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Research Article

Shintani Functions on $SL(3, \mathbf{R})$

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We investigate the Shintani functions attached to the spherical and nonspherical principal series representations of $SL(3, \mathbf{R})$. We give the explicit formulas of the radial part of Shintani functions and evaluate the dimension of the space of Shintani functions.

1. Introduction

Shintani function is originally introduced by Shintani for p-adic linear group $GL(n, k)$, where k is a finite extension of the p-adic field $\mathbf{Q}_p$ [1]. He defined some "Whittaker function" on $GL(n, k)$ and obtained the explicit formulas of them. Moreover, he proved the uniqueness of his function. Later, a more detail study of Shintani functions for $GL(n)$ was done by Murase and Sugano [2] (see also [3]). They obtained new kinds of integral formulas for the L-functions in terms of the global Shintani functions and proved the multiplicity one theorem of the local one at the finite primes.

On the other hand, the multiplicity and explicit formulas of the Archimedean Shintani functions were more recently investigated by some mathematicians. For example, Hirano studied the Shintani functions on $GL(2, \mathbf{R})$ [4] and $GL(2, \mathbf{C})$ [5], Tsuzuki on $SU(1, 1)$ [6] and $U(n, 1)$ [7], and Moriyama on $Sp(2, \mathbf{R})$ [8, 9]. They constructed the differential equations satisfied by the radial part of the Shintani functions and obtained the explicit formulas by solving them. Most of them are expressed by some linear combinations of the Gaussian hypergeometric functions. Moreover, the dimensions of the spaces of Shintani functions are obtained, which are sometimes bigger than 1.

In this paper, we investigate the Shintani functions on $G = SL(3, \mathbf{R})$, attached to the principal series representations of G. Now we explain the definition of the Shintani functions on G. We take

$$H = \left\{ \begin{pmatrix} H_1 & 0 \\ 0 & h_2 \end{pmatrix} \in G \mid (H_1, h_2) \in GL(2, \mathbf{R}) \times GL(1, \mathbf{R}) \right\} \tag{1.1}$$

as a subgroup of G and take $K = SO(3)$ as a maximal compact subgroup of G. Let π be an arbitrary irreducible unitary representation of G and (η, V_η) an irreducible unitary representation of H, and let $C_\eta^\infty(H\backslash G)$ be the space of smooth functions $F : G \to V_\eta$ satisfying $F(hg) = \eta(h)F(g)((h,g) \in H \times G)$. We consider the intertwining space $I_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbf{C}, K)}(\pi, C_\eta^\infty(H\backslash G))$ and its restriction

$$I_{\eta,\pi} \longrightarrow \mathrm{Hom}_K\Big(\tau, C_\eta^\infty(H \setminus G)\Big) \cong C_{\eta,\tau^*}(L) \tag{1.2}$$

to the minimal K-type (τ, V_τ) of π, where (τ^*, V_{τ^*}) is the contragredient representation of (τ, V_τ), $L := H\backslash G/K$ and $C_{\eta,\tau^*}^\infty(L)$ is the space of smooth functions $F : G \to V_\eta \otimes V_{\tau^*}$ satisfying $F(hgk) = (\eta(h) \otimes \tau^*(k^{-1}))F(g)$ for $(h, g, k) \in H \times G \times K$. The function which belongs to the image of above map is called the Shintani function. In this paper, we assume that π is the irreducible unitary principal series representation of G and η is the unitary character of H. The study of Shintani functions for the general unitary representation η of H is a further problem.

In Section 4, we investigate the Shintani functions attached to the spherical (or class one) principal series representations. These representations have unique K-fixed vector, and hence the minimal K-type is one-dimensional. In this case, the explicit formulas of Shintani functions are obtained by solving two Casimir equations which are characterized by the action of the center of universal enveloping algebra. We also obtain the necessary condition of the existence of nonzero Shintani functions and prove that the dimension of the space of Shintani functions is equal to or less than 1 (Theorem 4.8).

