorphism $\overline{h} : \alpha X \to h(X) \cup \{\mathbf{0}\}$.*

Proof. Let $\alpha X = X \cup \{\infty\}$ be the one-point Alexandroff compactification of (X,τ). We put

$$\overline{h}(x) = \begin{cases} h(x) & \text{for } x \in X, \\ \mathbf{0} & \text{for } x = \infty. \end{cases} \tag{3.16}$$

First, we will show that $\mathbf{0} \notin Y = h(X)$. Let $y \in Y$ and let $x = h^{-1}(y)$. Let $U \in \sigma$ be such that $x \in U$. Then $y(U) = h(x)(U) = 2$, while $\mathbf{0}(U) = 0$, so $y \neq \mathbf{0}$. Hence, $\mathbf{0} \notin Y$. Then $\overline{h} : \alpha X \to h(X) \cup \{\mathbf{0}\}$ is a bijection. Since X is an open set in αX, it suffices to verify the continuity of $\overline{h}$ at the point ∞. Let $V \subseteq \mathbb{K}_3^\sigma$ be an open set such that $\mathbf{0} \in V$. We will show that $h^{-1}(V \cap (Y \cup \{\mathbf{0}\}))$ is open in αX. Without loss of generality, we may assume that V is a subbasic set in the product topology of $\mathbb{K}_3^\sigma$, so $V = \pi_U^{-1}(k)$, where $k \in \{0,2\}$ and $U \in \sigma$. Since $\mathbf{0} \in V$, we need $\pi_U(\mathbf{0}) = \mathbf{0}(U) = 0 \in \{k\}$, so $k = 0$. Then,

$$\begin{aligned} \overline{h}^{-1}(V \cap (Y \cup \{\mathbf{0}\})) &= \overline{h}^{-1}(V \cap Y) \cup \overline{h}^{-1}(\{\mathbf{0}\}) = h^{-1}(V \cap Y) \cup \{\infty\} \\ &= h^{-1}(Y \cap \pi_U^{-1}(0)) \cup \{\infty\} = (X \setminus \operatorname{cl} U) \cup \{\infty\}, \end{aligned} \tag{3.17}$$

which is an open neighborhood of ∞ in the topology of αX. So $\overline{h}$ is continuous.

We will show that $\overline{h}$ is an open mapping. Let $W \subseteq \alpha X$ be open. Take any $y \in \overline{h}(W)$. Suppose that $y = \mathbf{0}$. Then $\infty = \overline{h}^{-1}(\mathbf{0}) \in W$, which means that $X \setminus W$ is closed and compact in (X,τ). There exist $U_1, U_2, \dots, U_k \in \sigma$ such that $X \setminus W \subseteq \bigcup_{i=1}^k U_i$. We put $V = (Y \cup \{\mathbf{0}\}) \cap (\bigcap \pi_{U_i}^{-1}(0))$. Clearly, $\mathbf{0} \in \pi_{U_i}^{-1}(0)$, which gives $y = \mathbf{0} \in V$. If $\mathbf{0} \neq t \in V$, then $t \in Y \cap (\bigcap_{i=1}^k \pi_{U_i}^{-1}(0)) = \bigcap_{i=1}^k (Y \cap \pi_{U_i}^{-1}(0))$. Then

$$\begin{aligned} \overline{h}^{-1}(t) = h^{-1}(t) &\in h^{-1}\left(\bigcap_{i=1}^k (Y \cap \pi_{U_i}^{-1}(0))\right) \\ &= \bigcap_{i=1}^k h^{-1}(Y \cap \pi_{U_i}^{-1}(0)) = \bigcap_{i=1}^k (X \setminus \operatorname{cl} U_i) \\ &= X \setminus \bigcup_{i=1}^k \operatorname{cl} U_i \subseteq X \setminus \bigcup_{i=1}^k U_i \subseteq W. \end{aligned} \tag{3.18}$$

This implies that $t \in \overline{h}(W)$, so we have $y = \mathbf{0} \in V \subseteq \overline{h}(W)$.

Let $y \neq \mathbf{0}$ and denote $x = \overline{h}^{-1}(y) = h^{-1}(y) \in W$. There exists $U \in \sigma$ such that $x \in U \subseteq W$. We put $V = (Y \cup \{\mathbf{0}\}) \cap \pi_U^{-1}(2)$. We have $h(x)(U) = 2$, so $y = h(x) \in \pi_U^{-1}(2)$, which gives $y \in V$. Let $t \in V$. Then $t \in \pi_U^{-1}(2)$, so $t(U) = 2$. In particular, $t \neq \mathbf{0}$, which implies that $t \in Y \cap \pi_U^{-1}(2)$, which means that $\overline{h}^{-1}(t) = h^{-1}(t) \in h^{-1}(Y \cap \pi_U^{-1}(2)) = U$. Then $\overline{h}^{-1}(t) \in U \subseteq W$, and so $t \in \overline{h}(W)$. Consequently, $y \in V \subseteq \overline{h}(W)$.

