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Research Article

Anti-*CC*-Groups and Anti-*PC*-Groups

Francesco Russo

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A group G has Černikov classes of conjugate subgroups if the quotient group $G/\mathrm{core}_G(N_G(H))$ is a Černikov group for each subgroup H of G. An anti-CC-group G is a group in which each nonfinitely generated subgroup K has the quotient group $G/\mathrm{core}_G(N_G(K))$ which is a Černikov group. Analogously, a group G has polycyclic-by-finite classes of conjugate subgroups if the quotient group $G/\mathrm{core}_G(N_G(H))$ is a polycyclic -by-finite group for each subgroup H of G. An anti-PC-group G is a group in which each nonfinitely generated subgroup K has the quotient group $G/\mathrm{core}_G(N_G(K))$ which is a polycyclic-by-finite group. Anti-CC-groups and anti-PC-groups are the subject of the present article.

1. Introduction

The groups in which each subgroup has only finitely many conjugates have been characterized by B. H. Neumann [1, Section 4, page 127] more than fifty years ago. A group G which has the center $Z(G)$ of finite index in G is called *central-by-finite*. B. H. Neumann showed that a group is *central-by-finite* if and only if each subgroup has only finitely many conjugates. A subgroup H of a group G is called *almost normal* in G if H has finitely many conjugates in G, that is, if H has finite index $|G : N_G(H)|$, where $N_G(H)$ is the normalizer of H in G. Therefore, Neumann's theorem [1, Section 4, page 127] shows that a central-by-finite group is characterized to have each subgroup, which is almost normal.

Neumann's theorem can be formulated in terms of classes of groups as follows. For a subgroup H of a group G, we write

$$N_G(Cl_G(H)) = \mathrm{core}_G(N_G(H)) = \bigcap_{x\in G} N_G(H)^x, \tag{1.1}$$

where $Cl_G(H)$ denotes the set of conjugates of H in G. Clearly, $\mathrm{core}_G(N_G(H))$ is a normal subgroup of G and

$$\bigcap_{x\in G} N_G(H)^x = \bigcap_{x\in G} N_G(H^x). \tag{1.2}$$

The index $|G : N_G(H)| = |Cl_G(H)|$ is finite if and only if the quotient group $G/\mathrm{core}_G(N_G(H))$ is finite. We will say that G has *finite classes of conjugate subgroups* if $G/\mathrm{core}_G(N_G(H))$ is a finite group for each subgroup H of G. Thus Neumann's theorem asserts that a group G has $G/\mathrm{core}_G(N_G(H))$, which is a finite group for each subgroup H of G if and only if G is central-by-finite [2, Introduction]. It is clear that H is almost normal in G if and only if $G/\mathrm{core}_G(N_G(H))$ is a finite group.

A first extension of the concept of group with finite classes of conjugate subgroups can be given as follows. A group G has *Černikov finite classes of conjugate subgroups* if $G/\mathrm{core}_G(N_G(H))$ is a Černikov group for each subgroup H of G (see [1, 3] for details about Černikov groups). This formulation has been recently introduced in [2], obtaining a satisfactory description as testified in [2, Main Theorem]. The initial work of Polovickiĭ [4] gave a description of a periodic group G with Černikov classes of conjugate subgroups by showing that G is *central-by-Černikov*, that is, G has $G/Z(G)$ which is a Černikov group. Since the class of Černikov groups extends the class of finite groups, Neumann's theorem can be found as a special situation in [2, Proposition 2.4].

A second extension of the concept of group with finite classes of conjugate subgroups can be given as follows. A group G has *polycyclic-by-finite classes of conjugate subgroups* if $G/\mathrm{core}_G(N_G(H))$ is a polycyclic-by-finite group for each subgroup H of G (see [1, 5] for details about polycyclic-by-finite groups). This formulation has been recently introduced in [6], obtaining a satisfactory description as testified in [6, Main Theorem]. Initially, [7, Theorem 5.5] describes a group G which is *central-by-*(*polycyclic-by-finite*), that is, G has $G/Z(G)$ which is a polycyclic-by-finite group. References [7, Theorem 5.5] and [6, Main Theorem] allow us to see Neumann's theorem as a special situation.