On the other hand, in Section 5, we investigate the Shintani functions attached to the nonspherical principal series representations, whose minimal K-type is three-dimensional representation of K. In this case, we construct two kinds of differential equations. One is the Casimir equation we used in Section 4, and the other is the gradient equation. The key point is as follows. We have three different nonspherical principal series with the same infinitesimal characters $Z(\mathfrak{g}) \to \mathbf{C}$. We cannot distinguish them only by the elements of $Z(\mathfrak{g})$. This is the reason we need the gradient operator which has distinct eigenvalues for different nonspherical principal series. By combining these equations, we obtain the explicit formulas of the Shintani functions, the necessary condition of the existence of nonzero Shintani functions, and prove that the dimension of the space of Shintani functions is equal to or less than 1 (Theorem 5.7).

As an application of the results of this paper, our explicit formulas for Shintani functions will be useful to compute the local zeta integral in the theory of Murase and Sugano ([2, 3]; (see also [10], in the case of $U(n,1)$). Furthermore, the author thinks these results are interesting themselves in view of the harmonic analysis on homogeneous spaces.

2. Preliminaries

2.1. Groups and Algebras

Let G be the real reductive Lie group $SL(3,\mathbf{R})$ and $\mathfrak{g} = \mathfrak{sl}(3,\mathbf{R})$ its Lie algebra. The Cartan involution $\theta : G \to G$ is defined by $\theta(g) = ({}^t g)^{-1}$ $(g \in G)$, and its differential $d\theta : \mathfrak{g} \to \mathfrak{g}$ is given by $d\theta(X) = -{}^t X$ $(X \in \mathfrak{g})$, where ^{t} means the transposition of matrices. Then the fixed subgroup of θ in G is equal to $K = SO(3)$, which is the maximal compact subgroup of G. Next, we define another involutive automorphism σ of G by $\sigma(g) = JgJ$ $(g \in G)$, where $J = \operatorname{diag}(-1,-1,1)$. Its differential $d\sigma : \mathfrak{g} \to \mathfrak{g}$ is given by $d\sigma(X) = JXJ$ $(X \in \mathfrak{g})$. The fixed subgroup H of σ in G is isomorphic to $GL(2,\mathbf{R})$, that is,

$$H = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \in G \right\} \cong GL(2,\mathbf{R}). \tag{2.1}$$

We define $+1$ and -1 eigenspaces of $d\theta$, $d\sigma$ in $\mathfrak{g}$ by

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} \mid d\theta(X) = X\}, & \mathfrak{p} &= \{X \in \mathfrak{g} \mid d\theta(X) = -X\}, \\ \mathfrak{h} &= \{X \in \mathfrak{g} \mid d\sigma(X) = X\}, & \mathfrak{q} &= \{X \in \mathfrak{g} \mid d\sigma(X) = -X\}. \end{aligned} \tag{2.2}$$

Then, $\mathfrak{k}, \mathfrak{h}$ are the Lie algebras of K, H, respectively. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$. Let $E_{ij} \in M(3,\mathbf{R})$ be the matrix whose (i,j)-component is 1 and the other components are 0 $(1 \le i, \ j \le 3)$. For $1 \le i < j \le 3$, we put $K_{ij} = E_{ij} - E_{ji}$, $X_{ij} = E_{ij} + E_{ji}$, $H_{ij} = E_{ii} - E_{jj}$. Then, we have

$$\mathfrak{k} = \bigoplus_{i<j} \mathbf{R} K_{ij}, \qquad \mathfrak{p} = \bigoplus_{i<j} \mathbf{R} X_{ij} \oplus \mathbf{R} H_{12} \oplus \mathbf{R} H_{23}. \tag{2.3}$$

Next, we take

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.4}$$

as a basis of $\mathfrak{h}$. We have $\mathfrak{p} \cap \mathfrak{q} = \mathbf{R} X_{13} \oplus \mathbf{R} X_{23}$, and we take $\mathfrak{a} = \mathbf{R} X_{13}$ as a maximal Abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. We define a subgroup A of G by

$$A = \exp(\mathfrak{a}) = \left\{ a_t := \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \mid t \in \mathbf{R} \right\}. \tag{2.5}$$

Then, G has a decomposition $G = HAK$. Throughout this paper, we put $L := H \backslash G / K$.

