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Research Article

Quasistatic Elastic Contact with Adhesion

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The aim of this paper is the variational study of the contact with adhesion between an elastic material and a rigid foundation in the quasistatic process where the deformations are supposed to be small. The behavior of this material is modelled by a nonlinear elastic law and the contact is modelled with Signorini's conditions and adhesion. The evolution of bonding field is described by a nonlinear differential equation. We derive a variational formulation of the mechanical problem, and we prove the existence and uniqueness of the weak solution using a theorem on variational inequalities, the theorem of Cauchy-Lipschitz, a lemma of Gronwall, as well as the fixed point of Banach.

1. Introduction

The phenomena of contact with or without friction between deformable bodies or between deformable and rigid bodies abound in industry and everyday life. The contact of the tires with the roads, the shoe with disc of break, pistons with skirts are current examples. Because of the importance of contact process in structural and mechanical systems, a considerable effort has been made in its mathematical modeling, mathematical analysis, and numerical simulations.

Process of adhesion is important in many industrial settings where parts, nonmetallic, are glued together. Recently, composite materials reached prominence, since they are very strong and light, and therefore, of considerable importance in aviation, space exploration, and in the automotive industry. Composite materials may undergo delamination under stress, a process in which different layers debond and move relatively to each other. To model the process when bonding is not permanent and debonding may take place, there is a need to add adhesion to the description of the contact process. For these reasons, adhesive contact between bodies when a glue is added to prevent the surfaces from relative motion, has recently received increased attention in the mathematical literature. In this paper we introduce an internal variable of surface, known as bonding field and denoted in this paper by β, which describes the fractional density of active bonds on the contact surface. When $\beta = 1$ at a point of contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction of the bonds is active. The problems of contact with adhesion were studied by several authors. Significant results on these problems can be found in [1–7] and references therein.

Here, the novelty consists in the introduction of the bonding field into the contact between a material elastic and a rigid foundation where the process is quasistatic and the material is modelled by a nonlinear elastic law. The main contribution of this study lies in the proof of existence and uniqueness of the weak solution of the mechanical problem.

This work is organized as follows. In Section 2 we present some notations and preliminaries. In Section 3 we state the mechanical models of elastic contact with adhesion, list the assumptions on the data of the mechanical problem and deduce its variational formulation. In Section 4 we state and prove the existence of a unique weak solution to the mechanical problem; the proof is based on arguments of evolutionary equations and Banach fixed point.

2. Notations and Preliminaries

In this section, we specify the standard notations used and we remind of some definitions and necessary results for the study of this mechanical problem.

We denote by S^N the space of second order symmetric tensors on $\mathbb{R}^N$ $(N = 2, 3)$ while "$\cdot$" and $|\cdot|$ represent the inner product and the Euclidean norm on $\mathbb{R}^N$ and S^N, respectively. Thus, for every $u, v \in \mathbb{R}^N$, and $\sigma, \tau \in S^N$ we have

$$u \cdot v = u_i v_i, \qquad |u| = (u \cdot u)^{1/2}, \qquad \sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \qquad |\sigma| = (\sigma \cdot \sigma)^{1/2}. \tag{2.1}$$

Here and below, the indices i, j run between 1, N and the summation convention over repeated indices is adopted.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. Moreover, we use also the spaces

$$\begin{gathered} H = \left\{ u = (u_i) / u_i \in L^2(\Omega) \right\}, \qquad \mathcal{H} = \left\{ \sigma = (\sigma_{ij}) / \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\}, \\ H_1 = \{ u \in H / \varepsilon(u) \in \mathcal{H} \}, \qquad \mathcal{H}_1 = \{ \sigma \in \mathcal{H} / \operatorname{Div} \sigma \in H \}, \end{gathered} \tag{2.2}$$

where $\varepsilon : H_1 \to \mathcal{H}$, $\operatorname{Div} : \mathcal{H} \to H$ are the deformation and the divergence operators, respectively, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \qquad \varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \qquad \operatorname{Div} \sigma = (\partial_j \sigma_{ij}). \tag{2.3}$$