Let χ be a property of subgroups in groups, and let $\underline{\underline{L}}$ be a family of subgroups of a given group G. It is a long standing line of research in Group Theory to study those groups in which all subgroups belonging to the family $\underline{\underline{L}}$ of subgroups have the property χ. The beginnings of this line reach back to works of Dedekind [8] and Miller and Moreno [9].

Examples of families of subgroups considered so far are the family $\underline{\underline{L}}_1$ of all proper subgroups, the family $\underline{\underline{L}}_2$ of all finite subgroups, $\underline{\underline{L}}_3$ of all infinite subgroups, $\underline{\underline{L}}_4$ of all abelian subgroups, $\underline{\underline{L}}_5$ of all nonabelian subgroups, and $\underline{\underline{L}}_6$ of all finitely generated subgroups; while subgroups properties considered are for instance to be normal, subnormal, and subnormal of bounded defect, complemented, supplemented, and almost normal, or to satisfy min, max, min-∞, and max-∞ (see [1, 10] for details). The references [11–21] show part of the literature which has been devoted to this topic during the last years.

We can often obtain a fairly good description of the group G if the family $\underline{\underline{L}}$ is not too distant from $\underline{\underline{L}}_1$. If, on the other hand, $\underline{\underline{L}}$ is not a small subfamily of $\underline{\underline{L}}_1$, the information "all subgroups of G belonging to $\underline{\underline{L}}$ have property χ" is rather restricted. We take the family $\underline{\underline{L}}_6$ as an example: the descriptions of groups, all of which finitely generated subgroups are subnormal (Baer-groups, see [1, Lemmas 2.34, 2.35]), almost normal (*FC*-groups, see [1]), or satisfying max (locally noetherian groups, see [1]), are rather unsatisfactory. An exception is the class of all groups, all of which finitely generated subgroups are normal. These are the Dedekind groups and they have been classified. Therefore it may be interesting to study groups in which a property χ is imposed on a *large* family of subgroups, for instance, on the family $\underline{\underline{L}}_7$ of all nonfinitely generated subgroups. Clearly, $\underline{\underline{L}}_7 = \underline{\underline{L}}_1/\underline{\underline{L}}_6$. For the property χ, we choose to have Černikov classes of conjugate subgroups.

So this article is devoted to groups G, satisfying either of the following properties:

(i) if the subgroup H of G is nonfinitely generated,
then $G/\mathrm{core}_G(N_G(H))$ is a Černikov group;

(ii) if the subgroup H of G is nonfinitely generated,
then $G/\mathrm{core}_G(N_G(H))$ is a polycyclic-by-finite group.

A group G which satisfies (i) is called anti-*CC*-group in analogy with the terminology which has been adopted in [13], where anti-*FC*-groups have been analyzed. An anti-*FC*-group G is a group in which each nonfinitely generated subgroup H is almost normal in G. A group G which satisfies (ii) is called anti-*PC*-group. From the previous considerations, it is clear that a group G is an anti-*FC*-group if and only if each nonfinitely generated subgroup H of G has $G/\mathrm{core}_G(N_G(H))$ which is a finite group. Therefore, the notions of the anti-*CC*-group and anti-*PC*-group extend the notion of the anti-*FC*-group so that most of the results in [13] can be found as special situations.

Section 2 is devoted to recall some preliminaries which help us to prove the main results. Our main results are contained in Sections 3 and 4. More precisely, Section 3 describes locally finite anti-*CC*-groups and anti-*PC*-groups. Section 4 describes locally nilpotent anti-*CC*-groups and anti-*PC*-groups.

Our notation is standard and can be found in [1]. The background has been referred to [1, Section 4.3] for *FC*-groups, to [4, 22, 23] for *CC*-groups, and to [7] for *PC*-groups. General information on locally finite and locally nilpotent groups can be found in [10, 14, 24].

