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Research Article

Some Identities on the Twisted (h, q)-Genocchi Numbers and Polynomials Associated with q-Bernstein Polynomials

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We give some interesting identities on the twisted (h, q)-Genocchi numbers and polynomials associated with q-Bernstein polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integer, $\mathbb{Q}_p$ denotes the ring of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N}\bigcup\{0\}$. Let $C_{p^n} = \{\zeta \mid \zeta^{p^n} = 1\}$ be the cyclic group of order p^n and let

$$\mathrm{T}_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n} = C_{p^\infty}. \tag{1.1}$$

The p-adic absolute value is defined by $|x| = 1/p^r$, where $x = p^r(s/t)$ ($r \in \mathbb{Q}$ and $s, t \in \mathbb{Z}$ with $(s,t) = (p,s) = (p,t) = 1$). In this paper we assume that $q \in \mathbb{C}_\mathrm{p}$ with $|q-1|_p < 1$ as an indeterminate.

The q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q} \tag{1.2}$$

(see [1–15]). Note that $\lim_{q\to 1}[x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.3}$$

(see [2–6, 8–15]). From (1.3), we note that

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} q^{\ell} f(\ell) \tag{1.4}$$

(see [4–6, 8–12]), where $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$. For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, Kim defined the q-Bernstein polynomials of the degree n as follows:

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k}, \tag{1.5}$$

(see [13–15]). For $h \in \mathbb{Z}$ and $\zeta \in T_p$, let us consider the twisted (h,q)-Genocchi polynomials as follows:

$$t\int_{\mathbb{Z}_p} e^{[x+y]_q t} \zeta^y q^{(h-1)y} d\mu_{-q}(y) = \sum_{n=0}^{\infty} G_{n,q,\zeta}^{(h)}(x) \frac{t^n}{n!}. \tag{1.6}$$

Then, $G_{n,q,\zeta}^{(h)}(x)$ is called nth twisted (h,q)-Genocchi polynomials.

In the special case, $x = 0$ and $G_{n,q,\zeta}^{(h)}(0) = G_{n,q,\zeta}^{(h)}$ are called the nth twisted (h,q)-Genocchi numbers.

In this paper, we give the fermionic p-adic integral representation of q-Bernstein polynomial, which are defined by Kim [13], associated with twisted (h,q)-Genocchi numbers and polynomials. And we construct some interesting properties of q-Bernstein polynomials associated with twisted (h,q)-Genocchi numbers and polynomials.

2. On the Twisted (h, q)-Genocchi Numbers and Polynomials

From (1.6), we note that

$$\begin{aligned}\frac{G^{(h)}_{n+1,q,\zeta}(x)}{n+1} &= \int_{\mathbb{Z}_p} [x+y]_q^n \zeta^y q^{(h-1)y}\, d\mu_{-q}(y)\\ &= \int_{\mathbb{Z}_p} \left([x]_q + q^x [y]_q\right)^n \zeta^y q^{(h-1)y}\, d\mu_{-q}(y)\\ &= \sum_{\ell=0}^{n} \binom{n}{\ell} [x]_q^{n-\ell} q^{\ell x} \int_{\mathbb{Z}_p} [y]_q^{\ell} \zeta^y q^{(h-1)y}\, d\mu_{-q}(y)\\ &= \sum_{\ell=0}^{n} \binom{n}{\ell} [x]_q^{n-\ell} q^{\ell x} \frac{G^{(h)}_{\ell+1,q,\zeta}}{\ell+1}.\end{aligned} \tag{2.1}$$

We also have

$$G^{(h)}_{n,q,\zeta}(x) = q^{-x} \sum_{\ell=0}^{n} \binom{n}{\ell} [x]_q^{n-\ell} q^{\ell x} G^{(h)}_{\ell,q,\zeta}. \tag{2.2}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, one has*

$$G^{(h)}_{n,q,\zeta}(x) = q^{-x}\left([x]_q + q^x G^{(h)}_{q,\zeta}\right)^n \tag{2.3}$$

with usual convention about replacing $(G^{(h)}_{q,\zeta})^n$ by $G^{h}_{n,q,\zeta}$.