The spaces H, $\mathcal{H}$, H_1, and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned}
\langle u, v\rangle_H &= \int_\Omega u_i v_i dx, \quad \forall u, v \in H, \\
\langle \sigma, \tau\rangle_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij} dx, \quad \forall \sigma, \tau \in \mathcal{H}, \\
\langle u, v\rangle_{H_1} &= \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \quad \forall u, v \in H, \\
\langle \sigma, \tau\rangle_{\mathcal{H}_1} &= \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle \operatorname{Div}\sigma, \operatorname{Div}\tau\rangle_H, \quad \forall \sigma, \tau \in \mathcal{H}.
\end{aligned} \tag{2.4}$$

The associated norms are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$, and $|\cdot|_{\mathcal{H}_1}$, respectively.

Since the boundary Γ is Lipschitz continuous, the unit outward normal vector ν on the boundary is defined almost everywhere for every vector field $u \in H_1$, we also use the notation u for the trace of u on Γ and we denote by u_ν and u_τ the normal and tangential components of u on the boundary Γ given by

$$u_\nu = u \cdot \nu, \qquad u_\tau = u - u_\nu \nu. \tag{2.5}$$

For a regular (say C^1) stress field σ, the application of its trace on the boundary to ν is the Cauchy stress vector $\sigma\nu$. We define, similarly, the normal and tangential components of the stress on the boundary Γ by

$$\sigma_\nu = (\sigma\nu) \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu. \tag{2.6}$$

And we recall that the following Green's formula holds

$$\langle \sigma, \varepsilon(u)\rangle_{\mathcal{H}} + \langle \operatorname{Div}\sigma, u\rangle_H = \int_\Gamma \sigma\nu u \, ds, \quad \forall u \in H_1. \tag{2.7}$$

Let Γ_1 be a measurable part of Γ such that meas $\Gamma_1 > 0$ and let V be the closed subset of H_1 defined by

$$V = \{v \in H_1 / v = 0 \text{ on } \Gamma_1\}. \tag{2.8}$$

Since $\operatorname{meas}\Gamma_1 > 0$, then Korn's inequality holds, and thus there exists a constant $c > 0$, depending only on Ω and Γ_1 such that

$$|\varepsilon(u)|_{\mathcal{H}} \geq c|u|_{H_1} \quad \forall u \in V. \tag{2.9}$$

A proof of Korn's inequality may be found in [8, page 79].

We define the inner product over the space V by

$$\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \quad \forall u, v \in V. \tag{2.10}$$

It follows from Korn's inequality that $|\cdot|_V$ and $|\cdot|_{H_1}$are equivalent norms on V. Therefore $(V, |\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a positive constant $c > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$|v|_{L^2(\Gamma_3)^N} \leq c|v|_V, \quad \forall v \in V. \tag{2.11}$$

Finally, we shall use the notation Q for the set defined by

$$Q = \left\{\beta \in C\Big(0, T; L^2(\Gamma_3)\Big) : 0 \leq \beta(t) \leq 1 \ \forall t \in [0, T] \text{ a.e. on } \Gamma_3, \ \beta(0) = \beta_0\right\}. \tag{2.12}$$

We end this preliminary with the following version of the classical theorem of Cauchy-Lipschitz which can be found in [9, page 60].

Theorem 2.1. *Assume that* $(X, |\cdot|_X)$ *is a real Banach space. Let* $F(t, \cdot) : X \rightarrow X$ *be an operator defined almost everywhere on* $]0, T[$*, satisfying the following conditions:*

(1) $|F(t, u) - F(t, v)|_X \leq L_F|u - v|_X \ \forall u, v \in X$ *a.e.* $t \in]0, T[$*, for some* L_F*,*

(2) $t \mapsto F(t, u) \in L^p([0, T]; X), \ \forall u \in X$*, and some* $p \geq 1$*.*

Then, for every $u_0 \in X$*, there exists a unique function* $u \in W^{1,p}([0, T]; X)$ *such that*

$$\begin{gathered} \dot{u}(t) = F(t, u(t)) \quad \text{a.e. } t \in]0, T[, \\ u(0) = u_0. \end{gathered} \tag{2.13}$$

This theorem will be used in Section 4 to prove the theorem of existence and uniqueness of weak solution of the mechanical problem.