2. Preliminary results

Let G be a group. An element x of G is called *FC-element* of G if $G/C_G(\langle x\rangle^G)$ is a finite group. The set $F(G)$ of all *FC*-elements of G is a characteristic subgroup of G, which is called *FC-center* of G [1, Section 4.3]. In a similar way, an element x of G is called *CC*-element of G if $G/C_G(\langle x\rangle^G)$ is a Černikov group. The set $C(G)$ of all *CC*-elements of G is a characteristic subgroup of G, which is called *CC-center* of G (see [25, Section 3]). In a similar way, an element x of G is called *PC*-element of G if $G/C_G(\langle x\rangle^G)$ is a polycyclic-by-finite group. The set $P(G)$ of all *PC*-elements of G is a characteristic subgroup of G, which is called *PC-center* of G (see [7]). Obviously, G is an *FC*-group if and only if $G = F(G)$. Similarly, G is a *CC*-group if and only if $G = C(G)$. Similarly, G is a *PC*-group if and only if $G = P(G)$.

The next result overlaps [25, Lemma 3.2] and it is shown only to the convenience of the reader.

Lemma 2.1. *Let G be a group and let n be a positive integer.*

(i) *G is an FC-group if and only if*

$$F(G) = \langle H = \langle h_1,\dots,h_n\rangle : G/\mathrm{core}_G(N_G(H)) \text{ is a finite group}\rangle. \tag{2.1}$$

(ii) *G is a CC-group if and only if*

$$C(G) = \langle H = \langle h_1,\dots,h_n\rangle : G/\mathrm{core}_G(N_G(H)) \text{ is a Černikov group}\rangle. \tag{2.2}$$

(iii) *G is a PC-group if and only if*

$$P(G) = \langle H = \langle h_1,\dots,h_n\rangle : G/\mathrm{core}_G(N_G(H)) \text{ is a polycyclic-by-finite-group}\rangle. \tag{2.3}$$

Proof. Assume that G is an FC-group, x is an FC-element of G and $K = \langle H = \langle h_1,\dots,h_n\rangle : G/\mathrm{core}_G(N_G(H))$ is finite$\rangle$. If $a \in C_G(\langle x\rangle^G)$, then $[b^y,a] = 1$ for each $b \in \langle x\rangle$ and $y \in G$, in particular,

$$a \in \bigcap_{g\in G} N_G(\langle x\rangle^g) = \mathrm{core}_G(N_G(\langle x\rangle)). \tag{2.4}$$

Therefore, $C_G(\langle x\rangle^G)$ is contained in $\mathrm{core}_G(N_G(\langle x\rangle))$ so that $G/\mathrm{core}_G(N_G(\langle x\rangle))$ is a finite group and x belongs to K. Then $F(G) \le K$, but $F(G) = G$ so that $G = K$. Conversely, assume that $F(G) = K$. Then each finitely generated subgroup of G is almost normal in G and this implies that G is an FC-group. Then (i) has been proved.

A similar argument shows (ii) and (iii). □

Reference [15] describes those groups in which each nonfinitely generated subgroup is subnormal. Such groups are called db-groups and they represent the dual class of the Baer groups (see [26], [1, Section 2.3]). Unfortunately, we cannot say that an anti-CC-group (resp., an anti-PC-group) is a db-group so that many results of [15] cannot be directly applied. However, it is possible to compare [13, Theorems 2.2, 2.11, 2.13, 3.6, 3.11, 3.16, 3.17, 4.6, 4.8, 4.11, 4.12, 4.15, 4.16] with [15, Theorems 1, 2, 3, 4, 5], noting that analogous situations happen for anti-CC-groups (resp., for anti-PC-groups). In particular, some methods which have been used in the present paper mime the methods which have been used in [13, 15].

We end this section, recalling two results which are fundamental in our investigations. The first result describes the structure of a group with Černikov classes of conjugate subgroups (see [2, Main Theorem], [4]).