For a Lie algebra $\mathfrak{l}$, we denote its complexification by $\mathfrak{l}_{\mathbf{C}}$, that is, $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$.

2.2. The Principal Series Representations

As a representation of G, we take the principal series representation defined as follows. Let P_0 be a minimal parabolic subgroup of G given by the upper triangular matrices in G and $P_0 = MA_{P_0}N$ the Langlands decomposition of P_0 with

$$A_{P_0} = \{\mathrm{diag}(a_1, a_2, a_3) \mid a_i > 0,\ a_1a_2a_3 = 1\},$$
$$M = K \cap \{\text{diagonals in } G\},$$
$$N = \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in G \mid x_1, x_2, x_3 \in \mathbf{R} \right\}. \tag{2.6}$$

To define a principal series representation with respect to the minimal parabolic subgroup P_0 of G, we firstly fix a character σ of M and a linear form $\nu \in \mathfrak{a}_{P_0}^* \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_{P_0}, \mathbf{C})$, where $\mathfrak{a}_{P_0}$ is the Lie algebra of A_{P_0}. We write

$$\nu(\mathrm{diag}(t_1, t_2, t_3)) = \nu_1 t_1 + \nu_2 t_2 \tag{2.7}$$

for $\mathrm{diag}(t_1, t_2, t_3) \in \mathfrak{a}_{P_0}$. Then, we can define a representation $\sigma \otimes a^{\nu}$ of MA_{P_0} and extend this to P_0 by the identification $P_0/N \simeq MA_{P_0}$, taking the trivial representation $\mathbf{1}_N$ as the representation of N. Then, the induced representation

$$\pi_{\sigma,\nu} := C^{\infty}\mathrm{Ind}_{P_0}^{G}(\sigma \otimes a^{\nu+\rho} \otimes 1_N) \tag{2.8}$$

is called the principal series representation of G. Here, ρ is the half sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ given by $a^{\rho} = a_1^2 a_2$, for $a = \mathrm{diag}(a_1, a_2, a_3) \in A_{P_0}$.

Concretely, the representation space is given by

$$C^{\infty}_{(M,\sigma)}(K) := \{f \in C^{\infty}(K) \mid f(mk) = \sigma(m)f(k),\ m \in M,\ k \in K\}, \tag{2.9}$$

and the action of G is defined by

$$(\pi_{\sigma,\nu}(x)f)(k) = a(kx)^{\nu+\rho} f(\kappa(kx)) \quad (x \in G, k \in K). \tag{2.10}$$

Here, for $g \in G$, $g = n(g)a(g)\kappa(g)$ $(n(g) \in N,\ a(g) \in A_{P_0},\ \kappa(g) \in K)$ is the Iwasawa decomposition. Throughout this paper, we assume that the representation $\pi_{\sigma,\nu}$ is irreducible. Moreover, we assume that ν_1, ν_2 are the elements of $\sqrt{-1}\mathbf{R}$. Then, this representation becomes unitary.

Next, we define characters σ_j $(j = 0, 1, 2, 3)$ of M as follows. The group M consisting of four elements is a finite Abelian group of $(2,2)$-type, and its elements except for the unity are given by

$$m_1 = \mathrm{diag}(1, -1, -1), \qquad m_2 = \mathrm{diag}(-1, 1, -1), \qquad m_3 = \mathrm{diag}(-1, -1, 1). \tag{2.11}$$

Table 1

	m_1	m_2	m_3
σ_1	1	−1	−1
σ_2	−1	1	−1
σ_3	−1	−1	1

The set $\widehat{M}$ consists of 4 characters $\{\sigma_j \mid j = 0,1,2,3\}$, where σ_0 is the trivial character of M and $\sigma_1, \sigma_2, \sigma_3$ are defined by Table 1.