From the two previous paragraphs, now it follows that for open $W \subseteq \alpha X$, $\overline{h}(W)$ is always an open set, which means that $\overline{h} : \alpha X \to h(X) \cup \{\mathbf{0}\}$ is an open mapping. Since it is also bijective and continuous, it is a homeomorphism and the proof is finished. □

Lemma 3.5. *Let (X,τ) be noncompact locally compact Hausdorff space with $w(X) = \mathfrak{m}$. Then there exist a closed subspace $Z \subseteq \mathbb{D}_3^{\mathfrak{m}}$ containing $\mathbf{0}$ and a continuous surjective mapping $f : Z \to \alpha X$ such that $f(Z \setminus \{\mathbf{0}\}) = X$.*

Proof. Let $\sigma \subseteq \tau$ be an open topology base of (X,τ) such that $|\sigma| = \mathfrak{m}$ and $\operatorname{cl} U$ is compact for every $U \in \sigma$. We denote $Y = h(X)$, $Y_0 = h(X) \cup \{\mathbf{0}\}$, where $h : X \to \mathbb{K}_3^{\sigma}$ is the canonical embedding described above. By Proposition 3.4, there exists a homeomorphism $\overline{h} : \alpha X \to Y_0$ such that $\overline{h}(\infty) = \mathbf{0}$ and $\overline{h}_{/X} = h$. We put $A = Y_0$ and $Z = \operatorname{cl}_{\mathbb{D}_3^{\sigma}} A$, where we consider the sets A, Z with the topologies induced from $\mathbb{D}_3^{\sigma}$. Since the topology of A is stronger than the topology of Y_0, which is induced from $\mathbb{K}_3^{\sigma}$, the identical mapping $\mathrm{id}_A : A \to Y_0$ is surjective and continuous. We will construct a continuous extension $g : Z \to Y_0$ of id_A.

Let us denote by $\operatorname{Fin}\sigma$ the family of all finite subsets of σ. The set $\operatorname{Fin}\sigma$ is naturally directed by the inclusion. For every $t \in Z$ and $K \in \operatorname{Fin}\sigma$, the set $\bigcap_{U\in K} \pi_U^{-1}(t(u))$ is an open neighborhood of t and so there exists $\varphi_t(K) \in A \cap [\bigcap_{U\in K} \pi_U^{-1}(t(u))] = Y_0 \cap [\bigcap_{U\in K} \pi_U^{-1}(t(u))]$. We will show that the net $\varphi_t(\operatorname{Fin}\sigma, \supseteq)$ has a unique cluster point in Y_0. Let $p,q \in Y_0$ be two distinct cluster points of $\varphi_t(\operatorname{Fin}\sigma, \supseteq)$.

Suppose first that $p \neq \mathbf{0} \neq q$. Then $p,q \in Y$, and so there exist $U_1, U_2 \in \sigma$ such that $h^{-1}(p) \in U_1$, $h^{-1}(q) \in U_2$ and $U_1 \cap U_2 = \varnothing$. We put $K = \{U_1, U_2\}$. There exist $L_1, L_2 \in \operatorname{Fin}\sigma$, $L_1 \supseteq K$, $L_2 \supseteq K$, such that $\varphi_t(L_1) \in h(U_1) = Y \cap \pi_{U_1}^{-1}(2)$, $\varphi_t(L_2) \in h(U_2) = Y \cap \pi_{U_2}^{-1}(2)$. In particular, $\varphi_t(L_1)(U_1) = \varphi_t(L_2)(U_2)$. At the same time, it holds that $\varphi_t(L_1) \in \bigcap_{U\in L_1} \pi_U^{-1}(t(U))$ and $\varphi_t(L_2) \in \bigcap_{U\in L_2} \pi_U^{-1}(t(U))$ by the definition of the net $\varphi_t(\operatorname{Fin}\sigma, \supseteq)$. Hence, we have

$$\begin{aligned} \varphi_t(L_1)(U_1) &= t(U_1), & \varphi_t(L_1)(U_2) &= t(U_2), \\ \varphi_t(L_2)(U_1) &= t(U_1), & \varphi_t(L_2)(U_2) &= t(U_2), \end{aligned} \tag{3.19}$$

which hold because $U_1, U_2 \in L_1, L_2$. We get $\varphi_t(L_1)(U_1) = \varphi_t(L_2)(U_2) = 2$, which gives $\varphi_t(L_1) \neq \mathbf{0}$. Then

$$\varphi_t(L_1) \in [Y \cap \pi_{U_1}^{-1}(2)] \cap [Y \cap \pi_{U_2}^{-1}(2)] = h(U_1) \cap h(U_2), \tag{3.20}$$

which contradict our former assumption that $U_1 \cap U_2 = \varnothing$.

Now, suppose that, for example and certainty, $p \neq \mathbf{0}$ and $q = \mathbf{0}$. Let $U \in \sigma$ such that $h^{-1}(p) \in U$. We put $K = \{U\}$. There exist $L_1, L_2 \in \operatorname{Fin}\sigma$, $L_1 \supseteq K$, $L_2 \supseteq K$, such that

$$\begin{gathered} \varphi_t(L_1) \in h(U) = Y \cap \pi_U^{-1}(2), \\ \varphi_t(L_2) \in \pi_U^{-1}(0). \end{gathered} \tag{3.21}$$

Then, by the definition of the net $\varphi_t(\operatorname{Fin}\sigma, \supseteq)$, we have

$$\begin{aligned} \varphi_t(L_1) &\in \bigcap_{V\in L_1} \pi_V^{-1}(t(V)), \\ \varphi_t(L_2) &\in \bigcap_{V\in L_2} \pi_V^{-1}(t(V)). \end{aligned} \tag{3.22}$$

Then $\varphi_t(L_1)(U) = t(U) = \varphi_t(L_2)(U)$, which is a contradiction.