Theorem 2.2 [2]. *Let G be a group with Černikov classes of conjugate subgroups. Then the following assertions hold:*

(i) *G has an abelian normal subgroup A such that G/A is a Černikov group;*
(ii) *if T is the torsion subgroup of A, then $G/C_G(T)$ is a finite group;*
(iii) *$[G,G]$ is a Černikov group;*
(iv) *if G is periodic, then G is a central-by-Černikov group.*

A group G which has an abelian normal subgroup A such that G/A is a Černikov group and is said to be *abelian-by-Černikov*. This situation happens in statement (i) of the preceding theorem.

The second result describes the structure of a group with polycyclic-by-finite classes of conjugate subgroups [6, Main Theorem].

THEOREM 2.3 [6]. *A group G has polycyclic-by-finite classes of a conjugate subgroups if and only if it is central-by-(polycyclic-by-finite).*

3. Locally finite case

The first two statements follow from the definitions and from Lemma 2.1, so the proofs have been omitted.

LEMMA 3.1. (i) *Subgroups and quotient groups of anti-CC-groups are anti-CC-groups.*
(ii) *Subgroups and quotient groups of anti-PC-groups are anti-PC-groups.*

LEMMA 3.2. (i) *If G is an anti-CC-group and $C(G) = G$, then G has Černikov classes of conjugate subgroups.*
(ii) *If G is an anti-PC-group and $P(G) = G$, then G has polycyclic-by-finite classes of conjugate subgroups.*

LEMMA 3.3. *Assume that x is an element of the anti-CC-group G. If $A = Dr_{i\in I}A_i$ is a subgroup of G consisting of $\langle x\rangle$-invariant nontrivial direct factors A_i, $i \in I$, with infinite index set I, then x belongs to $C(G)$.*

Proof. Consider $\langle x_1\rangle = \langle x\rangle \cap A$. Then $\operatorname{supp} x_1 = I_1$ is a finite subset of I, and $\langle x\rangle \cap Dr_{i\in M}A_i = 1$, where $M = I \setminus I_1$ is infinite. We choose two infinite subsets M_1 and M_2 of M such that $M_1 \cup M_2 = M$ and $M_1 \cap M_2 = \varnothing$. Obviously, $H_1 = \langle x\rangle Dr_{i\in M_1}A_i$ and $H_2 = \langle x\rangle Dr_{i\in M_2}A_i$ cannot be finitely generated, therefore, $G/\mathrm{core}_G(N_G(\langle H_1\rangle))$ and $G/\mathrm{core}_G(N_G(\langle H_2\rangle))$ are Černikov groups. Put $K_1 = \mathrm{core}_G(N_G(\langle H_1\rangle))$ and $K_2 = \mathrm{core}_G(N_G(\langle H_2\rangle))$. We note that

$$\begin{gathered} K_1 \cap K_2 \le \mathrm{core}_G(N_G(\langle H_1\rangle \cap \langle H_2\rangle)), \\ \mathrm{core}_G(N_G(\langle H_1\rangle \cap \langle H_2\rangle)) = \mathrm{core}_G(N_G(\langle x\rangle)). \end{gathered} \tag{3.1}$$

But

$$G/\mathrm{core}_G(N_G(\langle H_1\rangle \cap \langle H_2\rangle)) = G/\mathrm{core}_G(N_G(\langle x\rangle)) \tag{3.2}$$

is isomorphic to

$$\frac{G/K_1 \cap K_2}{\mathrm{core}_G(N_G(\langle H_1\rangle \cap \langle H_2\rangle))/K_1 \cap K_2}, \tag{3.3}$$

thanks to the well-known results of isomorphism between groups. $G/K_1 \cap K_2$ is a Černikov group because it is the subdirect product of the Černikov groups G/K_1 and G/K_2. Then $G/\mathrm{core}_G(N_G(\langle x\rangle))$ is a Černikov group, and so x belongs to $C(G)$. □

