ON EULERIAN EQUILIBRIA IN K-ORDER APPROXIMATION OF THE GYROSTAT IN THE THREE-BODY PROBLEM

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We consider the noncanonical Hamiltonian dynamics of a gyrostat in the three-body problem. By means of geometric-mechanics methods we study the approximate Poisson dynamics that arises when we develop the potential in series of Legendre and truncate this in an arbitrary order k. Working in the reduced problem, the existence and number of equilibria, that we denominate of Euler type in analogy with classic results on the topic, are considered. Necessary and sufficient conditions for their existence in an approximate dynamics of order k are obtained and we give explicit expressions of these equilibria, useful for the later study of the stability of the same ones. A complete study of the number of Eulerian equilibria is made in approximate dynamics of orders zero and one. We obtain the main result of this work, the number of Eulerian equilibria in an approximate dynamics of order k for $k \geq 1$ is independent of the order of truncation of the potential if the gyrostat S_0 is close to the sphere. The instability of Eulerian equilibria is proven for any approximate dynamics if the gyrostat is close to the sphere. In this way, we generalize the classical results on equilibria of the three-body problem and many of those obtained by other authors using more classic techniques for the case of rigid bodies.

1. Introduction

In the study of configurations of relative equilibria by differential geometry methods or by more classical ones, we will mention here the papers of Wang et al. [8], about the problem of a rigid body in a central Newtonian field; Maciejewski [3], about the problem of two rigid bodies in mutual Newtonian attraction. These papers have been generalized to the case of a gyrostat by Mondejar and Vigueras [4] to the case of two gyrostats in mutual Newtonian attraction.

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For the problem of three rigid bodies we would like to mention that Vidyakin [7] and Dubochine [1] proved the existence of Euler and Lagrange configurations of equilibria when the bodies possess symmetries; Zhuravlev and Petrutskii [9] made a review of the results up to 1990.

In Vera [5] and a recent paper of Vera and Vigueras [6] we study the noncanonical Hamiltonian dynamics of $n+1$ bodies in Newtonian attraction, where n of them are rigid bodies with spherical distribution of mass or material points and the other one is a triaxial gyrostat.

Let us remember that a gyrostat is a mechanical system S, composed of a rigid body S', and other bodies S'' (deformable or rigid) connected to it, in such a way that their relative motion with respect to its rigid part do not change the distribution of mass of the total system S, (see Leimanis [2] for details).

In this paper, we take $n = 2$ and as a first approach to the qualitative study of this system, we describe the approximate dynamics that arises in a natural way when we take the Legendre development of the potential function and truncate this until an arbitrary order. We give global conditions on the existence of relative equilibria and in analogy with classic results on the topic, we study the existence of relative equilibria that we will denominate of *Euler type* in the case in which S_1, S_2 are spherical or punctual bodies and S_0 is a gyrostat. Necessary and sufficient conditions for their existence in a approximate dynamics of order k are obtained and we give explicit expressions of these equilibria, useful for the later study of the stability of the same ones. A complete study of the number of Eulerian equilibria is made in approximate dynamics of orders zero and one. The number of Eulerian equilibria in an approximate dynamics of order k for $k > 1$ is independent of the order of truncation of the potential if the gyrostat S_0 is close to the sphere. The instability of Eulerian equilibria is proven for any approximate dynamics if the gyrostat is close to the sphere. The analysis is done in vectorial form avoiding the use of canonical variables and the tedious expressions associated with them.

We should notice that the studied system has potential interest both in astrodynamics (dealing with spacecraft) as well as in the understanding of the evolution of planetary systems recently found (and more to appear), where some of the planets may be modeled like a gyrostat rather than a rigid body. In fact, the equilibria reported might be well compared with the ones taken for the “parking areas” of the space missions (GENESIS, SOHO, DARWIN, etc.) around the Eulerian points of the Sun-Earth and the Earth-Moon systems.

To finish this introduction, we describe the structure of the article. The paper is organized in five sections, two appendices, and the bibliography. In these sections we study the equations of motion, Casimir function and integrals of the system, the relative equilibria and the existence of Eulerian equilibria in an approximate dynamics of order k, in particular the study of the bifurcations of Eulerian equilibria in an approximate dynamics of orders zero and one.

2. Equations of motion

Following the line of Vera and Vigueras [6] let S_0 be a gyrostat of mass m_0 and S_1, S_2 two spherical rigid bodies of masses m_1 and m_2. We use the following notation.

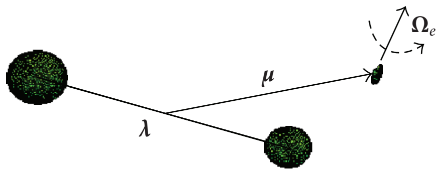

Figure 2.1. Gyrostat in the three-body problem.

For $\mathbf{u},\mathbf{v} \in \mathbb{R}^3$, $\mathbf{u}\cdot\mathbf{v}$ is the dot product, $|\mathbf{u}|$ is the Euclidean norm of the vector $\mathbf{u}$, and $\mathbf{u}\times\mathbf{v}$ is the cross-product. $\mathbf{I}_{\mathbb{R}^3}$ is the identity matrix and $\mathbf{0}$ is the zero matrix of order three. Let $\mathbf{z} = (\boldsymbol{\Pi},\boldsymbol{\lambda},\mathbf{p}_\lambda,\boldsymbol{\mu},\mathbf{p}_\mu) \in \mathbb{R}^{15}$ be a generic element of the twice reduced problem obtained using the symmetries of the system, where $\boldsymbol{\Pi} = \mathbb{I}\boldsymbol{\Omega} + \mathbf{l}_r$ is the total rotational angular momentum vector of the gyrostat, $\mathbb{I} = \operatorname{diag}(A,B,C)$ is the diagonal tensor of inertia of the gyrostat, and $\boldsymbol{\Omega}$ is the angular velocity of S_0 in the body frame, $\mathfrak{J}$, which is attached to its rigid part and whose axes have the direction of the principal axes of inertia of S_0. The vector $\mathbf{l}_r$ is the gyrostatic momentum that we suppose constant and is given by $\mathbf{l}_r = (0,0,l)$. The elements $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{p}_\lambda$, and $\mathbf{p}_\mu$ are, respectively, the barycentric coordinates and the linear momenta expressed in the body frame $\mathfrak{J}$ (see Figure 2.1).