The following proposition (see [11, Proposition 1.1]) gives the correspondence between the character σ of M and the minimal K-type of the principal series representation $\pi_{\sigma,\nu}$ of G.

Proposition 2.1. (1) *If σ is the trivial character of M, the representation $\pi_{\sigma,\nu}$ is spherical or class one. That is, it has a unique K-invariant vector in $H_{\sigma,\nu}$.*

(2) *If σ is not trivial, the minimal K-type of the restriction $\pi_{\sigma,\nu}|_K$ to K is a 3-dimensional representation of K, which is isomorphic to the unique standard one (τ_2, V_2). The multiplicity of this minimal K-type is one:*

$$\dim_{\mathbf{C}} \mathrm{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1. \tag{2.12}$$

3. The Space of Shintani Functions

3.1. The Definition of Shintani Functions

As a representation of H, we take the unitary character $\eta = \eta_{s,k} : H \to \mathbf{C}^\times$ ($s \in \sqrt{-1}\mathbf{R}$, $k \in \{0,1\}$) defined by

$$\eta\left(\left(\begin{array}{cc|c} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ \hline 0 & 0 & h_1 \end{array}\right)\right) = \det(H_1)^k |\det(H_1)|^{s-k}, \qquad H_1 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \in GL(2,\mathbf{R}). \tag{3.1}$$

Let $\eta = \eta_{s,k}$ be the unitary character of H defined previously. We consider the induced representation $C^\infty \mathrm{Ind}_H^G(\eta)$ with the representation space

$$C_\eta^\infty(H \setminus G) = \{F \in C^\infty(G) \mid F(hg) = \eta(h)F(g),\ \forall h \in H,\ \forall g \in G\}. \tag{3.2}$$

G acts on this space by right translation. Let $\pi_{\sigma,\nu}$ be the principal series representation of G. We consider the intertwining space

$$I_{\eta,\pi_{\sigma,\nu}} := \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}\left(\pi_{\sigma,\nu}, C_\eta^\infty(H \setminus G)\right). \tag{3.3}$$

We denote its image by $S_{\eta,\pi_{\sigma,\nu}}$, that is,

$$S_{\eta,\pi_{\sigma,\nu}} := \bigcup_{T\in I_{\eta,\pi_{\sigma,\nu}}} \mathrm{Image}(T). \tag{3.4}$$

We call the element of $S_{\eta,\pi_{\sigma,\nu}}$ the Shintani function of type $(\eta,\pi_{\sigma,\nu})$. Let (τ,V_τ) be the K-type of the principal series representation $\pi_{\sigma,\nu}$, and let $\iota:\tau\longrightarrow\pi_{\sigma,\nu}$ be the K-embedding of τ and ι^* the pullback via ι. Then, the map

$$\iota^* : I_{\eta,\pi_{\sigma,\nu}} \longrightarrow \mathrm{Hom}_K\left(\tau, C_\eta^\infty(H\setminus G)\right) \cong C_{\eta,\tau^*}^\infty(L) \tag{3.5}$$

gives the restriction of $T\in S_{\eta,\pi_{\sigma,\nu}}$ to τ, where τ^* is the contragredient representation of τ and the space $C_{\eta,\tau^*}^\infty(L)$ is defined by

$$C_{\eta,\tau^*}^\infty(L) = \left\{F : G \longrightarrow V_{\tau^*} \mid F\left(hgk^{-1}\right) = \eta(h)\tau^*(k)F(g),\ \forall(h,g,k)\in H\times G\times K\right\}. \tag{3.6}$$

We denote the image of $I_{\eta,\pi_{\sigma,\nu}}$ in $C_{\eta,\tau^*}^\infty(L)$ by $C_{\eta,\tau^*}^\infty(L)_{\pi_{\sigma,\nu}}$, and the element of this space is called the Shintani function of type $(\pi_{\sigma,\nu},\eta,\tau)$.