Lemma 3.4. *Assume that x is an element of the anti-PC-group G. If $A = Dr_{i\in I}A_i$ is a subgroup of G consisting of $\langle x\rangle$-invariant nontrivial direct factors A_i, $i \in I$, with infinite index set I, then x belongs to $P(G)$.*

Proof. We follow the argument of the previous proof, using polycyclic-by-finite groups instead of Černikov groups. □

Corollary 3.5. *Let G be an anti-CC-group and $A = Dr_{i\in I}A_i$ a subgroup of G consisting of infinitely many nontrivial direct factors. Then A is contained in $C(G)$.*

Corollary 3.6. *Let G be an anti-PC-group and $A = Dr_{i\in I}A_i$ a subgroup of G consisting of infinitely many nontrivial direct factors. Then A is contained in $P(G)$.*

Lemma 3.7. *Assume that g is an element of the anti-CC-group G and $A = Dr_{i\in I}A_i$ is a subgroup of G, with I as in Lemma 3.3. If $g \in N_G(A)$ and $g^n \in C_G(A)$ for some positive integer n, then g belongs to $C(G)$.*

Proof. We define two subsets of I, namely, $M_1 = \{i : Z(A_i) \neq 1\}$ and $M_2 = \{i : \gamma_n(A_i) \neq 1$ for every $n \in \mathbb{N}\}$. Obviously, $M_1 \cup M_2 = I$, so at least one of the two subsets is infinite.

Case 1 (M_2 is infinite). If $D_1,\dots,D_n$ are normal subgroups of a group F, then $[\dots[[D_1,D_2],D_3],\dots,D_n]$ is a normal subgroup of F, which is contained in $\bigcap_{i=1}^{n} D_i$, furthermore, $[D_i, D_jD_k] = [D_i,D_j][D_i,D_k]$.

Now $A = Dr_{i\in I}x^{-r}Ax^r$ for every positive integer r, where x is an element of G and we obtain that

$$T = Dr_{(i_1,\dots,i_n)\in I^n}A_{i_1} \cap x^{-1}A_{i_2}x \cap x^{-2}A_{i_3}x^2 \cap \cdots \cap x^{-n+1}A_{i_n}x^{n-1} \tag{3.4}$$

is a direct product of infinitely many nontrivial factors since $\gamma_n(A_i) \leq T$. By construction, x normalizes T and permutes the given direct factors of T. By combining the conjugates under x to one new factor, we have reduced the situation to that of Lemma 3.3, and find that x belongs to $C(G)$.

Case 2 (M_1 is infinite). Then the abelian group $Z(A)$ is normalized by x and centralized by x^n. Clearly, $Z(A)$ is of infinite rank. Denote by W the torsion subgroup of $Z(A)$. Again W is normalized by x. If the set of primes π occurring as orders of elements of W is infinite, we may define two subsets π_1, π_2 of π, both infinite such that $\pi_1 \cup \pi_2 = \pi$ and $\pi_1 \cap \pi_2 = \varnothing$. If W_1 and W_2 are the corresponding π_j-Sylow subgroups of W ($j = 1,2$), then $\langle x\rangle W_1$, $\langle x\rangle W_2$, and $\langle x\rangle W_1 \cap \langle x\rangle W_2 = \langle x\rangle$ belong to $C(G)$.

If M_1 is infinite and the torsion subgroup W is of a infinite rank but π is finite, there is a characteristic elementary abelian p-subgroup V of W which is of infinite rank. Again, V is the direct product of two infinite $\langle x\rangle$-invariant subgroups V_1 and V_2 such that $V_1 \cap \langle x\rangle V_2 = 1$. Again, $\langle x\rangle V_1$, $\langle x\rangle V_2$, and $\langle x\rangle V_1 \cap \langle x\rangle V_2 = \langle x\rangle$ belong to $C(G)$. If the torsion subgroup W is of finite rank, we can construct a torsion-free $\langle x\rangle$-invariant subgroup L of infinite rank in $Z(A)$. Again, $\langle x\rangle$-invariant subgroups of infinite rank L_1, L_2 can be chosen with $L_1 \cap \langle x\rangle L_2 = 1$, and $L_2L_1 = L$.