The twice reduced Hamiltonian of the system, obtained by the action of the group $\mathbf{SE}(3)$, has the following expression:

$$\mathcal{H}(\mathbf{z}) = \frac{|\mathbf{p}_\lambda|^2}{2g_1} + \frac{|\mathbf{p}_\mu|^2}{2g_2} + \frac{1}{2}\Pi\mathbb{I}^{-1}\Pi - \mathbf{l}_r\cdot\mathbb{I}^{-1}\Pi + \mathcal{V} \tag{2.1}$$

with

$$\begin{gathered} M_2 = m_1 + m_2, \qquad M_1 = m_1 + m_2 + m_0, \\ g_1 = \frac{m_1 m_2}{M_2}, \qquad g_2 = \frac{m_0 M_2}{M_1}, \end{gathered} \tag{2.2}$$

with $\mathcal{V}$ being the potential function of the system given by the formula

$$\mathcal{V}(\boldsymbol{\lambda},\boldsymbol{\mu}) = -\left(\frac{G m_1 m_2}{|\boldsymbol{\lambda}|} + \int_{S_0}\frac{G m_1\, dm(\mathbf{Q})}{|\mathbf{Q}+\boldsymbol{\mu}+(m_2/M_2)\boldsymbol{\lambda}|} + \int_{S_0}\frac{G m_2\, dm(\mathbf{Q})}{|\mathbf{Q}+\boldsymbol{\mu}-(m_1/M_2)\boldsymbol{\lambda}|}\right). \tag{2.3}$$

Let $\mathbf{M} = \mathbb{R}^{15}$, and we consider the manifold $(\mathbf{M},\{\cdot,\cdot\},\mathcal{H})$, with Poisson brackets $\{\cdot,\cdot\}$ defined by means of the Poisson tensor

$$\mathbf{B}(\mathbf{z}) = \begin{pmatrix} \widehat{\boldsymbol{\Pi}} & \widehat{\boldsymbol{\lambda}} & \widehat{\mathbf{p}_\lambda} & \widehat{\boldsymbol{\mu}} & \widehat{\mathbf{p}_\mu} \\ \widehat{\boldsymbol{\lambda}} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{p}_\lambda} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\boldsymbol{\mu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} \\ \widehat{\mathbf{p}_\mu} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} \end{pmatrix}. \tag{2.4}$$

In $\mathbf{B}(\mathbf{z})$, $\hat{\mathbf{v}}$ is considered to be the image of the vector $\mathbf{v} \in \mathbb{R}^3$ by the standard isomorphism between the Lie Algebras $\mathbb{R}^3$ and $\mathfrak{so}(3)$, that is,

$$\hat{\mathbf{v}} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}. \tag{2.5}$$

The equations of the motion are given by the following expression:

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, \mathcal{H}(\mathbf{z})\} \quad (\mathbf{z}) = \mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}(\mathbf{z}), \tag{2.6}$$

where $\nabla_{\mathbf{z}} f$ is the gradient of $f \in C^{\infty}(\mathbf{M})$ with respect to an arbitrary vector $\mathbf{z}$.

Developing $\{\mathbf{z}, \mathcal{H}(\mathbf{z})\}$, we obtain the following group of vectorial equations of the motion:

$$\begin{gathered} \frac{d\mathbf{\Pi}}{dt} = \mathbf{\Pi} \times \mathbf{\Omega} + \boldsymbol{\lambda} \times \nabla_{\lambda}\mathcal{V} + \boldsymbol{\mu} \times \nabla_{\mu}\mathcal{V}, \\ \frac{d\boldsymbol{\lambda}}{dt} = \frac{\mathbf{p}_{\lambda}}{g_1} + \boldsymbol{\lambda} \times \mathbf{\Omega}, \qquad \frac{d\mathbf{p}_{\lambda}}{dt} = \mathbf{p}_{\lambda} \times \mathbf{\Omega} - \nabla_{\lambda}\mathcal{V}, \\ \frac{d\boldsymbol{\mu}}{dt} = \frac{\mathbf{p}_{\mu}}{g_2} + \boldsymbol{\mu} \times \mathbf{\Omega}, \qquad \frac{d\mathbf{p}_{\mu}}{dt} = \mathbf{p}_{\mu} \times \mathbf{\Omega} - \nabla_{\mu}\mathcal{V}. \end{gathered} \tag{2.7}$$

Important elements of $\mathbf{B}(\mathbf{z})$ are the associate Casimir functions. We consider the total angular momentum $\mathbf{L}$ given by

$$\mathbf{L} = \mathbf{\Pi} + \boldsymbol{\lambda} \times \mathbf{p}_{\lambda} + \boldsymbol{\mu} \times \mathbf{p}_{\mu}. \tag{2.8}$$

Then the following result is verified (see Vera and Vigueras [6] for details).

Proposition 2.1. *If φ is a real smooth function not constant, then $\varphi(|\mathbf{L}|^2/2)$ is a Casimir function of the Poisson tensor $\mathbf{B}(\mathbf{z})$. Moreover* $\operatorname{Ker}\mathbf{B}(\mathbf{z}) = \langle \nabla_{\mathbf{z}}\varphi \rangle$. *Also, $d\mathbf{L}/dt = \mathbf{0}$, that is to say the total angular momentum vector remains constant.*

2.1. Approximate Poisson dynamics. To simplify the problem we assume that the gyrostat S_0 is symmetrical around the third axis of inertia OZ and with respect to the plane OXY being OX, OY, OZ are the coordinated axes of the body frame $\mathfrak{J}$. If the mutual distances are bigger than the individual dimensions of the bodies, then we can develop the potential in fast convergent series. Under these hypotheses, we will be able to carry out a study of equilibria in different approximate dynamics.

Applying the Legendre development of the potential, we have

$$\mathcal{V}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -\left(\frac{Gm_1m_2}{|\boldsymbol{\lambda}|} + \sum_{i=0}^{\infty} \frac{Gm_1A_{2i}}{|\boldsymbol{\mu} + (m_2/M_2)\boldsymbol{\lambda}|^{2i+1}} + \sum_{i=0}^{\infty} \frac{Gm_2A_{2i}}{|\boldsymbol{\mu} - (m_1/M_2)\boldsymbol{\lambda}|^{2i+1}} \right), \tag{2.9}$$

where $A_0 = m_0$, $A_2 = (C - A)/2$ and A_{2i} are certain coefficients related to the geometry of the gyrostat, see Vera and Vigueras [6] for details.

Definition 2.2. We call approximate potential of order k to the following expression:

$$\mathcal{V}_k(\boldsymbol{\lambda},\boldsymbol{\mu}) = -\left(\frac{Gm_1m_2}{|\boldsymbol{\lambda}|} + \sum_{i=0}^{k}\frac{Gm_1A_{2i}}{|\boldsymbol{\mu}+(m_2/M_2)\boldsymbol{\lambda}|^{2i+1}} + \sum_{i=0}^{k}\frac{Gm_2A_{2i}}{|\boldsymbol{\mu}-(m_1/M_2)\boldsymbol{\lambda}|^{2i+1}}\right). \tag{2.10}$$

It is easy to demonstrate the following lemmas.