3. Problem Statement and Variational Formulation

We consider an elastic material which occupies a bounded domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ and assume that its boundary Γ is regular and partitioned into three disjoint measurable parts Γ_1, Γ_2, and Γ_3 such that meas $\Gamma_1 > 0$. Let $[0, T]$ be the time interval of interest, where $T > 0$. The material is clamped on $\Gamma_1 \times]0, T[$ and therefore the displacement field vanishes there. We also assume that a volume force of density f_0 acts in $\Omega \times]0, T[$ and that a surface traction of density f_2 acts on $\Gamma_2 \times]0, T[$. On $\Gamma_3 \times]0, T[$ the material may come in contact with arigid foundation. Moreover, the process is quasistatic and the evolution of the bonding field is described by a nonlinear differential equation.

Under these conditions, the formulation of the mechanical problem is the following.

Problem P. Find a displacement field $u : \Omega \times [0,T] \rightarrow \mathbb{R}^N$, a stress field $\sigma : \Omega \times [0,T] \rightarrow S^N$ and a bonding field $\beta : \Gamma_3 \times [0,T] \rightarrow [0,1]$ such that

$$\sigma(t) = F\,(\varepsilon(u(t))) \quad \text{in } \Omega \times]0,T[, \tag{3.1}$$

$$\operatorname{Div} \sigma(t) + f_0(t) = 0 \quad \text{in } \Omega \times]0,T[, \tag{3.2}$$

$$u = 0, \quad \text{on } \Gamma_1 \times]0,T[, \tag{3.3}$$

$$\sigma \nu = f_2, \quad \text{on } \Gamma_2 \times]0,T[, \tag{3.4}$$

$$u_\nu \leq 0, \quad \sigma_\nu + \gamma_\nu R(u_\nu)\beta^2 \leq 0, \quad \left(\sigma_\nu + R(u_\nu)\beta^2\right) u_\nu = 0, \quad \text{on } \Gamma_3 \times]0,T[, \tag{3.5}$$

$$\sigma_\tau = 0, \quad \text{on } \Gamma_3 \times]0,T[, \tag{3.6}$$

$$\dot{\beta} = -\left(\gamma_\nu \beta [(-R(u_\nu))_+]^2 - \epsilon_a\right)_+, \quad \text{on } \Gamma_3 \times]0,T[, \tag{3.7}$$

$$\beta(0) = 0, \quad \text{on } \Gamma_3. \tag{3.8}$$

Here (3.1) is the nonlinear elastic constitutive law. Equation (3.2) represents the equilibrium (3.3) and (3.4) are the displacement-traction boundary conditions, respectively. Conditions (3.5) represent the Signorini conditions with adhesion where γ_ν is a given adhesion coefficient and R is the truncation operator defined by

$$R(\mathrm{s}) = \begin{cases} L, & \text{if } s \geq L, \\ s, & \text{if } |s| < L, \\ -L, & \text{if } s \leq -L. \end{cases} \tag{3.9}$$

Here $L > 0$ being the characteristic length of the bond, beyond which it does not offer additional traction. The introduction of the operator R, defined below, is motivated by the mathematical argument but is not restrictive in terms of application, since no restriction on the size of the parameter L is made in the sequel. Thus, by choosing L very large, we can assume that $R(u_\nu) = u_\nu$ and, therefore, from (3.5) we recover the contact conditions:

$$u_\nu \leq 0, \quad \sigma_\nu + \gamma_\nu u_\nu \beta^2 \leq 0, \quad \left(\sigma_\nu + u_\nu \beta^2\right) u_\nu = 0, \quad \text{on } \Gamma_3 \times]0,T[, \tag{3.10}$$

it follows from (3.5) that there is no penetration between the material and the foundation, since $u_\nu \leq 0$ during the process. Also, note that when the bonding vanishes, then the contact conditions (3.5) become the classical Signironi contact conditions with zero gap function, that is,

$$u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\tau = 0, \quad \sigma_\nu u_\nu = 0, \quad \text{on } \Gamma_3 \times]0,T[. \tag{3.11}$$

Condition (3.6) represents the frictionless contact and shows that the tangential stress vanishes on the contact surface during the process. Equation (3.7) describes the evolution of the bonding field with given material parameters γ_ν and ϵ_a. Also, the data β_0 in (3.8) is the given initial bonding field.