Hence, the net $\varphi_t(\operatorname{Fin}\sigma, \supseteq)$ has, for every $t \in Z$, the unique cluster point, say $y_t \in Y_0$. We put

$$g(t) = \begin{cases} t & \text{for } t \in A = Y_0, \\ y_t & \text{for } t \in Z \setminus A. \end{cases} \tag{3.23}$$

This defines a surjective mapping $g : Z \to Y_0$. We will check that $g^{-1}(\mathbf{0}) = \{\mathbf{0}\}$. Let $t \in g^{-1}(\mathbf{0})$. Then $g(t) = \mathbf{0}$. Suppose that $t \neq \mathbf{0}$. We have $t \notin A$, because g is injective in A. Then $\mathbf{0} = y_t$ is a cluster point of the net $\varphi_t(\operatorname{Fin}\sigma, \supseteq)$. Let us take any $U \in \sigma$ and let $K = \{U\}$. There exist $L \in \operatorname{Fin}\sigma$, $L \supseteq K$ such that $\varphi_t(L) \in \pi_U^{-1}(0)$, and so $\varphi_t(L)(U) = 0$. But, at the same time, we have $\varphi_t(L) \in \bigcap_{V\in L} \pi_V^{-1}(t(V))$, which means that $\varphi_t(L)(U) = t(U)$. Then $t(U) = 0$ for every $U \in \sigma$, which is a contradiction to our previous assumption that $t \neq \mathbf{0}$. So, $t = \mathbf{0}$ is the unique element of $g^{-1}(\mathbf{0})$.

We will verify that $g : Z \to Y_0$ is continuous at any point $t \in Z$. Let $t \in Z \setminus \{\mathbf{0}\}$. Then $g(t) \neq \mathbf{0}$, so $g(t) \in Y$. Let $W \subseteq \mathbb{K}_3^{\sigma}$ be an open set such that $g(t) \in W$. Then $Y \cap W$ is open in Y and $h^{-1}(Y \cap W)$ is open in (X, τ), while

$$x = h^{-1}\big(g(t)\big) \in h^{-1}(Y \cap W). \tag{3.24}$$

Then there exists $U \in \sigma$ such that

$$x = h^{-1}\big(g(t)\big) \in U \subseteq h^{-1}(Y \cap W). \tag{3.25}$$

Then

$$g(t) \in h(U) = Y \cap \pi_U^{-1}(2) = Y_0 \cap \pi_U^{-1}(2) \subseteq Y \cap W \subseteq W, \tag{3.26}$$

where we have used the fact that $\mathbf{0}(U) = 0$ and so $\mathbf{0} \notin \pi_U^{-1}(2)$.

Since (X, τ) is regular, there is some $V \in \sigma$ such that $x \in V \subseteq \operatorname{cl} V \subseteq U$. Then $h(x)(V) = g(t)(V) = 2$, so $g(t) \in Y \cap \pi_V^{-1}(2)$. If $t \in A$, then $t = g(t) \in Z \cap \pi_V^{-1}(2)$. Suppose that $t \in Z \setminus A$. Then $g(t)$ is a cluster point of the net $\varphi_t(\operatorname{Fin}\sigma, \supseteq)$. Let $K = \{V\}$. There exist $L \in \operatorname{Fin}\sigma$, $L \supseteq K$, such that $\varphi_t(L) \in \pi_V^{-1}(2)$, which means that $\varphi_t(L)(V) = 2$. Then, by the definition of the net $\varphi_t(\operatorname{Fin}\sigma, \supseteq)$, we have $\varphi_t(L) \in \bigcap_{S\in L} \pi_S^{-1}(t(S))$, which means that $\varphi_t(L)(V) = t(V) = 2$. Then $t \in Z \cap \pi_V^{-1}(2)$, which is an open neighborhood of t in the topology of Z.

We will show that $g(Z \cap \pi_V^{-1}(2)) \subseteq W$. Let $s \in Z \cap \pi_V^{-1}(2)$. Then $s(V) = 2$, which means that $s \neq \mathbf{0}$. If $s \in A = Y_0$, we have $g(s) = s \in Y_0 \cap \pi_V^{-1}(2) \subseteq \pi_U^{-1}(2) \subseteq W$. Let $s \in Z \setminus A$. Then $g(s) \in Y_0$, but it is not possible that $g(s) = \mathbf{0}$ since $\mathbf{0}$ has in g the unique preimage $\mathbf{0}$. Hence $g(s) \in Y$. Denote $z = h^{-1}(g(s)) \in X$ and suppose that $z \notin \operatorname{cl} V$. Then $h(z)(V) = g(s)(V) = 0$, that is, $g(s) \in \pi_V^{-1}(0)$. But $g(s)$ is the cluster point of the net $\varphi_s(\operatorname{Fin}\sigma, \supseteq)$, so there exist, for the open set $\pi_V^{-1}(0)$ and $K = \{V\} \in \operatorname{Fin}\sigma$, some $L \in \operatorname{Fin}\sigma$, $L \supseteq K$ such that $\varphi_s(L) \in \pi_V^{-1}(0)$. Then $\varphi_s(L)(V) = 0$. Further, $\varphi_s(L) \in \bigcap_{Q\in L} \pi_Q^{-1}(s(Q))$, which implies that $\varphi_s(L)(V) = s(V) = 0$. But this contradicts our previous assumption that $s \in Z \cap \pi_V^{-1}(2)$ which, in the contrary, gives $s(V) = 2$. Therefore, necessarily $z \in \operatorname{cl} V \subseteq U$. Then

$$g(s) = h(z) \in h(U) = Y \cap \pi_U^{-1}(2) \subseteq Y \cap W \subseteq W. \tag{3.27}$$