Lemma 2.3. *Given the approximate potential of order k,*

$$\begin{aligned}
\nabla_{\boldsymbol{\lambda}}\mathcal{V}_k &= \frac{Gm_1m_2\boldsymbol{\lambda}}{|\boldsymbol{\lambda}|^3} + \frac{Gm_1m_2}{M_2}\sum_{i=0}^{k}\frac{(\boldsymbol{\mu}+(m_2/M_2)\boldsymbol{\lambda})(2i+1)A_{2i}}{|\boldsymbol{\mu}+(m_2/M_2)\boldsymbol{\lambda}|^{2i+3}} \\
&\quad - \frac{Gm_1m_2}{M_2}\sum_{i=0}^{k}\frac{(\boldsymbol{\mu}-(m_1/M_2)\boldsymbol{\lambda})(2i+1)A_{2i}}{|\boldsymbol{\mu}-(m_1/M_2)\boldsymbol{\lambda}|^{2i+3}}, \\
\nabla_{\boldsymbol{\mu}}\mathcal{V}_k &= Gm_1\sum_{i=0}^{k}\frac{(\boldsymbol{\mu}+(m_2/M_2)\boldsymbol{\lambda})(2i+1)A_{2i}}{|\boldsymbol{\mu}+(m_2/M_2)\boldsymbol{\lambda}|^{2i+3}} + Gm_2\sum_{i=0}^{k}\frac{(\boldsymbol{\mu}-(m_1/M_2)\boldsymbol{\lambda})(2i+1)A_{2i}}{|\boldsymbol{\mu}-(m_1/M_2)\boldsymbol{\lambda}|^{2i+3}}.
\end{aligned} \tag{2.11}$$

The following identities are verified

$$\nabla_{\boldsymbol{\lambda}}\mathcal{V}_k = \widetilde{A}_{11}\boldsymbol{\lambda} + \widetilde{A}_{12}\boldsymbol{\mu}, \qquad \nabla_{\boldsymbol{\mu}}\mathcal{V}_k = \widetilde{A}_{21}\boldsymbol{\lambda} + \widetilde{A}_{22}\boldsymbol{\mu} \tag{2.12}$$

being

$$\begin{aligned}
\widetilde{A}_{11}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= \frac{Gm_1m_2}{|\boldsymbol{\lambda}|^3} + \frac{Gm_1m_2^2}{M_2^2}\left(\sum_{i=0}^{k}\frac{\beta_i}{|\boldsymbol{\mu}+(m_2/M_2)\boldsymbol{\lambda}|^{2i+3}}\right) \\
&\quad + \frac{Gm_1^2m_2}{M_2^2}\left(\sum_{i=0}^{k}\frac{\beta_i}{|\boldsymbol{\mu}-(m_1/M_2)\boldsymbol{\lambda}|^{2i+3}}\right), \\
\widetilde{A}_{12}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= \frac{Gm_1m_2}{M_2}\left(\sum_{i=0}^{k}\frac{\beta_i}{|\boldsymbol{\mu}+(m_2/M_2)\boldsymbol{\lambda}|^{2i+3}} - \sum_{i=0}^{k}\frac{\beta_i}{|\boldsymbol{\mu}-(m_1/M_2)\boldsymbol{\lambda}|^{2i+3}}\right), \\
\widetilde{A}_{22}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= Gm_1\left(\sum_{i=0}^{k}\frac{\beta_i}{|\boldsymbol{\mu}+(m_2/M_2)\boldsymbol{\lambda}|^{2i+3}}\right) + Gm_2\left(\sum_{i=0}^{k}\frac{\beta_i}{|\boldsymbol{\mu}-(m_1/M_2)\boldsymbol{\lambda}|^{2i+3}}\right), \\
\widetilde{A}_{21}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= \widetilde{A}_{12}(\boldsymbol{\lambda},\boldsymbol{\mu})
\end{aligned} \tag{2.13}$$

with coefficients $\beta_0 = m_0$, $\beta_1 = 3/2(C-A)$, $\beta_i = (2i+1)A_{2i}$ *for* $i \geqslant 1$.

Definition 2.4. Let $\mathbf{M} = \mathbb{R}^{15}$ and the manifold $(\mathbf{M}, \{\cdot,\cdot\}, \mathcal{H}_k)$, with Poisson brackets $\{\cdot,\cdot\}$, defined by means of the Poisson tensor (2.4). We call *approximate dynamics of order k* to the differential equations of motion given by the following expression:

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, \mathcal{H}_k(\mathbf{z})\}, \quad (\mathbf{z}) = \mathbf{B}(\mathbf{z})\boldsymbol{\nabla}_{\mathbf{z}}\mathcal{H}_k(\mathbf{z}) \tag{2.14}$$

being

$$\mathcal{H}_k(\mathbf{z}) = \frac{|\mathbf{p}_\lambda|^2}{2g_1} + \frac{|\mathbf{p}_\mu|^2}{2g_2} + \frac{1}{2}\Pi\mathbb{I}^{-1}\Pi - \mathbf{l}_r \cdot \mathbb{I}^{-1}\Pi + \mathcal{V}_k(\boldsymbol{\lambda}, \boldsymbol{\mu}). \tag{2.15}$$

2.1.1. Integrals of the system. On the other hand, it is easy to verify that

$$\nabla_{\mathbf{z}}(|\mathbf{\Pi}|^2)\mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}_0(\mathbf{z}) = 0 \tag{2.16}$$

and similarly when the gyrostat is of revolution

$$\nabla_{\mathbf{z}}(\mathbf{\Pi}_3)\mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}_k(\mathbf{z}) = 0, \tag{2.17}$$

where $\boldsymbol{\pi}_3$ is the third component of the rotational angular momentum of the gyrostat. It is verified the following result.

THEOREM 2.5. *In the approximate dynamics of order* 0, $|\mathbf{\Pi}|^2$ *is an integral of motion and also when the gyrostat is of revolution* $\boldsymbol{\pi}_3$ *is another integral of motion.*

2.2. Relative equilibria. The relative equilibria are the equilibria of the twice reduced problem whose Hamiltonian function is obtained in Vera and Vigueras [6] for the case $n = 2$. If we denote by $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}_\lambda^e, \boldsymbol{\mu}^e, \mathbf{p}_\mu^e)$ a generic relative equilibrium of an approximate dynamics of order k, then this verifies the equations

$$\begin{gathered}
\mathbf{\Pi}_e \times \mathbf{\Omega}_e + \boldsymbol{\lambda}^e \times (\boldsymbol{\nabla}_\lambda \mathcal{V}_k)_e + \boldsymbol{\mu}^e \times (\boldsymbol{\nabla}_\mu \mathcal{V}_k)_e = \mathbf{0}, \\
\frac{\mathbf{p}_\lambda^e}{g_1} + \boldsymbol{\lambda}^e \times \mathbf{\Omega}_e = \mathbf{0}, \qquad \mathbf{p}_\lambda^e \times \mathbf{\Omega}_e = (\boldsymbol{\nabla}_\lambda \mathcal{V}_k)_e, \\
\frac{\mathbf{p}_\mu^e}{g_2} + \boldsymbol{\mu}^e \times \mathbf{\Omega}_e = \mathbf{0}, \qquad \mathbf{p}_\mu^e \times \mathbf{\Omega}_e = (\boldsymbol{\nabla}_\mu \mathcal{V}_k)_e.
\end{gathered} \tag{2.18}$$

Also by virtue of the relationships obtained in Vera and Vigueras [6], we have the following result.