3.2. The A-Radial Part

Because G has the decomposition $G = HAK$, the element of $C_{\eta,\tau}^\infty(L)$ is characterized by its restriction to A. We denote the centralizer and the normalizer of A in $K\cap H$ by $Z_{K\cap H}(A)$, $N_{K\cap H}(A)$, respectively. It is easy to verify that $K\cap H$, $Z_{K\cap H}(A)$, and $N_{K\cap H}(A)$ are given as follows.

Lemma 3.1. *One has*

$$K\cap H = \left\{\left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ \hline 0 & 0 & 1\end{array}\right) \mid \theta\in\mathbf{R}\right\} \bigsqcup \left\{\left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0\\ \sin\theta & -\cos\theta & 0\\ \hline 0 & 0 & -1\end{array}\right) \mid \theta\in\mathbf{R}\right\},$$

$$Z_{K\cap H}(A) = \left\{I, \begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\right\}, \tag{3.7}$$

$$N_{K\cap H}(A) = \left\{I, \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\right\}.$$

Here, I is the unit element of $GL(3,\mathbf{R})$.

Let $w_0 := \mathrm{diag}(-1,-1,1)$. Then, $w_0 Z_{K\cap H}(A)$ is the unique nontrivial element of $W = N_{K\cap H}(A)/Z_{K\cap H}(A)$. Let $\eta : H \to \mathbf{C}$ be the unitary character of H and (τ, V_τ) the finite-dimensional representation of K. We denote by $C_W^\infty(A;\eta,\tau)$ the space of smooth functions $F : A \to V_\tau$ satisfying the following conditions:

$$\begin{aligned}
&(1)\ \eta(m)\tau(m)F(a) = F(a) \quad (\forall m \in Z_{K\cap H}(A),\ \forall a \in A),\\
&(2)\ \eta(w_0)\tau(w_0)F(a) = F\left(a^{-1}\right) \quad (\forall a \in A),\\
&(3)\ \eta(l)\tau(l)F(1) = F(1) \quad (\forall l \in K\cap H).
\end{aligned} \tag{3.8}$$

The following lemma is proved by Flensted and Jensen (see [12, Theorem 4.1]).

Lemma 3.2. *The restriction to A gives the following isomorphism:*

$$C_{\eta,\tau}^\infty(L) \cong C_W^\infty(A;\eta,\tau). \tag{3.9}$$

Through this isomorphism, we denote the image of $C_{\eta,\tau}^\infty(L)_{\pi_{\sigma,\nu}}$ in $C_W^\infty(A;\eta,\tau)$ by $C_W^\infty(A;\eta,\tau)_{\pi_{\sigma,\nu}}$. This is our target space in this paper. The following two lemmas are obvious.

Lemma 3.3. *For $t \neq 0$, one has*

$$\mathfrak{g} = \mathrm{Ad}\left(a_t^{-1}\right)\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{k}. \tag{3.10}$$

Lemma 3.4. *Let $F \in C_{\eta,\tau}^\infty(L)$. For $X \in \mathfrak{k}$, $Y \in \mathfrak{h}$, $Z \in \mathfrak{a}$, one has*

$$R\left(\left(\mathrm{Ad}\left(a_t^{-1}\right)Y\right)ZX\right)F(a_t) = \eta(Y)\tau(-X)(Zf)(a_t). \tag{3.11}$$

4. Shintani Functions Attached to the Spherical Principal Series Representations

Throughout this section, as a character of M, we take the trivial character $\sigma = \sigma_0$. Then, the principal series representation $\pi_{\sigma_0,\nu}$ is the spherical or class one principal series representation whose minimal K-type is the one-dimensional trivial representation $(\mathbf{1}, V_{\mathbf{1}})$ of K, which occurs of multiplicity one in $\pi_{\sigma_0,\nu}|_K$ (Proposition 2.1).