Assumptions. For the variational study of the mechanical problem, we assume that the operator F satisfies the following conditions:

$$F : \Omega \times S^N \longrightarrow S^N \text{ such that } \exists m > 0 \quad \text{such that } (F(x,\varepsilon_1) - F(x,\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m|\varepsilon_1 - \varepsilon_2|^2$$
$$\text{a.e. } x \in \Omega \quad \forall \varepsilon_1, \varepsilon_2 \in S^N, \tag{3.12a}$$

$$\exists L > 0 \text{ such that } |F(x,\varepsilon_1) - F(x,\varepsilon_2)| \leq L|\varepsilon_1 - \varepsilon_2|$$
$$\text{a.e. } x \in \Omega \quad \forall \varepsilon_1, \varepsilon_2 \in S^N, \tag{3.12b}$$

$$\text{The mapping } x \longmapsto F(x,\varepsilon) \text{ is Lebesgue measurable}$$
$$\text{a.e. } x \in \Omega, \quad \forall \varepsilon \in S^N, \tag{3.12c}$$

$$\text{The mapping } x \longmapsto F(x,0) \in \mathscr{H}. \tag{3.12d}$$

We suppose that the adhesion coefficients satisfy

$$\gamma_\nu \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \epsilon_a \geq 0 \text{ a.e. on } \Gamma_3. \tag{3.13}$$

And the body forces and surface traction have the regularity:

$$f_0 \in W^{1,\infty}(0,T;H), \quad f_2 \in W^{1,\infty}\left(0,T;L^2(\Gamma_2)^N\right). \tag{3.14}$$

The initial data satisfy

$$\beta_0 \in Q. \tag{3.15}$$

We use the convex subset of admissible displacements defined by

$$U = \{u \in V \text{ such that } u_\nu \leq 0 \text{ on } \Gamma_3\}. \tag{3.16}$$

It follows from (3.14) and Riesz-Frechet representation theorem that there exists a unique function $f : [0,T] \rightarrow V$ such that:

$$\langle f(t), v\rangle_V = \langle f_0(t), v\rangle_H + \langle f_2(t), v\rangle_{L^2(\Gamma_2)^N} \quad \forall v \in V,\ t \in]0,T[. \tag{3.17}$$

Moreover, we note that the conditions (3.14) imply

$$f \in W^{1,\infty}(0,T;V). \tag{3.18}$$

Finally, we define the adhesion functional $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j(\beta, u, v) = -\int_{\Gamma_3} \gamma_\nu \beta^2 (-R(u_\nu))_+ v_\nu ds. \tag{3.19}$$

3.1. Variational Formulation

By applying Green's formula, and using the equilibrium equation and the boundary conditions, we easily deduce the following variational formulation of the mechanical Problem P.

Problem PV. Find a displacement $u : [0,T] \to V$, and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that:

$$u(t) \in U, \quad \langle F(\varepsilon(u(t))), \varepsilon(v - u(t)) \rangle_{\mathcal{H}} + j(\beta(t), u(t), v - u(t)) \geq \langle f(t), v - u(t) \rangle_V \quad \forall v \in U,\ t \in]0,T[, \tag{3.20}$$

$$\dot{\beta}(t) = -\left(\gamma_\nu \beta(t) [(-R(u_\nu(t)))_+]^2 - \epsilon_a\right)_+ \text{ a.e. } t \in]0,T[, \tag{3.21}$$

$$\beta(0) = \beta_0. \tag{3.22}$$

Note that the variational Problem PV is formulated in terms of displacement and bonding field, since the stress field was eliminated. However, if the solution (u, β) of these variational problems is known, the corresponding stress field σ can be easily obtained by using the nonlinear elastic law (3.1).