LEMMA 2.6. *If* $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}_\lambda^e, \boldsymbol{\mu}^e, \mathbf{p}_\mu^e)$ *is a relative equilibrium of an approximate dynamics of order k, the following relationships are verified:*

$$\begin{aligned}
|\mathbf{\Omega}_e|^2|\boldsymbol{\lambda}^e|^2 - (\boldsymbol{\lambda}^e \cdot \mathbf{\Omega}_e)^2 &= \frac{1}{g_1}(\boldsymbol{\lambda}^e \cdot (\boldsymbol{\nabla}_\lambda \mathcal{V}_k)_e), \\
|\mathbf{\Omega}_e|^2|\boldsymbol{\mu}^e|^2 - (\boldsymbol{\mu}^e \cdot \mathbf{\Omega}_e)^2 &= \frac{1}{g_2}(\boldsymbol{\mu}^e \cdot (\boldsymbol{\nabla}_\mu \mathcal{V}_k)_e).
\end{aligned} \tag{2.19}$$

The last two previous identities will be used to obtain necessary conditions for the existence of relative equilibria in this approximate dynamics.

We will study certain relative equilibria in the approximate dynamics supposing that the vectors $\mathbf{\Omega}_e$, $\boldsymbol{\lambda}^e$, $\boldsymbol{\mu}^e$ satisfy special geometric properties.

Definition 2.7. $\mathbf{z}_e$ is an *Eulerian relative equilibrium* in an approximate dynamics of order k when $\boldsymbol{\lambda}^e$ and $\boldsymbol{\mu}^e$ are proportional and $\mathbf{\Omega}_e$ is perpendicular to the straight line that they generate.

Remark 2.8. The previous hypotheses simplify the conditions of Lemma 2.6. In a next paper we will study the possible "inclined" relative equilibria, in which $\mathbf{\Omega}_e$ form an angle $\alpha \neq 0$ and $\pi/2$ with the vector $\boldsymbol{\lambda}^e$.

From the equations of motion, the following property is deduced.

Proposition 2.9. *In a Eulerian relative equilibrium for any approximate dynamics of arbitrary order, moments are not exercised on the gyrostat.*

Next we obtain necessary and sufficient conditions for the existence of Eulerian relative equilibria.

3. Relative equilibria of Euler type

According to the relative position of the gyrostat S_0 with respect to S_1 and S_2, there are three possible equilibrium configurations (see Figure 3.1): (a) $S_0S_2S_1$, (b) $S_2S_0S_1$, and (c) $S_2S_1S_0$.

3.1. Necessary condition of existence

Lemma 3.1. *If* $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}_\lambda^e, \boldsymbol{\mu}^e, \mathbf{p}_\mu^e)$ *is a relative equilibrium of Euler type, then for the configuration* $S_0S_2S_1$,

$$\left|\boldsymbol{\mu}^e + \frac{m_1}{M_2}\boldsymbol{\lambda}^e\right| = |\boldsymbol{\lambda}^e| + \left|\boldsymbol{\mu}^e - \frac{m_2}{M_2}\boldsymbol{\lambda}^e\right|. \tag{3.1}$$

In a similar way, for the configuration $S_2S_0S_1$,

$$|\boldsymbol{\lambda}^e| = \left|\boldsymbol{\mu}^e - \frac{m_1}{M_2}\boldsymbol{\lambda}^e\right| + \left|\boldsymbol{\mu}^e + \frac{m_2}{M_2}\boldsymbol{\lambda}^e\right|. \tag{3.2}$$

Finally, for the configuration $S_2S_1S_0$,

$$\left|\boldsymbol{\mu}^e - \frac{m_2}{M_2}\boldsymbol{\lambda}^e\right| = \left|\boldsymbol{\mu}^e + \frac{m_1}{M_2}\boldsymbol{\lambda}^e\right| + |\boldsymbol{\lambda}^e|. \tag{3.3}$$

Next we study necessary and sufficient conditions for the existence of relative equilibria of Euler type for the configuration $S_0S_2S_1$; the other configurations are studied in a similar way. If $\mathbf{z}_e$ is a relative equilibrium of Euler type, in the configuration $S_0S_2S_1$ in an

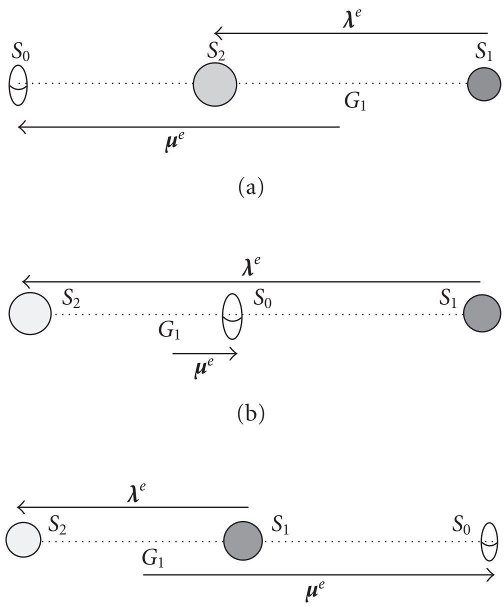

Figure 3.1. Eulerian configurations.

approximate dynamics of order k, we have

$$g_1|\boldsymbol{\Omega}_e|^2|\boldsymbol{\lambda}^e|^2=\boldsymbol{\lambda}^e\cdot(\nabla_{\boldsymbol{\lambda}}\mathcal{V}_k)_e,\qquad g_2|\boldsymbol{\Omega}_e|^2|\boldsymbol{\mu}^e|^2=\boldsymbol{\mu}^e\cdot(\nabla_{\boldsymbol{\mu}}\mathcal{V}_k)_e,$$
$$\boldsymbol{\mu}^e-\frac{m_1}{M_2}\boldsymbol{\lambda}^e=\rho\boldsymbol{\lambda}^e,\qquad \boldsymbol{\mu}^e+\frac{m_2}{M_2}\boldsymbol{\lambda}^e=(1+\rho)\boldsymbol{\lambda}^e,\qquad \boldsymbol{\mu}^e=\frac{((1+\rho)m_1+\rho m_2)}{M_2}\boldsymbol{\lambda}^e, \tag{3.4}$$

where $\rho\in(0,+\infty)$ in the case (a), $\rho\in(-1,0)$ in the case (b), and $\rho\in(-\infty,-1)$ in the case (c). And it is possible to obtain the following expressions:

$$(\nabla_{\boldsymbol{\lambda}}\mathcal{V}_k)_e=f_1(\rho)\boldsymbol{\lambda}^e,\qquad (\nabla_{\boldsymbol{\mu}}\mathcal{V}_k)_e=f_2(\rho)\boldsymbol{\lambda}^e, \tag{3.5}$$

where

$$f_1(\rho)=\frac{Gm_1m_2}{|\boldsymbol{\lambda}^e|^3}+\frac{Gm_1m_2}{M_2}\left(\sum_{i=0}^{k}\frac{\beta_i}{|\boldsymbol{\lambda}^e|^{2i+3}}\left(\frac{1+\rho}{|1+\rho|^{2i+3}}-\frac{\rho}{|\rho|^{2i+3}}\right)\right),$$
$$f_2(\rho)=\sum_{i=0}^{k}\frac{G\beta_i}{|\boldsymbol{\lambda}^e|^{2i+3}}\left(\frac{m_1(1+\rho)}{|1+\rho|^{2i+3}}+\frac{m_2\rho}{|\rho|^{2i+3}}\right). \tag{3.6}$$