4.1. The Capelli Elements

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. $Z(\mathfrak{g})$ has two independent generators, and they are obtained as the Capelli elements because $\mathfrak{g} = \mathfrak{sl}_3$ is of type A_2 (see [13]). For $i = 1,2,3$, we put

$$E'_{ii} = E_{ii} - \frac{1}{3}\left(\sum_{k=1}^{3} E_{kk}\right). \tag{4.1}$$

The following proposition gives the explicit description of the independent generators of $Z(\mathfrak{g})$ (see [11]).

Proposition 4.1. *The independent generators* $\{Cp_2, Cp_3\}$ *of* $Z(\mathfrak{g})$ *are given as follows:*

$$\begin{aligned} Cp_2 &= (E'_{11}-1)E'_{22} + E'_{22}(E'_{33}+1) + (E'_{11}-1)(E'_{33}+1) - E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21}, \\ Cp_3 &= (E'_{11}-1)E'_{22}(E'_{33}+1) + E_{12}E_{23}E_{31} + E_{13}E_{21}E_{32} - (E'_{11}-1)E_{23}E_{32} - E_{13}E'_{22}E_{31} \\ &\quad - E_{12}E_{21}(E'_{33}+1). \end{aligned} \tag{4.2}$$

Since $Cp_2,\ Cp_3$ are the elements of $Z(\mathfrak{g})$, they act on $\pi_{\sigma_0,\nu}$ as the scalar operators. And since the space of Shintani functions is the image of the $(\mathfrak{g}_{\mathbb{C}}, K)$-homomorphism of $\pi_{\sigma_0,\nu}$, they act on the space of Shintani functions as the same scalar operators, respectively.

4.2. Eigenvalues of Cp_2, Cp_3

In order to construct the partial differential equations satisfied by spherical functions attached to the spherical principal series, we have to compute the eigenvalues of the actions of the Capelli elements Cp_2, Cp_3. For the spherical principal series representation, $\sigma = \sigma_0$ is the trivial character of M. Let f_0 be the generator of the minimal K-type in $H_{\sigma_0,\nu}$ normalized such that $f_0 \mid K \equiv 1$. The actions of Cp_2, Cp_3 on f_0 are computed in [11], and the result is as follows.

Proposition 4.2. *The Capelli elements* $Cp_2,\ Cp_3$ *act on* f_0 *by scalar multiples, and the eigenvalues are given as follows:*

$$\begin{aligned} Cp_2 f_0 &= -\frac{1}{3}\left(\nu_1^2 - \nu_1\nu_2 + \nu_2^2\right) f_0, \\ Cp_3 f_0 &= -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2) f_0. \end{aligned} \tag{4.3}$$

4.3. Construction of the Casimir Equations

Next, we compute the actions of Cp_2, Cp_3 on $F(a_t) \in C^{\infty}_{\eta,1}(L)|_A$. Here, $\eta = \eta_{s,k}$ is the unitary character of H.

Lemma 4.3. *For* $F(a_t) \in C^{\infty}_{\eta,\tau}(L)|_A$, *one has*

$$\begin{aligned} R\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_i\right)F(a_t) &= sF(a_t) \quad (i = 1, 4), \\ R\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_i\right)F(a_t) &= 0 \quad (i = 2, 3), \\ R(X_{13})F(a_t) &= \frac{dF}{dt}(a_t). \end{aligned} \tag{4.4}$$

Proof. By definition, we have

$$\begin{aligned} R\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_i\right)F(a_t) &= \frac{d}{du}F\left(a_t\exp\left(u\,\mathrm{Ad}\left(a_t^{-1}\right)Y_i\right)\right)\Big|_{u=0} \\ &= \frac{d}{du}F(\exp(uY_i)a_t)\Big|_{u=0} \\ &= \frac{d}{du}\eta(\exp(uY_i))\Big|_{u=0}F(a_t). \end{aligned} \tag{4.5}$$

Since $\exp(uY_1) = \mathrm{diag}(e^u, 1, e^{-u})$, we have $\eta((\exp(uY_1)) = e^{ku}\cdot e^{(s-k)u} = e^{su}$.