Below in this subsection β, β_1, and β_2 denote various elements of Q, and u, u_1, u_2, and v represent elements of V, and $c > 0$ represent constants which may depend on Ω, Γ_1, Γ_3, γ_ν, and L. Note that the adhesion functional j is linear with respect to the last argument and therefore

$$j(\beta, u, -v) = -j(\beta, u, v). \tag{3.23}$$

Next using (3.19), as well as the properties (3.9) of the truncation operator R, we find

$$j(\beta_1, u_1, u_2 - u_1) + j(\beta_2, u_2, u_1 - u_2) \leq c \int_{\Gamma_3} |\beta_1 - \beta_2| |u_1 - u_2| ds. \tag{3.24}$$

And by the Sobolev theorem trace we obtain

$$j(\beta_1, u_1, u_2 - u_1) + j(\beta_2, u_2, u_1 - u_2) \leq c |\beta_1 - \beta_2|_{L^2(\Gamma_3)} |u_1 - u_2|_V. \tag{3.25}$$

We now take $\beta = \beta_1 = \beta_2$ in (3.25) to deduce

$$j(\beta, u_1, u_2 - u_1) + j(\beta, u_2, u_1 - u_2) \leq 0. \tag{3.26}$$

Similar computations, based on the Lipschitz continuity of R, show that the following inequality also holds

$$\left| j(\beta, u_1, v) - j(\beta, u_2, v) \right| \leq c|u_1 - u_2|_V |v|_V. \tag{3.27}$$

We take $u_1 = v$ and $u_2 = 0$ in (3.26) then we use equality $R(0) = 0$ and (3.23) to obtain

$$j(\beta, v, v) \geq 0. \tag{3.28}$$

The inequalities (3.25)–(3.28) will be used in various places to prove the theorem of existence and uniqueness of the weak solution.

4. Existence and Uniqueness of Weak Solution

Theorem 4.1. *Assume that* (3.12a)–(3.15) *hold. Then there exists a unique solution* (u, β) *to Problem PV and it satisfies*

$$\begin{aligned} &u \in W^{1,\infty}(0, T; V), \\ &\beta \in W^{1,\infty}(0, T; L^\infty(\Gamma_3)). \end{aligned} \tag{4.1}$$

A triple (u, σ, β) which satisfies (3.1) and (3.20)–(3.22) is called weak solution of the mechanical Problem P. We conclude that under the stated assumptions, the mechanical problem has a unique weak solution. The regularity of the weak solution in terms of stress is given by

$$\sigma \in W^{1,\infty}(0, T; \mathcal{H}_1). \tag{4.2}$$

Indeed, taking $v = \varphi \in D^\infty(\Omega)$ in (3.20) and using (3.1), (3.17) we find:

$$\operatorname{Div} \sigma(t) + f_0(t) = 0. \tag{4.3}$$

Now (3.14) and (4.3) imply that $\operatorname{Div} \sigma \in W^{1,\infty}(0, T; H)$, which in its turn implies $\sigma \in W^{1,\infty}(0, T; \mathcal{H}_1)$.

The proof of Theorem 4.1 will be carried out in several steps. In the first step we consider the following problem in which $\beta \in Q$, is given.

Problem 1. Find a displacement $u_\beta : [0, T] \to V$ such that

$$\begin{aligned} u_\beta(t) \in U, \quad F\left\langle \left(\varepsilon(u_\beta(t))\right), \varepsilon(v - u_\beta(t)) \right\rangle_{\mathcal{H}} + j(\beta(t), u_\beta(t), v - u_\beta(t)) \geq \left\langle f(t), v - u_\beta(t) \right\rangle_V \\ \forall v \in U,\ t \in]0, T[. \end{aligned} \tag{4.4}$$

We have the following result.