Now it is clear that $g(Z \cap \pi_V^{-1}(2)) \subseteq W$, which means that g is continuous at every point $t \in Z \setminus \{\mathbf{0}\}$.

Finally, we will check the continuity of g at $t = \mathbf{0}$. Let $\mathbf{0} \in W \subseteq \mathbb{K}_3^\sigma$, where W is open in the topology of $\mathbb{K}_3^\sigma$. Then $Y_0 \cap W$ is open in Y_0, and so $\overline{h}^{-1}(Y_0 \cap W)$ is open in αX, while

$$\infty \in \overline{h}^{-1}(\mathbf{0}) \subseteq \overline{h}^{-1}(Y_0 \cap W) = (X \setminus C) \cup \{\infty\}, \tag{3.28}$$

where $C \subseteq X$ is compact and closed in (X,τ). Hence, there exist $U_1, U_2, \ldots, U_k \in \sigma$ such that $C \subseteq \bigcup_{i=1}^k U_i$. Then $t = \mathbf{0} \in Z \cap (\bigcap_{i=1}^k \pi_{U_i}^{-1}(0))$, while the set $Z \cap (\bigcap_{i=1}^k \pi_{U_i}^{-1}(0))$ is open in Z.

We will prove that

$$g\left(Z \cap \left(\bigcap_{i=1}^k \pi_{U_i}^{-1}(0)\right)\right) \subseteq W. \tag{3.29}$$

Let $s \in Z \cap (\bigcap_{i=1}^k \pi_{U_i}^{-1}(0))$ and denote $z = \overline{h}^{-1}(g(s)) \in \alpha X$. First, we will show that $z \in (X \setminus C) \cup \{\infty\}$. Clearly, $s(U_i) = 0$ for $i \in \{1,2,\ldots,k\}$. If $s = \mathbf{0}$, then $z = \infty$ and we are done. So suppose that $s \neq \mathbf{0}$. It follows that $g(s) \neq \mathbf{0}$, so $g(s) \in Y$, which means that $z = h^{-1}(g(s))$.

If $s \in A$, then $g(s) = s \in \bigcap_{i=1}^k \pi_{U_i}^{-1}(0)$, and so $h(z)(U_i) = g(s)(U_i) = 0$ for every $i \in \{1,2,\ldots,k\}$. Then $z \notin \operatorname{cl} U_i$ for every $i \in \{1,2,\ldots,k\}$, which means that $z \notin C$, which yields $z \in (X \setminus C) \cup \{\infty\}$.

Consider now the second possibility, that is, suppose that $s \in Z \setminus A$. Let $z \in U_i$ for some $i \in \{1,2,\ldots,k\}$. Then $h(z)(U_i) = g(s)(U_i) = 2$. But $g(s)$ is a cluster point of the net $\varphi_s(\operatorname{Fin}\sigma, \supseteq)$ and so, for the open set $\pi_{U_i}^{-1}(2) \subseteq \mathbb{K}_3^\sigma$ and for $K = \{U_i\} \in \operatorname{Fin}\sigma$, there exist $L \in \operatorname{Fin}\sigma$, $L \supseteq K$ such that $\varphi_s(L) \in \pi_{U_i}^{-1}(2)$. Then $\varphi_s(L)(U_i) = 2$. We also have $\varphi_s(L) \in \bigcap_{Q\in L} \pi_Q^{-1}(s(Q))$, which together gives $\varphi_s(L)(U_i) = s(U_i) = 2$. But this contradicts the choice of the element s, for which we have $s(U_j) = 0$ for every $j \in \{1,2,\ldots,k\}$. Therefore, $z \notin \bigcup_{i=1}^k U_i$, which implies that $z \notin C$, and so $z \in (X \setminus C) \cup \{\infty\}$. Overall, we have $z = \overline{h}^{-1}(g(s)) \in (X \setminus C) \cup \{\infty\}$ for every $z \in Z \cap (\bigcap_{i=1}^k \pi_{U_i}^{-1}(0))$. Then

$$g(s) = \overline{h}(z) \in \overline{h}((X \setminus C) \cup \{\infty\}) = Y_0 \cap W \subseteq W. \tag{3.30}$$

Hence,

$$g\left(Z\cap\left(\bigcap_{i=1}^{k}\pi_{U_i}^{-1}(0)\right)\right)\subseteq W, \tag{3.31}$$

which means that g is continuous at the point $\mathbf{0}\in Z$.