Restricting us to the case (a) we have

$$f_1(\rho) = \frac{Gm_1m_2}{|\boldsymbol{\lambda}^e|^3} + \frac{Gm_1m_2}{M_2}\left(\sum_{i=0}^{k}\frac{\beta_i}{|\boldsymbol{\lambda}^e|^{2i+3}}\left(\frac{1}{(1+\rho)^{2i+2}} - \frac{1}{\rho^{2i+2}}\right)\right),$$
$$f_2(\rho) = \sum_{i=0}^{k}\frac{G\beta_i}{|\boldsymbol{\lambda}^e|^{2i+3}}\left(\frac{m_1}{(1+\rho)^{2i+2}} + \frac{m_2}{\rho^{2i+2}}\right). \tag{3.7}$$

Now, from the identities

$$\boldsymbol{\lambda}^e \cdot (\nabla_{\lambda}\mathcal{V}_k)_e = |\boldsymbol{\lambda}^e|^2 f_1(\rho), \qquad \boldsymbol{\mu}^e \cdot (\nabla_{\boldsymbol{\mu}}\mathcal{V}_k)_e = \frac{((1+\rho)m_1+\rho m_2)}{M_2}|\boldsymbol{\lambda}^e|^2 f_2(\rho) \tag{3.8}$$

we deduce the following equations:

$$|\boldsymbol{\Omega}_e|^2 = \frac{f_1(\rho)}{g_1}, \qquad |\boldsymbol{\Omega}_e|^2 = \frac{M_2 f_2(\rho)}{g_2((1+\rho)m_1+\rho m_2)}. \tag{3.9}$$

Then for a relative equilibrium of Euler type ρ must be a positive real root of the following equation:

$$m_0(m_1+m_2)((1+\rho)m_1+\rho m_2)f_1(\rho) = m_1m_2(m_0+m_1+m_2)f_2(\rho). \tag{3.10}$$

We summarize all these results in the following proposition.

Proposition 3.2. *If $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}_\lambda^e, \boldsymbol{\mu}^e, \mathbf{p}_\mu^e)$ is an Eulerian relative equilibrium in the configuration $S_0S_2S_1$, (3.10) has, at least, a positive real root; where the functions $f_1(\rho)$ and $f_2(\rho)$ are given by (3.7) and the modulus of the angular velocity of the gyrostat is*

$$|\boldsymbol{\Omega}_e|^2 = \frac{f_1(\rho)}{g_1}. \tag{3.11}$$

Remark 3.3. If a solution of relative equilibrium of Euler type exists, in an approximate dynamics of order k, fixing $|\boldsymbol{\lambda}_e|$, (3.10) has positive real solutions. The number of real roots of (3.10) will depend, obviously, on the numerous parameters that exist in our system. Similar results would be obtained for the other two cases.

3.2. Sufficient condition of existence. The following proposition indicates how to find solutions of (2.18).

Proposition 3.4. *Fixing $|\boldsymbol{\lambda}^e|$, let ρ be a solution of (3.10) where the functions $f_1(\rho)$ and $f_2(\rho)$ are given for the case (a) as (3.7) then $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}_\lambda^e, \boldsymbol{\mu}^e, \mathbf{p}_\mu^e)$, given by*

$$\boldsymbol{\lambda}^e = (\lambda^e, 0, 0), \qquad \boldsymbol{\mu}^e = (\mu^e, 0, 0), \qquad \boldsymbol{\Omega}_e = (0, 0, \omega_e),$$
$$\mathbf{p}_\lambda^e = (0, g_1\omega_e\lambda^e, 0), \qquad \mathbf{p}_\mu^e = (0, g_2\omega_e\mu^e, 0), \qquad \boldsymbol{\Pi}_e = (0, 0, C\omega_e + l), \tag{3.12}$$

where

$$\boldsymbol{\mu}^e = \frac{((1+\rho)m_1 + \rho m_2)}{M_2}\boldsymbol{\lambda}^e, \qquad \omega_e^2 = \frac{f_1(\rho)}{g_1}, \tag{3.13}$$

is a solution of relative equilibrium of Euler type, in an approximate dynamics of order k in the configuration $S_0S_2S_1$. The total angular momentum of the system is given by

$$\mathbf{L} = (0, 0, C\omega_e + l + g_1\omega_e\lambda^e + g_2\omega_e\mu^e), \tag{3.14}$$

where l is the gyrostatic momentum.

Let us see the existence and number of solutions for the approximate dynamics of orders zero and one, respectively. For superior order it is possible to use a similar technical.

4. Eulerian equilibria in an approximate dynamics of orders zero and one

For the configuration $S_0S_2S_1$, in an approximate dynamics of order zero, we have

$$\begin{aligned} f_1(\rho) &= \frac{Gm_1m_2}{|\boldsymbol{\lambda}_e|^3}\left(1 + \frac{m_0}{M_2}\left(\frac{1}{(1+\rho)^2} - \frac{1}{\rho^2}\right)\right), \\ f_2(\rho) &= \frac{Gm_0}{|\boldsymbol{\lambda}_e|^3}\left(\frac{m_1}{(1+\rho)^2} + \frac{m_2}{\rho^2}\right). \end{aligned} \tag{4.1}$$

Equation (3.10) is equivalent to the following polynomial equation:

$$\begin{aligned} &(m_1 + m_2)\rho^5 + (3m_1 + 2m_2)\rho^4 + (3m_1 + m_2)\rho^3 \\ &\quad - (3m_0 + m_2)\rho^2 - (3m_0 + 2m_2)\rho - (m_0 + m_2) = 0. \end{aligned} \tag{4.2}$$

This equation has an unique positive real solution, then in this case for the approximate dynamics of order zero, there exists a unique relative equilibrium of Euler type.

On the other hand, one has

$$|\boldsymbol{\Omega}_e|^2 = \frac{G(m_1 + m_2)}{|\boldsymbol{\lambda}^e|^3}\left(1 + \frac{m_0}{m_1 + m_2}\left(\frac{1}{(1+\rho)^2} - \frac{1}{\rho^2}\right)\right), \tag{4.3}$$

ρ being the only one positive solution of (4.2).

The following proposition gathers the results about relative equilibria of Euler type in an approximate dynamics of order zero in any of the previously mentioned cases (a), (b), or (c).

Proposition 4.1. (1) *If ρ is the unique positive root of (4.2) with $|\boldsymbol{\Omega}_e|^2$ being expressed as (4.3), then $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}_\lambda^e, \boldsymbol{\mu}^e, \mathbf{p}_\mu^e)$, given by*

$$\begin{aligned} \boldsymbol{\lambda}^e &= (\lambda^e, 0, 0), \qquad \boldsymbol{\mu}^e = (\mu^e, 0, 0), \qquad \boldsymbol{\Omega}_e = (0, 0, \omega_e), \\ \mathbf{p}_\lambda^e &= (0, g_1\omega_e\lambda^e, 0), \qquad \mathbf{p}_\mu^e = (0, g_2\omega_e\mu^e, 0), \qquad \boldsymbol{\Pi}_e = (0, 0, C\omega_e + l), \end{aligned} \tag{4.4}$$

is the unique solution of relative equilibrium of Euler type in the configuration $S_0S_2S_1$.