Now $\langle x\rangle L_1$, $\langle x\rangle L_2$, and $\langle x\rangle L_1 \cap \langle x\rangle L_2 = \langle x\rangle$ belong to $C(G)$. This completes Case 2, and the result follows. □

Lemma 3.8. *Assume that g is an element of the anti-PC-group G and* $A = Dr_{i\in I}A_i$ *is a subgroup of G, with I as in Lemma 3.4. If* $g \in N_G(A)$ *and* $g^n \in C_G(A)$ *for some positive integer n, then g belongs to* $P(G)$.

Proof. We follow the argument of the previous proof, using polycyclic-by-finite groups instead of Černikov groups and Lemma 3.4 instead of Lemma 3.3. □

Corollary 3.9. *If the anti-CC-group G has an abelian torsion subgroup that does not satisfy the minimal condition on its subgroups, then all elements of finite order belong to* $C(G)$.

Proof. Denote the torsion subgroup of $C(G)$ by T. We deduce from Corollary 3.5 that T does not satisfy min-*ab*. Choose an element x of finite order in G. A result of Zaĭtsev [21] implies that T possesses an abelian $\langle x\rangle$-invariant subgroup A that does not satisfy min-*ab*. From Lemma 3.7, x belongs to $C(G)$. □

Corollary 3.10. *If the anti-PC-group G has an abelian torsion subgroup that does not satisfy the minimal condition on its subgroups, then all elements of finite order belong to* $P(G)$.

Proof. We follow the argument of the previous proof, using polycyclic-by-finite groups instead of Černikov groups, Corollary 3.6 instead of Corollary 3.5, and Lemma 3.8 instead of Lemma 3.7. □

Theorem 3.11. *If G is a locally finite anti-CC-group, then either G has Černikov classes of conjugate subgroups or G is a Černikov group.*

Proof. If G does not satisfy min-*ab*, then $G = C(G)$ by Corollary 3.9. From Lemma 3.2, G has Černikov classes of conjugate subgroups. If G satisfies min-*ab*, then a famous result of Shunkov [1, page 98] implies that G is a Černikov group. □

Theorem 3.12. *If G is a locally finite anti-PC-group, then either G has finite classes of conjugate subgroups or G is a Černikov group.*

Proof. If G does not satisfy min-*ab*, then $G = P(G)$ by Corollary 3.10. From Lemma 3.2, G has polycyclic-by-finite classes of conjugate subgroups. Then Theorem 2.3 implies that $G/Z(G)$ is a polycyclic-by-finite group. Since G is periodic, $G/Z(G)$ is a finite group. If G satisfies min-*ab*, then a famous result of Shunkov [1, page 98] implies that G is a Černikov group. □

Corollary 3.13. *If G is a locally finite anti-CC-group, then either G is central-by-Černikov or G is a Černikov group.*

Proof. From Theorem 3.11, either G has Černikov classes of conjugate subgroups or G is a Černikov group. In the first case, we may apply (iv) of Theorem 2.2 so that the result follows. □

Corollary 3.14. *If G is a locally finite anti-PC-group, then either G is central-by-finite or G is a Černikov group.*

Proof. From Theorem 3.12, either G has finite classes of conjugate subgroups or G is a Černikov group. In the first case, we recall that this is a different formulation of the Neumann's theorem, as mentioned in the introduction of the present paper. Then the result follows. □

It seems opportune to note that Theorems 3.11 and 3.12 include [13, Theorem 2.2] as a special case, and agree with [15, Theorem 1].

Now the classification of the locally finite anti-CC-group is easy to see.

Theorem 3.15. *The infinite locally finite group G which is not a Černikov group is an anti-CC-group if and only if G is central-by-Černikov.*

Proof. If G is not a Černikov group, then the result follows from Corollary 3.13. □

In a similar way, the classification of the locally finite anti-PC-group is easy to see.