Consequently, g is continuous at every point of Z, so it is continuous. Further, it is surjective and $g(Z\setminus\{\mathbf{0}\})=Y$. We put $f=h^{-1}\circ g$. Now it is clear that f has all the required properties. □

Lemma 3.6. *For every cardinal* $\mathfrak{m}\geqslant\aleph_0$, *there exists a continuous surjective mapping* $f:\mathbb{D}^{\mathfrak{m}}\to\mathbb{D}_3^{\mathfrak{m}}$ *such that* $f(\mathbb{D}^m\setminus\{\mathbf{0}\})=\mathbb{D}_3^{\mathfrak{m}}\setminus\{\mathbf{0}\}$.

Proof. We put, for every $\alpha\in\mathfrak{m}$, $X_\alpha=\mathbb{D}\times\mathbb{D}$, $Y_\alpha=\mathbb{D}_3$, $f_\alpha(0,0)=0$, $f_\alpha(0,1)=1$, $f_\alpha(1,0)=1$, $f_\alpha(1,1)=2$. The mapping $f_\alpha:X_\alpha\to Y_\alpha$ is continuous and surjective. We put $f(t)(\alpha)=f_\alpha(t(\alpha))$ for every $t\in\prod_{\alpha\in\mathfrak{m}}X_\alpha$. Since $\prod_{\alpha\in\mathfrak{m}}X_\alpha$ is homeomorphic to $\mathbb{D}^{\mathfrak{m}}$, we can identify these spaces. Thus, the mapping $f:\prod_{\alpha\in\mathfrak{m}}X_\alpha=D^{\mathfrak{m}}\to\prod_{\alpha\in\mathfrak{m}}Y_\alpha=\mathbb{D}_3^{\mathfrak{m}}$ has the desired properties. □

Remark 3.7. By a less elementary argument, the spaces $\mathbb{D}^{\mathfrak{m}}$ and $\mathbb{D}_3^{\mathfrak{m}}$ are actually homeomorphic. Up to homeomorphism, the Cantor cube $\mathbb{D}^{\aleph_0}$ is a unique nonempty compact zero-dimensional space of countable weight without isolated points (see [11, page 135, Theorem 4.2.5]). Since the spaces $\mathbb{D}^{\mathfrak{m}}$ and $\mathbb{D}_3^{\mathfrak{m}}$ are also homogeneous, without loss of generality we may immediately have that $f(\mathbf{0})=\mathbf{0}$.

We combine the previous results to a proposition.

Proposition 3.8. *Let* (X,τ) *be a noncompact locally compact Hausdorff space with* $w(X)=\mathfrak{m}$, *where* $\mathfrak{m}\geqslant\aleph_0$. *Then there exists a closed subspace* $Z\subseteq\mathbb{D}^{\mathfrak{m}}$, *such that* $\mathbf{0}\in Z$ *and a continuous surjective mapping* $f:Z\to\alpha X$ *such that* $f(Z\setminus\{\mathbf{0}\})=X$.

Proof. Denote by $f_3:Z_3\to\alpha X$ the mapping whose existence is ensured by Lemma 3.5 and by $f_2:\mathbb{D}^m\to\mathbb{D}_3^{\mathfrak{m}}$ the mapping described in Lemma 3.6, where $Z_3\subseteq\mathbb{D}_3^{\mathfrak{m}}$ is the closed subspace of $\mathbb{D}_3^{\mathfrak{m}}$ whose existence is also ensured by Lemma 3.5. We put $Z=f_2^{-1}(Z_3)$ and $f=f_3\circ f_2$, where we consider the composition $f_3\circ f_2$ to be restricted to its domain Z. □

The next proposition shows that one space can be replaced, under some conditions, by the other space in mutual compactificability, if it is a continuous image of that space.

Proposition 3.9. *Let* (X,τ_X) *be a locally compact space, let* (Y,τ_Y), (Z,τ_Z) *be topological spaces such that* $X\cap Y=\varnothing$, $Y\cap Z=\varnothing$. *Suppose that there exists a compact topology* τ_K *on* $K=X\cup Y$ *such that the sets* X, Y *are pointwise separated in* (K,τ_K) *and the topologies induced on the sets* X, Y *by* τ_K *coincide with their original topologies. Let* $f:Z\to X$ *be a surjective continuous mapping such that for any ultrafilter* $\mathcal{U}$ *on* Z *having no cluster point, the ultrafilter* $f(\mathcal{U})$ *has no cluster point in* X.

Then there exists a compact topology τ_L *on the set* $L=Y\cup Z$ *such that the sets* Y, Z *are pointwise separated, and the topologies on* Y, Z *induced by* τ_L *coincide with their original topologies.*

Proof. Denote $F = \mathrm{cl}_K X \setminus X \subseteq Y$. We will define a topology on L by setting its open subbase. We will define two types of neighborhoods.

(i) A neighborhood of type 1 is the set of the form $U \cup (W \cap Y)$, where $U \in \tau_Z$, $W \in \tau_K$, $W \cap F = \varnothing$, $U \subseteq f^{-1}(W \cap X)$.

(ii) A neighborhood of type 2 is the set of the form $f^{-1}(W \cap X) \cup (W \cap Y)$, where $W \in \tau_K$, $W \cap F \neq \varnothing$.

The reader may check that the neighborhoods of type 1 and of type 2 together constitute a base of some topology, say τ_L, on the set $L = Y \cup Z$, but the subbase features of these neighborhoods are fully sufficient.