(2) *If* $\rho \in (-1,0)$ *is the unique root of the equation*

$$\begin{aligned}(m_1+m_2)\rho^5+(3m_1+2m_2)\rho^4+(3m_1+m_2)\rho^3 \\ +(3m_0+2m_1+m_2)\rho^2+(3m_0+2m_2)\rho+(m_0+m_2)=0\end{aligned} \tag{4.5}$$

with

$$|\boldsymbol{\Omega}_e|^2=\frac{G(m_1+m_2)}{|\boldsymbol{\lambda}_e|^3}\left(1+\frac{m_0}{m_1+m_2}\left(\frac{1}{\rho^2}-\frac{1}{(1+\rho)^2}\right)\right), \tag{4.6}$$

then $\mathbf{z}_e=(\boldsymbol{\Pi}_e,\boldsymbol{\lambda}^e,\mathbf{p}_\lambda^e,\boldsymbol{\mu}^e,\mathbf{p}_\mu^e)$, *given by*

$$\begin{aligned}\boldsymbol{\lambda}^e&=(\lambda^e,0,0), & \boldsymbol{\mu}^e&=(\mu^e,0,0), & \boldsymbol{\Omega}_e&=(0,0,\omega_e),\\ \mathbf{p}_\lambda^e&=(0,g_1\omega_e\lambda^e,0), & \mathbf{p}_\mu^e&=(0,g_2\omega_e\mu^e,0), & \boldsymbol{\Pi}_e&=(0,0,C\omega_e+l),\end{aligned} \tag{4.7}$$

is the unique solution of relative equilibrium of Euler type in the configuration $S_2S_0S_1$.

(3) *If* $\rho \in (-\infty,-1)$ *is the unique root of the equation*

$$\begin{aligned}(m_1+m_2)\rho^5+(3m_1+2m_2)\rho^4+(2m_0+3m_1+m_2)\rho^3 \\ +(3m_0+m_2)\rho^2+(3m_0+2m_2)\rho+(m_0+m_2)=0\end{aligned} \tag{4.8}$$

with

$$|\boldsymbol{\Omega}_e|^2=\frac{G(m_1+m_2)}{|\boldsymbol{\lambda}_e|^3}\left(1+\frac{m_0}{m_1+m_2}\left(\frac{1}{\rho^2}+\frac{1}{(1+\rho)^2}\right)\right), \tag{4.9}$$

then $\mathbf{z}_e=(\boldsymbol{\Pi}_e,\boldsymbol{\lambda}^e,\mathbf{p}_\lambda^e,\boldsymbol{\mu}^e,\mathbf{p}_\mu^e)$, *given by*

$$\begin{aligned}\boldsymbol{\lambda}^e&=(\lambda^e,0,0), & \boldsymbol{\mu}^e&=(\mu^e,0,0), & \boldsymbol{\Omega}_e&=(0,0,\omega_e),\\ \mathbf{p}_\lambda^e&=(0,g_1\omega_e\lambda^e,0), & \mathbf{p}_\mu^e&=(0,g_2\omega_e\mu^e,0), & \boldsymbol{\Pi}_e&=(0,0,C\omega_e+l),\end{aligned} \tag{4.10}$$

is the unique solution of relative equilibrium of Euler type in the configuration $S_2S_1S_0$.

Remark 4.2. If $m_0 \to 0$, then $|\boldsymbol{\Omega}_e|^2=G(m_1+m_2)/|\boldsymbol{\lambda}_e|^3$ and the equations that determine the Eulerian equilibria are the same ones of the restricted three-body problem.

4.1. Number of Eulerian equilibria in an approximate dynamics of order one. For the approximate dynamics of order one, after carrying out the appropriate calculations, (3.10) corresponding to the configuration $S_0S_2S_1$ is reduced to the study of the positive

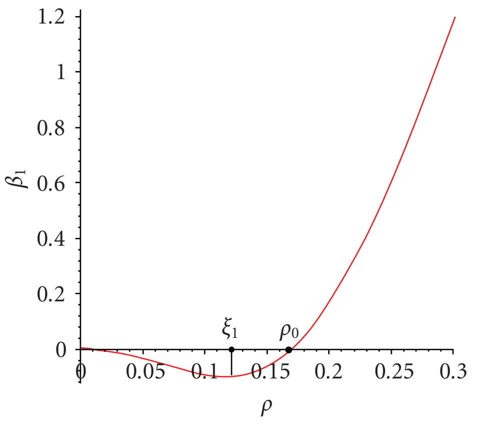

Figure 4.1. Function $R_1(\rho)$.

real roots of the polynomial

$$\begin{aligned} p_1(\rho) = {} & m_0a^2(m_1+m_2)\rho^9 + m_0a^2(5m_1+4m_2)\rho^8 + m_0a^2(10m_1+6m_2)\rho^7 \\ & + 3m_0a^2(3m_1+m_2-m_0)\rho^6 + 3m_0a^2(m_1-m_2-3m_0)\rho^5 \\ & - (6m_0m_2a^2+10m_0^2a^2+\beta_1(m_1+m_2+5m_0))\rho^4 \\ & - (4m_0m_2a^2+5m_0^2a^2+\beta_1(10m_0+4m_2))\rho^3 \\ & - (m_0m_2a^2+m_0^2a^2+\beta_1(6m_2+10m_0))\rho^2 \\ & - \beta_1(5m_0+4m_2)\rho - \beta_1(m_0+m_2), \end{aligned} \tag{4.11}$$

where $a = |\boldsymbol{\lambda}_e|$ and $\beta_1 = 3(C-A)/2$, C and A being the principal moments of inertia of the gyrostat.

To study the positive real roots of this equation, after a detailed analysis of the same one, it can be expressed in the following way:

$$\beta_1 = R_1(\rho) = \frac{m_0a^2\rho^2(\rho+1)^2p_0(\rho)}{q_0(\rho)}, \tag{4.12}$$

Where $\beta_1 = 3(C-A)/2$, p_0 is the polynomial of grade five that determines the relative equilibria in the approximate dynamics of order 0, that is given by formula (4.2), and the polynomial q_0 comes determined by the following expression:

$$\begin{aligned} q_0(\rho) = {} & (m_1+m_2+5m_0)\rho^4 + (4m_2+10m_0)\rho^3 \\ & + (6m_2+10m_0)\rho^2 + (4m_2+5m_0)\rho + (m_0+m_2). \end{aligned} \tag{4.13}$$

The rational function $R_1(\rho)$, for any value of m_0, m_1, m_2, always presents a minimum ξ_1 located among 0 and ρ_0, with this last value being the only one positive zero of the polynomial $p_0(\rho)$.

By virtue of these statements, the following result is obtained (see Figure 4.1).