Theorem 3.16. *The infinite locally finite group G which is not a Černikov group is an anti-PC-group if and only if G is central-by-finite.*

Proof. If G is not a Černikov group, then the result follows from Corollary 3.14. □

4. Locally nilpotent case

A group G is called *soluble-by-finite* if it has a normal soluble subgroup S whose index $|G:S|$ is finite. We recall that a group G has *finite abelian section rank* if it has no infinite elementary abelian p sections for every prime p (see [1, volume II, Section 10]). Following [1, 13], a soluble-by-finite group G is an $\mathcal{S}_1$-group if it has finite abelian section rank and the set of prime divisors of orders of elements of G is finite. Literature on $\mathcal{S}_1$-groups can be found, for instance, in [1, volume II]. Finally, we recall the notion of rank of a group, following the well-known terminology of Prüfer (see [1]). If A is an abelian group, the *torsion-free rank* of A is the rank of the factor group $A/T(A)$, where $T(A)$ denotes the set of all elements of finite order in A. The torsion-free rank of A is denoted by $r_0(A)$. The *total rank* of A is the sum $r_0(A)+\sum_p r_p(A)$, where $r_p(A)$ is the *rank* of the p components of A for each prime number p.

Theorem 4.1. *Let G be an anti-CC-group having an ascending series whose factors are either locally nilpotent or locally finite. Then G has Černikov classes of conjugate subgroups or is a soluble-by-finite $\mathcal{S}_1$-group.*

Proof. G possesses an ascending normal series whose factors are either locally nilpotent or locally finite [1, Theorem 2.31]. Let K be the largest radical normal subgroup of G. It follows from Corollary 3.13 that the largest locally finite normal subgroup T/K of G/K is either central-by-Černikov or a Černikov group. On the other hand, the factor group G/K has no nontrivial locally nilpotent normal subgroups, and hence T/K is a Černikov group. If H/T is a locally nilpotent normal subgroup of G/T, then the centralizer $C_{H/K}(T/K)$ is a locally nilpotent normal subgroup of G/K so that $C_{H/K}(T/K)=1$ and H/K is a Černikov group. It follows that $T=G$ so that G has a normal radical subgroup K such that T/K is a Černikov group (in this situation, G is said to be a radical-by-Černikov group). Assume that G has Černikov classes of conjugate subgroups. Then every abelian subgroup of G

has finite total rank by Corollary 3.5. A result of Charin (see [1, Theorem 6.36]) implies that K is a soluble $\mathcal{S}_1$-group. We conclude that G has a normal soluble $\mathcal{S}_1$-subgroup K such that G/K is a Černikov group. Therefore, G is an extension of a soluble $\mathcal{S}_1$-group by an abelian group with min by a finite group. An abelian group with min is clearly an $\mathcal{S}_1$-group and the class of $\mathcal{S}_1$-groups is closed with respect to extensions of two of its members (see [1, 15]). Therefore, G is a soluble-by-finite $\mathcal{S}_1$-group. □

Theorem 4.2. *Let G be an anti-PC-group having an ascending series whose factors are either locally nilpotent or locally finite. Then G has finite classes of conjugate subgroups or is a soluble-by-finite $\mathcal{S}_1$-group.*

Proof. We repeat the argument of the previous proof so that it is shown only for the convenience of the reader.