By (1.6) *and* (2.1) *one gets*

$$\begin{aligned}\frac{G^{(h)}_{n+1,q^{-1},\zeta^{-1}}(1-x)}{n+1} &= \int_{\mathbb{Z}_p} [1-x+y]_{q^{-1}}^n \zeta^{-y} q^{-(h-1)y}\, d\mu_{-q^{-1}}(y)\\ &= \frac{[2]_q}{(1-q^{-1})^n} \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^n q^{h-1} \zeta \frac{q^{\ell x}}{1+q^{h+\ell}\zeta}\\ &= (-1)^n q^{n+h-1} \zeta \left(\frac{[2]_q}{(1-q)^n} \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^{\ell} \frac{q^{\ell x}}{1+q^{h+\ell}\zeta} \right)\\ &= (-1)^n \zeta q^{n+h-1} \frac{G^{(h)}_{n+1,q,\zeta}(x)}{n+1}.\end{aligned} \tag{2.4}$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, one has*

$$G_{n,q^{-1},\zeta^{-1}}^{(h)}(1-x) = (-1)^{n-1}\zeta q^{n+h-2} G_{n,q,\zeta}^{(h)}. \tag{2.5}$$

From (1.5), *one gets the following recurrence formula:*

$$q^h \zeta G_{n,q,\zeta}^{(h)}(1) + G_{n,q,\zeta}^{(h)} = \begin{cases} [2]_q & if\ n=1, \\ 0 & if\ n>1. \end{cases} \tag{2.6}$$

Therefore, we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, one has*

$$G_{0,q,\zeta} = 0, \qquad q^{h-1}\zeta\left(qG_{q,\zeta}^{(h)}+1\right)^n + G_{n,q,\zeta}^{(h)} = \begin{cases} [2]_q & if\ n=1, \\ 0 & if\ n>1 \end{cases} \tag{2.7}$$

with usual convention about replacing $(G_{q,\zeta}^{(h)})^n$ by $G_{n,q,\zeta}^h$.

From Theorem 2.3, we note that

$$\begin{aligned} q^{2h}\zeta^2 G_{n,q,\zeta}^{(h)}(2) - q^h \zeta n[2]_q &= -q^{h-1}\zeta \sum_{\ell=0}^{n} \binom{n}{\ell} q^{\ell} G_{\ell,q,\zeta}^{(h)} \\ &= -q^{h-1}\zeta\left(qG_{q,\zeta}^{(h)}+1\right)^n \\ &= G_{n,q,\zeta}^{(h)} \quad \text{if } n>1. \end{aligned} \tag{2.8}$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, one has*

$$q^{2h}\zeta^2 G_{n,q,\zeta}^{(h)}(2) = G_{n,q,\zeta}^{(h)} + nq^h \zeta [2]_q. \tag{2.9}$$

Remark 2.5. We note that Theorem 2.4 also can be proved by using fermionic integral equation (1.4) in case of $n = 2$.

By (2.4) and Theorem 2.2, we get

$$\begin{aligned}\frac{G^{(h)}_{n+1,q^{-1},\zeta^{-1}}(2)}{n+1} &= (-1)^n q^{n+h-1}\zeta\frac{G^{(h)}_{n+1,q,\zeta}(-1)}{n+1}\\ &= (-1)^n q^{n+h-1}\zeta\int_{\mathbb{Z}_p}[x-1]^n_q\zeta^x q^{(h-1)x}\,d\mu_{-q}(x)\\ &= q^{h-1}\zeta\int_{\mathbb{Z}_p}[1-x]^n_{q^{-1}}\zeta^x q^{(h-1)x}\,d\mu_{-q}(x).\end{aligned} \tag{2.10}$$

Therefore, we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, one has*

$$(n+1)q^{h-1}\zeta\int_{\mathbb{Z}_p}[1-x]^n_{q^{-1}}\zeta^x q^{(h-1)x}\,d\mu_{-q}(x) = G^{(h)}_{n+1,q^{-1},\zeta^{-1}}(2). \tag{2.11}$$

Let $n \in \mathbb{N}$. By Theorems 2.4 and 2.6, we get

$$(n+1)q^{h-1}\zeta\int_{\mathbb{Z}_p}[1-x]^n_{q^{-1}}\zeta^x q^{(h-1)x}\,d\mu_{-q}(x) = q^{2h}\zeta^2 G^{(h)}_{n+1,q^{-1},\zeta^{-1}} + (n+1)q^{h-1}\zeta[2]_q. \tag{2.12}$$

Therefore, we obtain the following corollary.