The intersection of a neighborhood of any type with the sets Y or Z, respectively, is open in the original topology on Y or Z, respectively, so the induced topologies are weaker or equal to the original topologies on the sets Y, Z, respectively. Let $V \in \tau_Y$. There exists $W \in \tau_K$ such that $V = W \cap Y$. Then $W \cap X \in \tau_K$, so $U = f^{-1}(W \cap X) \in \tau_Z$. The set $P = U \cup V = U \cup (W \cap Y) = f^{-1}(W \cap X) \cup (W \cap Y)$ is a neighborhood of type 1 or type 2 depending on the emptiness of $W \cap F$, and we have $V = Y \cap P$. Hence, V can be induced by τ_L, so τ_Y and the topology induced on Y from (L, τ_L) are equal. Let $U \in \tau_Z$. The set X is locally compact and θ-regular, so X is an open subspace of $\mathrm{cl}_K X$. Then, there exists $W \in \tau_K$ such that $X = W \cap \mathrm{cl}_K X$. Then $W \cap X = X$, so $f^{-1}(W \cap X) = Z$, and, consequently, $U \subseteq f^{-1}(W \cap X)$. Moreover, $W \cap F = W \cap F \cap \mathrm{cl}_K X = (W \cap \mathrm{cl}_K X) \cap F = X \cap (\mathrm{cl}_K X \setminus X) = \varnothing$. We put $Q = U \cup (W \cap Y)$, which is then a neighborhood of type 1, and so $Q \in \tau_L$. Since $U = Q \cap Z$, U can be induced by τ_L. Hence, τ_Z and the topology on Z induced from (L, τ_L) coincide.

We will prove that Y, Z are pointwise separated in (L, τ_L). Let $y \in Y$ and $z \in Z$. Then $x = f(z) \in X$, so there exist $W_1, W_2 \in \tau_K$ such that $y \in W_1$, $x \in W_2$, and $W_1 \cap W_2 = \varnothing$. Let $U_1 = f^{-1}(W_1 \cap X)$ and $U_2 = f^{-1}(W_2 \cap X)$. We put

$$\begin{aligned} P &= U_1 \cup (W_1 \cap Y) = f^{-1}(W_1 \cap X) \cup (W_1 \cap Y), \\ P &= U_2 \cup (W_2 \cap Y) = f^{-1}(W_2 \cap X) \cup (W_2 \cap Y). \end{aligned} \tag{3.32}$$

Clearly, P, Q, respectively, are neighborhoods of type 1 or type 2, depending on the emptiness of $W_1 \cap F$, $W_2 \cap F$, respectively. We have $y \in P$, $z \in Q$, and $P \cap Q = \varnothing$. Indeed, if $t \in P \cap Q$, then $t \in U_1 \cap U_2$, since $W_1 \cap W_2 = \varnothing$. But then, $f(t) \in W_1 \cap W_2$, which is a contradiction. Hence, Y and Z are pointwise separated in (L, τ_L).

Finally, it remains to show that (L, τ_L) is compact. Let $\mathcal{U}$ be an ultrafilter in L. Then either $Y \in \mathcal{U}$ or $Z \in \mathcal{U}$. Suppose that $Y \in \mathcal{U}$. If $\mathcal{U}$ has a cluster point in Y, we are done. So, suppose that $\mathcal{U}$ has no cluster point in Y. Since K is compact, the ultrafilter $\mathcal{U}_Y = \{A \cap Y \mid A \in \mathcal{U}\}$ on Y has a cluster point in K, say $x \in X$. The mapping $f : Z \to X$ is surjective, so there is $z \in Z$ such that $f(z) = x$. Let P be a neighborhood of type 1 or type 2 containing z. Then $P = U \cup (W \cap Y)$, where $U \in \tau_Z$, $W \in \tau_K$, and $U \subseteq f^{-1}(W \cap X)$. We have $z \in U \subseteq f^{-1}(W \cap X)$, so $x = f(z) \in W \cap X \subseteq W$. Let $A \in \mathcal{U}$. Then $A \cap Y \cap W \neq \varnothing$ since x is a cluster point of $\mathcal{U}_Y$ and W is its open neighborhood in (K, τ_K), but then also $A \cap P \neq \varnothing$. Hence, z is a cluster point of $\mathcal{U}$ in (L, τ_L).

Suppose that $Z \in \mathcal{U}$. If $\mathcal{U}$ has a cluster point, we are done. So, suppose that $\mathcal{U}$ has no cluster point in Z. Then, by the assumption, the ultrafilter $\mathcal{F} = \{f(A \cap Z) \mid A \in \mathcal{U}\}$

on X has no cluster point in X. Since (K, τ_K) is compact, $\mathcal{F}$ has a cluster point in K, say $y \in Y$. Let $Q \in \tau_L$ be a neighborhood of y of type 1 or type 2. Since the elements of $\mathcal{F}$ are subsets of X, we have $y \in F = \mathrm{cl}_K X \setminus X$. Then, Q cannot be a neighborhood of type 1 because the neighborhoods of type 1 cannot meet F. Hence, $Q = f^{-1}(W \cap X) \cup (W \cap Y)$, where $W \in \tau_K$ and $W \cap F \neq \varnothing$. Let $A \in \mathcal{U}$. From $y \in Q$, it follows that $y \in W$, so $f(A \cap Z) \cap W \neq \varnothing$. Then there exists $t \in A \cap Z$ such that $f(t) \in W \cap X$. Consequently, $t \in f^{-1}(W \cap X) \cap A \subseteq Q \cap A$, which means that y is a cluster point of $\mathcal{U}$.