G possesses an ascending normal series whose factors are either locally nilpotent or locally finite [1, Theorem 2.31]. Let K be the largest radical normal subgroup of G. It follows from Corollary 3.14 that the largest locally finite normal subgroup T/K of G/K is either central-by-finite or a Černikov group. From then, we repeat exactly the corresponding part in the proof of Theorem 4.1, using Corollary 3.6 instead of Corollary 3.5. It follows that G is a soluble-by-finite $\mathcal{S}_1$-group. □

Corollary 4.3. *Let G be an anti-CC-group having an ascending series whose factors are either locally nilpotent or locally finite. Then G is abelian-by-Černikov or a soluble-by-finite $\mathcal{S}_1$-group.*

Proof. This follows from Theorems 4.1 and 2.2. □

Corollary 4.4. *Let G be an anti-PC-group having an ascending series whose factors are either locally nilpotent or locally finite. Then G is central-by-finite or a soluble-by-finite $\mathcal{S}_1$-group.*

Proof. This follows from Theorem 4.2 and the formulation of Neumann's theorem as in the introduction. □

It is well known that a locally nilpotent group G has its torsion subgroup T which is locally finite and the quotient group G/T which is torsion-free (see [1]). Then it is enough to investigate the structure of a torsion-free locally nilpotent anti-CC-group (resp., anti-PC-group) in order to have a satisfactory description of a locally nilpotent anti-CC-group (resp., anti-PC-group).

Proposition 4.5. *Let G be a torsion-free locally nilpotent anti-CC-group. If G is neither finitely generated nor abelian, then it is nilpotent of class 2.*

Proof. Assume from Theorem 4.1 that G has Černikov classes of conjugate subgroups. $[G,G]$ should be a Černikov group from Theorem 2.2 and this cannot be. Then we may assume that G is a soluble-by-finite $\mathcal{S}_1$-group, since G is nonfinitely generated, also its center $Z(G)$ is nonfinitely generated from [27, Lemma 2.6]. Let $X/Z(G)$ be a subgroup of $G/Z(G)$. Then X is nonfinitely generated, and hence $G/\text{core}_G(N_G(X))$ is a Černikov group. But every subgroup of $G/Z(G)$ has such property so that $G/Z(G)$ has Černikov classes of conjugate subgroups. Now $G/Z(G)$ satisfies Theorem 2.2 so that its derived subgroup $[G/Z(G), G/Z(G)]$ is a Černikov group. We note that $T(G/Z(G)) = T(G)Z(G)/Z(G)$ and $T(G) = 1$, then $T(G/Z(G)) = 1$ and $G/Z(G)$ is a torsion-free group. Now $[G/Z(G), G/Z(G)] = 1$ so that $G/Z(G)$ is abelian, and G is nilpotent of class 2. □

Proposition 4.6. *Let G be a torsion-free locally nilpotent anti-PC-group. If G is neither finitely generated nor abelian, then it is nilpotent of class 2.*

Proof. We may repeat the argument of the preceding proof, consider the corresponding statements for anti-PC-groups. □

Theorem 4.7. *Assume that G is a locally nilpotent anti-CC-group with torsion subgroup T. Then*

(i) *T is either central-by-Černikov or a Černikov group;*
(ii) *G/T is torsion-free nilpotent of class 2, whenever it is neither finitely generated nor abelian.*

Proof. (i) follows from Corollary 3.13. (ii) follows from Proposition 4.5. □

Theorem 4.8. *Assume that G is a locally nilpotent anti-PC-group with torsion subgroup T. Then*

(i) *T is either central-by-finite or a Černikov group;*
(ii) *G/T is torsion-free nilpotent of class 2, whenever it is neither finitely generated nor abelian.*

Proof. (i) follows from Corollary 3.14. (ii) follows from Proposition 4.6. □

5. Examples

Example 5.1. Each anti-FC-group is an anti-CC-group as testified by definitions. Examples of anti-FC-groups can be found in [13, page 44, lines 1–13] or [13, Example 3.12]. Of course, each anti-FC-group is an anti-PC-group.

Example 5.2. The Example which has been described in [2, Section 4] is a nonperiodic group with Černikov classes of conjugate subgroups. This example is an anti-CC-group. Each central-by-(polycyclic-by-finite) group is an anti-PC-group thanks to Theorem 2.3.

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Francesco Russo: Department of Mathematics, Faculty of Mathematics, University of Naples,
Via Cinthia, 80126 Naples, Italy
Email address: francesco.russo@dma.unina.it