Therefore, we have

$$R\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_1\right)F(a_t) = sF(a_t). \tag{4.6}$$

Next, since

$$\exp(uY_2) = \begin{pmatrix} \cosh u & \sinh u & 0 \\ \sinh u & \cosh u & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.7}$$

$\eta(\exp(uY_2)) = 1^k\cdot|1|^{s-k} = 1$. Therefore, we have

$$R\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_2\right)F(a_t) = 0. \tag{4.8}$$

The computations of the actions of $\mathrm{Ad}(a_t^{-1})(Y_i)$ $(i = 3, 4)$ are similar. Finally, since $\exp(uX_{13}) = a_u$, we have

$$R(X_{13})F(a_t) = \frac{d}{du}F(a_{t+u})\Big|_{u=0} = \frac{dF}{dt}(a_t). \tag{4.9}$$

□

By simple computations of matrices, we have the following expressions of the elements in $M(3,\mathbf{R})$.

Lemma 4.4. *One has*

$$E'_{11} = \frac{\cosh^2 t + 1}{3\cosh(2t)}\mathrm{Ad}\left(a_t^{-1}\right)Y_1 - \frac{1}{3}\mathrm{Ad}\left(a_t^{-1}\right)Y_4 - \frac{1}{2}\tanh(2t)K_{13},$$

$$E'_{22} = \frac{1}{3}(-H_{12} + H_{23}),$$

$$E'_{33} = \frac{\sinh^2 t - 1}{3\cosh(2t)}\mathrm{Ad}\left(a_t^{-1}\right)Y_1 - \frac{1}{3}\mathrm{Ad}\left(a_t^{-1}\right)Y_4 + \frac{1}{2}\tanh(2t)K_{13},$$

$$E_{23} = \frac{1}{2\sinh t}\mathrm{Ad}\left(a_t^{-1}\right)Y_3 + \frac{1}{2\tanh t}K_{12} + \frac{1}{2}K_{23},$$

$$E_{13} = \frac{1}{2}X_{13} + \frac{1}{2}K_{13},$$
$$E_{12} = \frac{1}{2\cosh t}\mathrm{Ad}\left(a_t^{-1}\right)Y_2 - \frac{1}{2}\tanh t K_{23} + \frac{1}{2}K_{12}. \tag{4.10}$$

To make use of Lemma 3.4, we have to rewrite Cp_2, Cp_3 in the form of linear combinations of the elements in $(\mathrm{Ad}(a_t^{-1})\mathfrak{h})\mathfrak{a}\mathfrak{k}$. To do this, we use the following formulas which can be obtained by direct computation.