Corollary 2.7. *For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, one has*

$$\int_{\mathbb{Z}_p}[1-x]^n_{q^{-1}}\zeta^x q^{(h-1)x}\,d\mu_{-q}(x) = q^{h+1}\zeta\frac{G^{(h)}_{n+1,q^{-1},\zeta^{-1}}}{n+1} + [2]_q. \tag{2.13}$$

By (1.5), *we get the symmetry of q-Bernstein polynomials as follows:*

$$B_{k,n}(x,q) = B_{n-k,n}\left(1-x,q^{-1}\right) \tag{2.14}$$

(see [11]).

Thus, by Corollary 2.7 and (2.14), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x,q) q^{(h-1)x}\zeta^x \, d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}\left(1-x,q^{-1}\right) q^{(h-1)x}\zeta^x \, d\mu_{-q}(x) \\ &= \binom{n}{k}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{k-\ell}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n-\ell} q^{(h-1)x}\zeta^x \, d\mu_{-1}(x) \\ &= \binom{n}{k}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{k-\ell}\left(q^{h+1}\zeta\frac{G^{(h)}_{n-\ell+1,q^{-1},\zeta^{-1}}}{n-\ell+1}+[2]_q\right) \\ &= \begin{cases} q^{h+1}\zeta\dfrac{G^{(h)}_{n+1,q^{-1},\zeta^{-1}}}{n+1}+[2]_q & \text{if } k=0, \\ q^{h+1}\zeta\dbinom{n}{k}\displaystyle\sum_{\ell=0}^{k}\dbinom{k}{\ell}(-1)^{k-\ell}\dfrac{G^{(h)}_{n-\ell+1,q^{-1},\zeta^{-1}}}{n-\ell+1} & \text{if } k>0. \end{cases}\end{aligned} \tag{2.15}$$

From (2.15), we have the following theorem.

Theorem 2.8. *For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, one has*

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q) q^{(h-1)x}\zeta^x \, d\mu_{-q}(x) = \begin{cases} q^{h+1}\zeta\dfrac{G^{(h)}_{n+1,q^{-1},\zeta^{-1}}}{n+1}+[2]_q & \text{if } k=0, \\ q^{h+1}\zeta\dbinom{n}{k}\displaystyle\sum_{\ell=0}^{k}\dbinom{k}{\ell}(-1)^{k-\ell}\dfrac{G^{(h)}_{n-\ell+1,q^{-1},\zeta^{-1}}}{n-\ell+1} & \text{if } k>0. \end{cases} \tag{2.16}$$

For $n,k \in \mathbb{Z}_+$ with $n > k$, fermionic p-adic invariant integral for multiplication of two q-Bernstein polynomials on $\mathbb{Z}_p$ can be given by the following:

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x,q) q^{(h-1)x}\zeta^x \, d\mu_{-q}(x) &= \int_{\mathbb{Z}_p}\binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k} q^{(h-1)x}\zeta^x \, d\mu_{-q}(x) \\ &= \int_{\mathbb{Z}_p}\binom{n}{k}[x]_q^k\left(1-[x]_q\right)^{n-k} q^{(h-1)x}\zeta^x \, d\mu_{-1}(x) \\ &= \binom{n}{k}\sum_{\ell=0}^{n-k}\binom{n-k}{\ell}(-1)^{\ell}\int_{\mathbb{Z}_p}[x]_q^{k+\ell} q^{(h-1)x}\zeta^x \, d\mu_{-1}(x).\end{aligned} \tag{2.17}$$

From Theorem 2.8 and (2.17), we have the following corollary.