Proposition 4.3. *In the approximate dynamics of order one, if the gyrostat* S_0 *is prolate* ($\beta_1 < 0$)*, the following hold:*

(1) $\beta_1 < R_1(\xi_1)$*, then relative equilibria of Euler type do not exist.*

(2) $\beta_1 = R_1(\xi_1)$*, then there exists an only relative equilibrium of Euler type.*

(3) $R_1(\xi_1) < \beta_1 < 0$*, then two 1-parametric families of relative equilibria of Euler type exist.*

If S_0 *is oblate* ($\beta_1 > 0$)*, then there exists a unique* 1*-parametric family of relative equilibria of Euler type.*

Similarly for the configuration $S_2S_0S_1$ we obtain the following result (see Figures A.1 and A.2).

Proposition 4.4. *In the approximate dynamics of order one, if* $m_1 \neq m_2$ *and the gyrostat* S_0 *is oblate, then there exists an unique* 1*-parametric family of relative equilibria of Euler type; on the other hand, if the gyrostat* S_0 *is prolate and we have:*

(1) $\beta_1 < R_1(\xi_1)$*, then there exists an unique* 1*-parametric family of relative equilibria of Euler type.*

(2) $\beta_1 = R_1(\xi_1)$*, then two relative equilibria of Euler type exist.*

(3) $R_1(\xi_1) < \beta_1 < 0$*, then three* 1*-parametric families of relative equilibria of Euler type exist. If* $m_1 = m_2$ *and* S_0 *is oblate, then relative equilibria of Euler type do not exist; but if* S_0 *is prolate we have:*

(4) $R_1(-1/2) < \beta_1 < 0$*, then two 1-parametric families of relative equilibria of Euler type exist.*

(5) $\beta_1 = R_1(-1/2)$*, there exists an only equilibrium of Euler type.*

(6) $\beta_1 < R_1(-1/2)$*, then relative equilibria of Euler type do not exist.*

The results for the configuration $S_2S_1S_0$ are similar to that of the configuration $S_0S_2S_1$.

4.1.1. Number of Eulerian equilibria in an approximate dynamics of order k. In the approximate dynamics of order k, the polynomial $p_k(\rho)$, that determines the Eulerian equilibria has degree $5+4k$. Similar results to the previous ones show

$$p_k(\rho) = m_0a^2\rho^2(\rho+1)^2p_{k-1}(\rho) + \beta_k q_{k-1}(\rho) \tag{4.14}$$

with $q_{k-1}(\rho)$ being a positive polynomial. In general, for usual celestial bodies, $\beta_k \approx 0$ for $k > 1$. Using a recurrent reasoning and applying the implicit function theorem, the number of Eulerian equilibria in the approximate dynamics of order k is the same as that of the approximate dynamics of order one. If β_k is not close to zero, for certain k , a particular analysis of the equation $p_k(\rho)$ should be made.

4.2. Stability of Eulerian relative equilibria. The tangent flow of (2.7) in the equilibrium $\mathbf{z}_e$ comes given by

$$\frac{d\delta\mathbf{z}}{dt} = \mathfrak{U}(\mathbf{z}_e)\delta\mathbf{z} \tag{4.15}$$

with $\delta\mathbf{z} = \mathbf{z} - \mathbf{z}_e$ and $\mathfrak{U}(\mathbf{z}_e)$ is the Jacobian matrix of (2.7) in $\mathbf{z}_e$.

The characteristic polynomial $\mathfrak{U}(\mathbf{z}_e)$ has the following expression:

$$p = \lambda(\lambda^2 + \Phi^2)(\lambda^4 + m\lambda^2 + n)(\lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s) \tag{4.16}$$

with $\Phi = ((C - A)\omega_e + l)/A$, where the coefficients that intervene in the previous polynomial are functions of the parameters of the problem and ρ being ρ the root of (3.10).

4.2.1. Order-zero approximate dynamics. The characteristic polynomial (4.16) of $\mathfrak{U}(\mathbf{z}_e)$ simplifies to

$$p = \lambda^3(\lambda^2 + \Phi^2)(\lambda^2 + \omega_e^2)^2(\lambda^2 + p)(\lambda^4 + q\lambda^2 + r) \tag{4.17}$$

with coefficients expressed in Appendix B.

If $p \geq 0$, $q \geq 0$, $r \geq 0$, $q^2 - 4r \geq 0$, then $\mathbf{z}_e$ is spectrally stable. These conditions are not verified since $r < 0$.

Proposition 4.5. *If $\mathbf{z}_e$ is the only relative equilibrium in the configuration $S_0S_2S_1$ of the zero-order approximate dynamics, then this is unstable.*

4.2.2. Order-one approximate dynamics. We will analyze the case in which the gyrostat is close to sphere. In this case $C - A \approx 0$, then applying the implicit function theorem, $\mathbf{z}_e$ is unstable.

If $C - A$ is not close to zero, the coefficients of the polynomial (4.16) have very complicated expressions. Numeric calculations prove that there exist, for certain values of the parameter $C - A$, linear stable Eulerian relative equilibria (see Vera [5] for details).

These results are equally valid for the configurations $S_2S_0S_1$ and $S_2S_1S_0$.

5. Conclusions and future works

The approximate Poisson dynamics of a gyrostat (or rigid body) in Newtonian interaction with two spherical or punctual rigid bodies is considered. We give global conditions on the existence of Eulerian equilibria and in analogy with classic results on the topic, we study the existence of equilibria that we denominate of Euler type in the case in which S_1, S_2 are spherical or punctual bodies and S_0 is a gyrostat. Necessary and sufficient conditions for their existence in a approximate dynamics of order k are obtained and we give explicit expressions of these equilibria, useful for the later study of the stability of the same ones. A complete study of the number of Eulerian equilibria is made in approximate dynamics of orders zero and one. The number of Eulerian equilibria in an approximate dynamics of order k for $k > 1$ is independent of the order of truncation of the potential if the gyrostat S_0 is close to the sphere. The instability of Eulerian equilibria is proven for any approximate dynamics if the gyrostat is close to the sphere

The methods employed in this work are susceptible of being used in similar problems. Numerous problems are open, and among them it is necessary to consider the study of the possible existence of the "inclined" relative equilibria, in which $\mathbf{\Omega}_e$ form an angle $\alpha \neq 0$ and $\pi/2$ with the vector $\boldsymbol{\lambda}^e$.

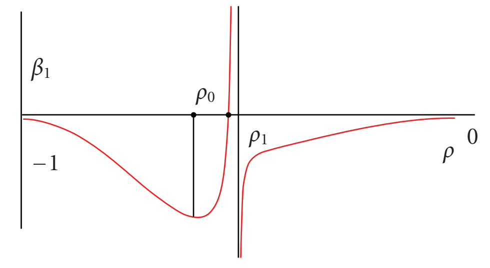

Figure A.1. Function $R_1(\rho)$ for $m_1 \neq m_2$.

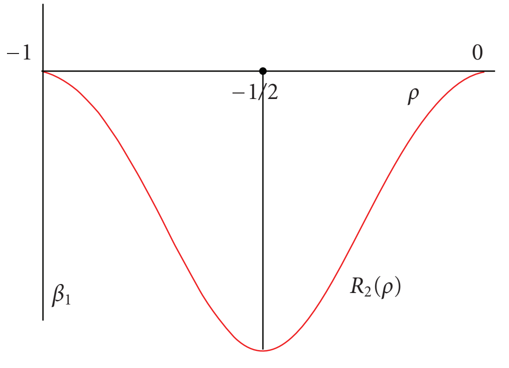

Figure A.2. Function $R_1(\rho)$ for $m_1 = m_2$.