Lemma 4.2. *There exists a unique solution to Problem 1 which satisfies*

$$u_\beta \in C(0,T;V). \tag{4.5}$$

Proof. Let $t \in [0,T]$. We consider the operator $A_t : V \to V$ defined by

$$\langle A_t u, v\rangle_V = \langle F(\varepsilon(u(t))), \varepsilon(v)\rangle_{\mathscr{H}} + j(\beta(t), u(t), v) \quad \forall u, v \in V. \tag{4.6}$$

Let $u_1, u_2 \in V$. We have

$$\begin{aligned}\langle A_t u_1 - A_t u_2, u_1 - u_2\rangle_V = {} & \langle F(\varepsilon(u_1)) - F(\varepsilon(u_2)), \varepsilon(u_1 - u_2)\rangle_{\mathscr{H}} \\ & - j(\beta, u_1, u_2 - u_1) - j(\beta, u_2, u_1 - u_2).\end{aligned} \tag{4.7}$$

We use (3.26), (3.12a), Korn's inequality and the equivalence of the two norms $|\cdot|_{H_1}$ and $|\cdot|_V$ to show that

$$\langle A_t u_1 - A_t u_2, u_1 - u_2\rangle_V \geq c|u_1 - u_2|_V^2. \tag{4.8}$$

The operator A_t is, therefore, strongly monotone.
Let $u_1, u_2, v \in V$. We have

$$\begin{aligned}|\langle A_t u_1 - A_t u_2, v\rangle_V| \leq {} & |\langle F(\varepsilon(u_1)) - F(\varepsilon(u_2)), \varepsilon(v)\rangle_{\mathscr{H}}| \\ & + \left| j(\beta, u_1, v) - j(\beta, u_2, v)\right|.\end{aligned} \tag{4.9}$$

Using Cauchy-Schwartz's inequality, (3.12b) and (3.27), one obtains

$$|A_t u_1 - A_t u_2|_V \leq c|u_1 - u_2|_V. \tag{4.10}$$

Therefore, A_t is a continuous Lipschitz operator. Since U is a nonempty convex closed subset of V, it follows from the standard results on elliptic variational inequalities that there exists a unique element u_β, such that

$$u_\beta \in U, \quad \langle A_t u_\beta, v - u_\beta\rangle \geq \langle f, v - u_\beta\rangle_V \quad \forall v \in U. \tag{4.11}$$

Using (4.6) we get

$$u_\beta(t) \in U, \quad \langle F(\varepsilon(u_\beta(t))), \varepsilon(v - u_\beta(t))\rangle_{\mathscr{H}} + j(\beta(t), u_\beta(t), v - u_\beta(t)) \geq \langle f(t), v - u_\beta(t)\rangle_V \quad \forall v \in U, \ t \in]0,T[. \tag{4.12}$$

We now show that $u_\beta \in C(0,T;V)$.

Let $t_1, t_2 \in [0,T]$. Denote $u_\beta(t_i)$, $\beta(t_i)$, and $f(t_i)$ by u_i, β_i and f_i (for $i = 1,2$), respectively. Then we have

$$\begin{aligned}\langle F(\varepsilon(u_1)) - F(\varepsilon(u_2)), \varepsilon(u_1 - u_2)\rangle_{\mathcal{H}} \leq \langle f_1 - f_2, u_1 - u_2\rangle_V \\ + j(\beta_1, u_1, u_2 - u_1) + j(\beta_2, u_2, u_1 - u_2).\end{aligned} \tag{4.13}$$

Using (3.12a) we obtain

$$m|\varepsilon(u_1 - u_2)|^2_{\mathcal{H}} \leq \langle f_1 - f_2, u_1 - u_2\rangle_V + j(\beta_1, u_1, u_2 - u_1) + j(\beta_2, u_2, u_1 - u_2). \tag{4.14}$$

Using Korn's inequality and the fact that $|\cdot|_V$, $|\cdot|_{H_1}$ are equivalent norms on V, Cauchy-Schwartz's inequality and (3.25) we get

$$|u_1 - u_2|_V \leq c\Big(|f_1 - f_2|_V + |\beta_1 - \beta_2|_{L^2(\Gamma_3)}\Big). \tag{4.15}$$

From the previous inequality, the fact that $\beta \in \mathcal{Q}$ and the regularity of the function f given by (3.18), it follows that $u_\beta \in C(0,T;V)$. □

In the second step we use the displacement field u_β, obtained in Lemma 4.2 and we consider the following auxiliary problem.