Corollary 2.9. *For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, one has*

$$\sum_{\ell=0}^{n-k}\binom{n-k}{\ell}(-1)^{\ell}\frac{G^{(h)}_{k+\ell+1,q,\zeta}}{k+\ell+1}=\begin{cases} q^{h+1}\zeta\dfrac{G^{(h)}_{n+1,q^{-1},\zeta^{-1}}}{n+1}+[2]_q & \text{if } k=0,\\ q^{h+1}\zeta\displaystyle\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{k-\ell}\frac{G^{(h)}_{n-\ell+1,q^{-1},\zeta^{-1}}}{n-\ell+1} & \text{if } k>0.\end{cases} \tag{2.18}$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1+n_2>2k$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)B_{k,n_2}(x,q)q^{(h-1)x}\zeta^x\,d\mu_{-q}(x)\\ &\quad=\binom{n_1}{k}\binom{n_2}{k}\int_{\mathbb{Z}_p}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(-1)^{2k-\ell}[1-x]_{q^{-1}}^{n_1+n_2-\ell}q^{(h-1)x}\zeta^x\,d\mu_{-q}(x)\\ &\quad=\binom{n_1}{k}\binom{n_2}{k}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(-1)^{2k-\ell}\left(\frac{G^{(h)}_{n_1+n_2-\ell+1,q^{-1},\zeta^{-1}}}{n_1+n_2-\ell+1}q^{h+1}\zeta+[2]_q\right).\end{aligned} \tag{2.19}$$

From (2.19), we have the following theorem.

Theorem 2.10. *For $n \in \mathbb{Z}_+$ and $\zeta \in T_p$, one has*

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)B_{k,n_2}(x,q)q^{(h-1)x}\zeta^x\,d\mu_{-q}(x)\\ &\quad=\begin{cases} q^{h+1}\zeta\dfrac{G^{(h)}_{n_1+n_2+1,q^{-1},\zeta^{-1}}}{n_1+n_2+1}+[2]_q & \text{if } k=0,\\ \displaystyle\binom{n_1}{k}\binom{n_2}{k}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(-1)^{2k-\ell}\frac{G^{(h)}_{n_1+n_2-\ell+1,q^{-1},\zeta^{-1}}}{n_1+n_2-\ell+1} & \text{if } k>0.\end{cases}\end{aligned} \tag{2.20}$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1+n_2>2k$, fermionic p-adic invariant integral for multiplication of two q-Bernstein polynomials on $\mathbb{Z}_p$ can be given by the following:

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x,q)B_{k,n_2}(x,q)q^{(h-1)x}\zeta^x\,d\mu_{-q}(x)\\ &\quad=\binom{n_1}{k}\binom{n_2}{k}\int_{\mathbb{Z}_p}\sum_{\ell=0}^{n_1+n_2-2k}(-1)^{\ell}\binom{n_1+n_2-2k}{\ell}[x]_q^{2k+\ell}q^{(h-1)x}\zeta^x\,d\mu_{-q}(x)\\ &\quad=\binom{n_1}{k}\binom{n_2}{k}\sum_{\ell=0}^{n_1+n_2-2k}(-1)^{\ell}\binom{n_1+n_2-2k}{\ell}\frac{G^{(h)}_{2k+\ell+1,q,\zeta}}{2k+\ell+1}.\end{aligned} \tag{2.21}$$

From Theorem 2.10 and (2.21), we have the following corollary.

Corollary 2.11. *For $n_1, n_2, k \in \mathbb{Z}_+$ and $n_1 + n_2 > 2k$, one has*

$$\sum_{\ell=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{\ell} (-1)^{\ell} \frac{G_{2k+\ell+1,q,\zeta}^{(h)}}{2k+\ell+1} = \begin{cases} q^{h+1}\zeta \dfrac{G_{n_1+n_2+1,q^{-1},\zeta^{-1}}^{(h)}}{n_1+n_2+1} + [2]_q & \text{if } k = 0, \\ \displaystyle\sum_{\ell=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{\ell} (-1)^{2k-\ell} \frac{G_{n_1+n_2-\ell+1,q^{-1},\zeta^{-1}}^{(h)}}{n_1+n_2-\ell+1} & \text{if } k > 0. \end{cases} \tag{2.22}$$

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