Appendices

A. The function $R_1(\rho)$ in approximate dynamics of order one for the configuration $S_2S_0S_1$

B. Coefficients of the characteristic polynomial in Eulerian relative equilibria

The coefficients of the characteristic polynomial (4.17) are

$$\omega_e^2 = \frac{G((m_2+m_1)\rho^4 + (2m_1+2m_2)\rho^3 + (m_2+m_1)\rho^2 - 2m_0\rho - m_0)}{\lambda_e^3(1+\rho)^2\rho^2},$$

$$p = \frac{G((m_2+4m_0+m_1)\rho^3 + (3m_2+6m_0)\rho^2 + (4m_0+3m_2)\rho + m_0 + m_2)}{(1+\rho)^3\rho^3\lambda_e^3},$$

$$\begin{aligned} q = G(&(-2m_1\rho^4 m_2 + (-2m_0m_1 + m_1^2 + m_2^2 - 2m_1m_2 - 2m_0m_2)\rho^3 \\ &+ (3m_2^2 + m_1m_2 - 6m_0m_1)\rho^2 + (-m_1m_2 + 3m_2^2 + 2m_0m_2 - 4m_0m_1)\rho \\ &+ m_2^2 - m_0m_1 + m_0m_2 - m_1m_2))/((1+\rho)^3\rho^3\lambda_e^3), \end{aligned} \tag{B.1}$$

$$r = \frac{G^2(a_1\rho^4 + a_2\rho^4 + a_3\rho^2 + a_4\rho + a_5)}{((1+\rho)^8\rho^8\lambda_e^9)}.$$

B.1. Coefficients a_i $(i = 1,\ldots,5)$

$$
\begin{aligned}
a_1 = &-42m_2^7m_1 - 48m_2^7m_0 - 147m_2^6m_1^2 - 336m_2^6m_1m_0 - 129m_2^6m_0^2\\
&- 207m_2^5m_1^3 - 782m_2^5m_1^2m_0 - 673m_2^5m_1m_0^2 - 81m_2^5m_0^3 - 150m_2^4m_1^4\\
&- 869m_2^4m_1^3m_0 - 1325m_2^4m_1^2m_0^2 - 378m_2^4m_1m_0^4 - 64m_2^3m_1^5\\
&- 513m_2^3m_1^4m_0 - 1270m_2^3m_1^3m_0^2 - 702m_2^3m_1^2m_0^3 - 14m_2^2m_1^6\\
&- 165m_2^2m_1^5m_0 - 610m_2^2m_1^4m_0^2 - 648m_2^2m_1^3m_0^3 - 24m_2m_1^6m_0\\
&- 119m_2m_1^5m_0^2 - 297m_2m_1^4m_0^3 + 2m_1^6m_0^2 - 54m_1^5m_0^3.\\
a_2 = &-60m_2^7m_1 - 54m_2^7m_0 - 243m_2^6m_1^2 - 474m_2^6m_1m_0 - 173m_2^6m_0^2\\
&- 399m_2^5m_1^3 - 1345m_2^5m_1^2m_0 - 999m_2^5m_1m_0^2 - 135m_2^5m_0^3 - 329m_2^4m_1^4\\
&- 1846m_2^4m_1^3m_0 - 2223m_2^4m_1^2m_0^2 - 648m_2^4m_1m_0^3 - 138m_2^3m_1^5\\
&- 1364m_2^3m_1^4m_0 - 2506m_2^3m_1^3m_0^2 - 1242m_2^3m_1^2m_0^3 - 24m_2^2m_1^6\\
&- 536m_2^2m_1^5m_0 - 1530m_2^2m_1^4m_0^2 - 1188m_2^2m_1^3m_0^3 - 90m_2m_1^6m_0\\
&- 477m_2m_1^5m_0^2 - 567m_2m_1^4m_0^3 - 56m_1^6m_0^2 - 108m_1^5m_0^3.\\
a_3 = &-42m_2^7m_1 - 36m_2^7m_0 - 183m_2^6m_1^2 - 342m_2^6m_1m_0 - 93m_2^6m_0^2\\
&- 349m_2^5m_1^3 - 1097m_2^5m_1^2m_0 - 630m_2^5m_1m_0^2 - 81m_2^5m_0^3 - 358m_2^4m_1^4\\
&- 1776m_2^4m_1^3m_0 - 166m_2^4m_1^2m_0^2 - 405m_2^4m_1m_0^3 - 189m_2^3m_1^5\\
&- 1614m_2^3m_1^4m_0 - 2256m_2^3m_1^3m_0^2 - 810m_2^3m_1^2m_0^3 - 31m_2^2m_1^6\\
&- 827m_2^2m_1^5m_0 - 1683m_2^2m_1^4m_0^2 - 810m_2^2m_1^3m_0^3 - 6m_2m_1^7\\
&- 228m_2m_1^6m_0 - 666m_2m_1^5m_0^2 - 405m_2m_1^4m_0^3 - 30m_1^7m_0\\
&- 81m_1^5m_0^3 - 111m_1^6m_0^2.\\
a_4 = &-12m_2^7m_1 - 12m_2^7m_0 - 56m_2^6m_1^2 - 114m_2^6m_1m_0 - 24m_2^6m_0^2\\
&- 130m_2^5m_1^3 - 387m_2^5m_1^2m_0 - 162m_2^5m_1m_0^2 - 179m_2^4m_1^4\\
&- 687m_2^4m_1^3m_0 - 432m_2^4m_1^2m_0^2 - 140m_2^3m_1^5 - 693m_2^3m_1^4m_0\\
&- 588m_2^3m_1^3m_0^2 - 52m_2^2m_1^6 - 387m_2^2m_1^5m_0 - 432m_2^2m_1^4m_0^2 - 6m_2m_1^7\\
&- 108m_2m_1^6m_0 - 162m_2m_1^5m_0^2 - 12m_1^7m_0 - 24m_1^6m_0^2.\\
a_5 = &-(m_0+m_2)\big(18m_0m_2^6 + 12m_1m_2^6 + 94m_2^5m_0m_1 + 36m_1^2m_2^5\\
&\quad + 81m_2^4m_0^2m_1 + 168m_2^4m_0m_1^2 + 42m_2^4m_1^3 + 128m_2^3m_0m_1^3\\
&\quad + 27m_2^3m_1^4 + 15m_2^2m_1^5 + 31m_2^2m_0m_1^4 + 126m_2^2m_0^2m_1^3 + 18m_0^2m_2^5\\
&\quad + 54m_2m_0^2m_1^4 + 12m_2m_0m_1^5 + 5m_2m_1^6 + 7m_1^6m_0 + 9m_0^2m_1^5\\
&\quad + 144m_2^3m_0^2m_1^2\big).
\end{aligned}
\qquad (B.2)
$$

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