Hence, in any case, $\mathcal{U}$ has a cluster point in L, so (L, τ_L) is compact. This completes the proof. □

Now, we can state the second theorem.

Theorem 3.10. *Let (X, τ) be a locally compact Hausdorff space with $w(X) = \mathfrak{m}$, where $\mathfrak{m} \geqslant \aleph_0$. Then $\mathcal{C}(X) \succcurlyeq \mathcal{C}(\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\})$.*

Proof. By Proposition 3.8, there exist a closed subspace $Z \subseteq \mathbb{D}^{\mathfrak{m}}$ containing $\mathbf{0}$ and a continuous surjective mapping $f : Z \to \alpha X$ such that $f(Z \setminus \{\mathbf{0}\}) = X$. For a given ultrafilter $\mathcal{U}$ in $Z \setminus \{\mathbf{0}\}$ which has no cluster point in $Z \setminus \{\mathbf{0}\}$, $f(\mathcal{U})$ has no cluster point in X. Indeed, Z is compact, so $\mathbf{0}$ must be the cluster point of $\mathcal{U}$. Since f is continuous, $f(\mathbf{0}) = \infty$ must be the cluster point of $f(\mathcal{U})$. However, the Alexandroff compactification of (X, τ) is Hausdorff, so $f(\mathcal{U})$ cannot have another cluster point in αX. It follows from Proposition 3.9 that $\mathcal{C}(X) \succcurlyeq \mathcal{C}(Z \setminus \{\mathbf{0}\})$. But $Z \setminus \{\mathbf{0}\}$ is a closed subspace of $\mathbb{D}^m \setminus \{\mathbf{0}\}$, so we may use Proposition 3.1 similarly as in the proof of Theorem 3.2 and show that $\mathcal{C}(Z \setminus \{\mathbf{0}\}) \succcurlyeq \mathcal{C}(\mathbb{D}^m \setminus \{\mathbf{0}\})$. Then $\mathcal{C}(X) \succcurlyeq \mathcal{C}(\mathbb{D}^m \setminus \{\mathbf{0}\})$. □

Combining our two theorems, we have a tool for determining some compactificability classes practically. The next proposition is an immediate consequence of Theorems 3.2 and 3.10.

Proposition 3.11. *Let (X, τ) be a locally compact Hausdorff space with $w(X) = \mathfrak{m}$, where $\mathfrak{m} \geqslant \aleph_0$. If X contains a closed subspace Y with $\mathcal{C}(\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\}) \succcurlyeq \mathcal{C}(Y)$, then $\mathcal{C}(X) = \mathcal{C}(Y) = \mathcal{C}(\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\})$.*

Since the Cantor cube $\mathbb{D}^{\mathfrak{m}}$ is a closed subspace of the Tichonov cube $\mathbb{I}^{\mathfrak{m}}$, which has the weight $\mathfrak{m}$, we have the following result.

Corollary 3.12. *For any $\mathfrak{m} \geqslant \aleph_0$, it holds that $\mathcal{C}(\mathbb{I}^{\mathfrak{m}} \setminus \{\mathbf{0}\}) = \mathcal{C}(\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\})$.*

Proof. $\mathbb{D}^{\mathfrak{m}}$ is a closed subspace in $\mathbb{I}^{\mathfrak{m}}$, so $\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\}$ is a closed subspace of $\mathbb{I}^{\mathfrak{m}} \setminus \{\mathbf{0}\}$. Both of these spaces have the weight $\mathfrak{m}$ and they are locally compact and Hausdorff. It follows from Theorem 3.2 that $\mathcal{C}(\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\}) \succcurlyeq \mathcal{C}(\mathbb{I}^{\mathfrak{m}} \setminus \{\mathbf{0}\})$. From Theorem 3.10, it follows that $\mathcal{C}(\mathbb{I}^{\mathfrak{m}} \setminus \{\mathbf{0}\}) \succcurlyeq \mathcal{C}(\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\})$, so together we have $\mathcal{C}(\mathbb{I}^{\mathfrak{m}} \setminus \{\mathbf{0}\}) = \mathcal{C}(\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\})$. □

Remark 3.13. Note that another, but less elementary argument proving Theorem 3.10 and Corollary 3.12 follows from the fact that $\mathbb{I}^{\aleph_0}$ is a homogeneous space (see [11, page 254, Theorem 6.1.6]). Then $\mathbb{I}^{\mathfrak{m}}$ is also homogeneous for $\mathfrak{m} \geqslant \aleph_0$ (as a product of $\mathfrak{m}$ copies of $\mathbb{I}^{\aleph_0}$). Let (X, τ) be a locally compact Hausdorff space with $w(X) = \mathfrak{m} \geqslant \aleph_0$. It is well known that there is an embedding $f : \alpha X \to \mathbb{I}^{\mathfrak{m}}$ such that $f(\alpha X)$ is a closed subspace of $\mathbb{I}^{\mathfrak{m}}$ (see [3, page 115, Theorem 2.3.23]). The homogeneity of $\mathbb{I}^{\mathfrak{m}}$ allows to have $f(\infty) = \mathbf{0}$. Then, by