Problem 2. Find a bonding field $\theta_\beta : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\dot{\theta}_\beta(t) = -\Big(\gamma_\nu \theta_\beta(t)\big[(-R(u_{\beta\nu}(t)))_+\big]^2 - \epsilon_a\Big)_+ \quad \text{a.e.} \quad t \in]0,T[, \tag{4.16}$$

$$\theta_\beta(0) = \beta_0. \tag{4.17}$$

We have the following result.

Lemma 4.3. *There exists a unique solution to Problem 2 which satisfies*

$$\theta_\beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_3)) \cap \mathcal{Q}. \tag{4.18}$$

Proof. Consider the mapping $F_\beta : [0,T] \times L^\infty(\Gamma_3) \to L^\infty(\Gamma_3)$ defined by

$$F_\beta(t, \theta_\beta) = -\Big(\gamma_\nu \theta_\beta(t)\big[(-R(u_{\beta\nu}(t)))_+\big]^2 - \epsilon_a\Big)_+. \tag{4.19}$$

It follows from the properties of the truncation operator R, that F_β is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta_\beta \in L^\infty(\Gamma_3)$, the mapping

$t \mapsto F_\beta(t, \theta_\beta)$ belongs to $L^\infty(0,T;L^\infty(\Gamma_3))$. Then from Theorem 2.1, we deduce the existence of a unique function $\theta_\beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_3))$, which satisfies (4.16)-(4.17). The regularity $\theta_\beta \in \mathcal{Q}$, follows from (4.16)-(4.17) and assumption $0 \leq \beta_0 \leq 1$ a.e. on Γ_3. Indeed, (4.16) implies that for a.e. $x \in \Gamma_3$, the function $t \to \theta_\beta(x,t)$ is decreasing and its derivative vanishes when $\gamma_\nu \theta_\beta(t)[(-R(u_{\beta\nu}(t)))_+]^2 \le \epsilon_a$. Combining these properties with the inequality $0 \le \beta_0 \le 1$ we deduce that $0 \le \theta_\beta(t) \le 1$, for all $t \in [0,t]$, a.e. on Γ_3, which shows that $\theta_\beta \in Q$. □

In the third step we denote by u_β the solution of Problem 1 and θ_β the solution of Problem 2, for every $\beta \in Q$. Moreover, we define the operator $\Lambda : Q \to Q$ by

$$\Lambda\beta = \theta_\beta. \tag{4.20}$$

Lemma 4.4. *The operator Λ has a unique fixed point β^*.*

Proof. We show that for a positive integer p the mapping Λ^p is a contraction on Q. For this, suppose that β_i are two functions of Q and we denote by u_i, θ_i the functions obtained in Lemmas 4.2 and 4.3, respectively, for $\beta = \beta_i$, $(i = 1,2)$.

Let $t \in [0,T]$. We use (4.4) and (3.25) to deduce that

$$\langle F(\varepsilon(u_1(t)) - F(\varepsilon(u_2(t)))), \varepsilon(u_1(t) - u_2(t))\rangle_{\mathcal{H}} \le c\big|\beta_1(t) - \beta_2(t)\big|_{L^2(\Gamma_3)}|u_1(t) - u_2(t)|_V. \tag{4.21}$$

Using the fact that F is a strongly monotone, Korn's inequality and $|\cdot|_V$, $|\cdot|_{H_1}$ are equivalent norms on V, we get

$$|u_1(t) - u_2(t)|_V \le c\big|\beta_1(t) - \beta_2(t)\big|_{L^2(\Gamma_3)}, \tag{4.22}$$

which implies

$$\int_0^t |u_1(s) - u_2(s)|_V ds \le c\int_0^t \big|\beta_1(s) - \beta_2(s)\big|_{L^2(\Gamma_3)} ds. \tag{4.23}$$

On the other hand, it follows from (4.16) and (4.17) that

$$\theta_i(t) = \beta_0 - \int_0^t \Big(\gamma_\nu \theta_i(s)[(-R(u_{i\nu}(s)))_+]^2 - \epsilon_a\Big)_+ ds \qquad i = 1,2. \tag{4.24}$$

And then

$$|\theta_1(t) - \theta_2(t)|_{L^2(\Gamma_3)} \le c\int_0^t \Big|\theta_1(s)[(-R(u_{1\nu}(s)))_+]^2 - \theta_2(s)[(-R(u_{2\nu}(s)))_+]^2\Big|_{L^2(\Gamma_3)} ds. \tag{4.25}$$