Theorem 3.2, we get $\mathscr{C}(X) \succcurlyeq \mathscr{C}(\mathbb{I}^{\mathfrak{m}} \setminus \{\mathbf{0}\})$. Of course, similarly we have $\mathscr{C}(\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\}) \succcurlyeq \mathscr{C}(\mathbb{I}^{\mathfrak{m}} \setminus \{\mathbf{0}\})$. On the other hand, it is also well known that there exists a continuous surjection $g : \mathbb{D}^{\mathfrak{m}} \to \mathbb{I}^{\mathfrak{m}}$ such that $g^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\}$ (adjust [3, Exercise 3.2.B, page 193]). Now, Proposition 3.9 yields $\mathscr{C}(\mathbb{I}^{\mathfrak{m}} \setminus \{\mathbf{0}\}) \succcurlyeq \mathscr{C}(\mathbb{D}^{\mathfrak{m}} \setminus \{\mathbf{0}\})$, which completes the proof.

The Cantor space $\mathbb{D}^{\aleph_0}$ is homeomorphic to a closed subspace of $\mathbb{I}$. Applying Theorem 3.2 several times, we obtain the next corollary.

Corollary 3.14. *For any $k, n \in \mathbb{N}$, the spaces $\mathbb{A}^k$, $\mathbb{R}^n$, $\mathbb{A}^k \times \mathbb{R}^n$, $\mathbb{I}^{\aleph_0} \setminus \{\mathbf{0}\}$, $\mathbb{D}^{\aleph_0} \setminus \{\mathbf{0}\}$ are of the same class of mutual compactificability.*

Proof. All the spaces mentioned in the corollary are Hausdorff locally compact having the weight $\aleph_0$. The mapping $h : \mathbb{D}^{\aleph_0} \to \mathbb{I} = [0,1]$ defined as $h(x) = \sum_{i=1}^{\infty}(2x_i/3^i)$, where $x = (x_i)$, is a homeomorphic embedding in which $h(\mathbf{0}) = 0$. Then $\mathbb{D}^{\aleph_0} \setminus \{\mathbf{0}\}$ is homeomorphic to a closed subspace of the space $(0,1]$, which is homeomorphic with the real ray $\mathbb{A}$. Hence, $\mathscr{C}(\mathbb{D}^{\aleph_0} \setminus \{\mathbf{0}\}) \succcurlyeq \mathscr{C}(\mathbb{A})$. Further, $\mathbb{A}$ is homeomorphic with a closed subspace of $\mathbb{R}^n$, $\mathbb{R}^n$ is a closed subspace of $\mathbb{A} \times \mathbb{R}^n$, which is again a closed subspace of $\mathbb{R}^{n+1}$. By Theorem 3.10, $\mathscr{C}(\mathbb{R}^{n+1}) \succcurlyeq \mathscr{C}(\mathbb{D}^{\aleph_0} \setminus \{\mathbf{0}\})$. Then, by Theorem 3.2, we have

$$\mathscr{C}(\mathbb{A}) \succcurlyeq \mathscr{C}(\mathbb{R}^n) \succcurlyeq \mathscr{C}(\mathbb{A} \times \mathbb{R}^n) \succcurlyeq \mathscr{C}(\mathbb{R}^{n+1}) \succcurlyeq \mathscr{C}(\mathbb{D}^{\aleph_0} \setminus \{\mathbf{0}\}) = \mathscr{C}(\mathbb{I}^{\aleph_0} \setminus \{\mathbf{0}\}) \succcurlyeq \mathscr{C}(\mathbb{A}). \tag{3.33}$$

□

Recall that a space is said to be a *generalized continuum* if it is a locally compact connected Hausdorff space. We also have the following corollary.

Corollary 3.15. *Let (X, τ) be a noncompact locally connected metrizable generalized continuum. Then $\mathscr{C}(X) = \mathscr{C}(\mathbb{R})$.*

Proof. If (X, τ) is a noncompact locally connected metrizable generalized continuum, then it is second countable and contains a closed copy of $\mathbb{A}$ (see, e.g., [14]). Then $w(X) = \aleph_0$, so by Proposition 3.11 and Corollary 3.14, we get $\mathscr{C}(X) = \mathscr{C}(\mathbb{A}) = \mathscr{C}(\mathbb{D}^{\aleph_0} \setminus \{\mathbf{0}\}) = \mathscr{C}(\mathbb{R})$. □

In the previous results, we have found the compactificability classes of various spaces constructed or derived in some way from the real line. These spaces are usually uncountable, second countable, locally compact, and Hausdorff. Therefore, one may state a natural question if it is true that every such a space must be of the same class of compactificability as $\mathbb{R}$. It is not difficult to prove that the space which is a disjoint union of $\mathbb{N}$ with the discrete topology and any uncountable second-countable compact space is a proper counterexample. After some further investigation, the reader may find out that the behavior at infinity of a space is a more determining property for the classes of the mutual compactificability than its cardinality or its separation properties. But these considerations will be a subject of our next, forthcoming, paper.

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Martin Maria Kovár: Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, Technická 8, Brno 616 69, Czech Republic
E-mail address: kovar@feec.vutbr.cz


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