Using the definition (3.9) and writing $\theta_1 = \theta_1 - \theta_2 + \theta_2$, we get

$$|\theta_1(t) - \theta_2(t)|_{L^2(\Gamma_3)} \le c\int_0^t |\theta_1(s) - \theta_2(s)|_{L^2(\Gamma_3)} ds + c\int_0^t |u_{1\nu}(s) - u_{2\nu}(s)|_{L^2(\Gamma_3)} ds. \tag{4.26}$$

By Gronwall's inequality and the Sobolev trace theorem, it follows that:

$$|\theta_1(t) - \theta_2(t)|_{L^2(\Gamma_3)} \leq c \int_0^t |u_1(s) - u_2(s)|_V ds. \tag{4.27}$$

Using (4.20) and (4.27), we obtain

$$\left|\Lambda\beta_1(t) - \Lambda\beta_2(t)\right|_{L^2(\Gamma_3)} \leq c \int_0^t |u_1(s) - u_2(s)|_V ds. \tag{4.28}$$

We now combine (4.23) and (4.28) to see that

$$\left|\Lambda\beta_1(t) - \Lambda\beta_2(t)\right|_{L^2(\Gamma_3)} \leq c \int_0^t \left|\beta_1(s) - \beta_2(s)\right|_{L^2(\Gamma_3)} ds. \tag{4.29}$$

And reiterating this inequality p times, yields

$$\left|\Lambda^p\beta_1 - \Lambda^p\beta_2\right|_{C(0,T;L^2(\Gamma_3))} \leq \frac{c^p T^p}{p!} \left|\beta_1 - \beta_2\right|_{C(0,T;L^2(\Gamma_3))} \quad \forall p \in N. \tag{4.30}$$

Recall that Q is a nonempty closed set in the Banach space $C(0,T;L^2(\Gamma_3))$ and note that inequality (4.30) shows that for p sufficiently large $\Lambda^p : Q \to Q$, then Λ has a unique fixed point $\beta^* \in Q$. □

Now, the proof of the Theorem 4.1, is a consequence of the previous lemmas.

Proof. Existence. Let $\beta^* \in Q$ be the fixed point of Λ and u^* be the solution of Problem 1 for $\beta = \beta^*$, that is, $u^* = u_{\beta^*}$. Arguments similar to those used in proof of (4.22) lead to

$$|u^*(t_1) - u^*(t_2)|_V \leq c\left|\beta^*(t_1) - \beta^*(t_2)\right|_{L^2(\Gamma_3)} \quad \forall t_1, t_2 \in [0,T]. \tag{4.31}$$

Since $\beta^* = \theta_{\beta^*}$ it follows from Lemma 4.3 that $\beta^* \in W^{1,\infty}(0,T;L^\infty(\Gamma_3))$ and therefore (4.31) implies that $u^* \in W^{1,\infty}(0,T;V)$. From (4.4), (4.16) and (4.17) we conclude that (u^*, β^*) is a solution of the Problem PV which satisfies (4.1).

Uniqueness. The uniqueness of the solution is a consequence of the fixed point of the operator Λ defined by (4.20). Indeed, let (u, β) be a solution of Problem PV which satisfies (4.1). Since $\beta \in Q$, it follows from (3.20) that u is solution to Problem 1, but the Lemma 4.2 implies that this problem has a unique solution denoted u_β, we get

$$u = u_\beta. \tag{4.32}$$

We put $u = u_\beta$ in (3.21) and using the initial condition (3.22) then we can see that β is a solution to Problem 2, but the Lemma 4.3 implies that this last problem has a unique solution denoted θ_β, we get

$$\beta = \theta_\beta. \tag{4.33}$$

We use now (4.20) and (4.33) to see that $\Lambda\beta = \beta$, that is, β is a fixed point of the operator Λ. It follows now from Lemma 4.4 that

$$\beta = \beta^*. \tag{4.34}$$

The uniqueness of the solution is now a consequence of (4.32) and (4.34). □

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