Lemma 4.5. *One has*

$$\left[K_{13}, \mathrm{Ad}\left(a_t^{-1}\right)Y_1\right] = -2\cosh(2t)X_{13}, \qquad \left[K_{13}, \mathrm{Ad}\left(a_t^{-1}\right)Y_4\right] = -\cosh(2t)X_{13},$$
$$\left[K_{12}, \mathrm{Ad}\left(a_t^{-1}\right)Y_3\right] = \sinh t X_{13},$$
$$\left[K_{23}, \mathrm{Ad}\left(a_t^{-1}\right)Y_3\right] = -\frac{2\sinh^3 t}{\cosh(2t)}\mathrm{Ad}\left(a_t^{-1}\right)Y_1 + 2\sinh t\,\mathrm{Ad}\left(a_t^{-1}\right)Y_4 + \frac{\cosh t}{\cosh(2t)}K_{13},$$
$$[K_{13}, X_{13}] = \frac{2}{\cosh(2t)}\mathrm{Ad}\left(a_t^{-1}\right)Y_1 - 2\tanh(2t)K_{13},$$
$$\left[K_{23}, \mathrm{Ad}\left(a_t^{-1}\right)Y_2\right] = -\cosh t X_{13},$$
$$\left[K_{12}, \mathrm{Ad}\left(a_t^{-1}\right)Y_2\right] = \frac{2\cosh^3 t}{\cosh(2t)}\mathrm{Ad}\left(a_t^{-1}\right)Y_1 - 2\cosh t\,\mathrm{Ad}\left(a_t^{-1}\right)Y_4 - \frac{\sinh t}{\cosh(2t)}K_{13},$$
$$[K_{12}, X_{13}] = -\frac{1}{\sinh t}\mathrm{Ad}\left(a_t^{-1}\right)Y_3 - \frac{1}{\tanh t}K_{12},$$
$$[K_{23}, X_{13}] = \frac{1}{\cosh t}\mathrm{Ad}\left(a_t^{-1}\right)Y_2 - \tanh t K_{23},$$
$$\left[X_{13}, \mathrm{Ad}\left(a_t^{-1}\right)Y_2\right] = -\tanh t\,\mathrm{Ad}\left(a_t^{-1}\right)Y_2 - \frac{1}{\cosh t}K_{23},$$
$$\left[K_{13}, \mathrm{Ad}\left(a_t^{-1}\right)Y_2\right] = -\frac{1}{\tanh t}\mathrm{Ad}\left(a_t^{-1}\right)Y_3 - \frac{\cosh(2t)}{\sinh t}K_{12},$$
$$\left[K_{13}, \mathrm{Ad}\left(a_t^{-1}\right)Y_3\right] = \tanh t\,\mathrm{Ad}\left(a_t^{-1}\right)Y_2 - \frac{\cosh(2t)}{\cosh t}K_{23},$$
$$\left[X_{13}, \mathrm{Ad}\left(a_t^{-1}\right)Y_1\right] = -2\tanh(2t)\mathrm{Ad}\left(a_t^{-1}\right)Y_1 - \frac{2}{\cosh(2t)}K_{13},$$
$$\left[X_{13}, \mathrm{Ad}\left(a_t^{-1}\right)Y_4\right] = -\tanh(2t)\mathrm{Ad}\left(a_t^{-1}\right)Y_1 - \frac{1}{\cosh(2t)}K_{13},$$
$$\left[K_{12}, \mathrm{Ad}\left(a_t^{-1}\right)Y_1\right] = -\frac{\cosh(2t)}{\cosh t}\mathrm{Ad}\left(a_t^{-1}\right)Y_2 - \tanh t K_{23},$$
$$\left[K_{12}, \mathrm{Ad}\left(a_t^{-1}\right)Y_4\right] = \frac{1-\sinh^2 t}{\cosh t}\mathrm{Ad}\left(a_t^{-1}\right)Y_2 - 2\tanh t K_{23},$$

$$\left[K_{23}, \mathrm{Ad}\left(a_t^{-1}\right)Y_1\right] = -\frac{\cosh(2t)}{\sinh t}\mathrm{Ad}\left(a_t^{-1}\right)Y_3 - \frac{1}{\tanh t}K_{12},$$

$$\left[K_{23}, \mathrm{Ad}\left(a_t^{-1}\right)Y_4\right] = -\frac{1+\cosh^2 t}{\sinh t}\mathrm{Ad}\left(a_t^{-1}\right)Y_3 - \frac{2}{\tanh t}K_{12}. \tag{4.11}$$

Here, $[X, Y] := XY - YX$ *is the Lie bracket on* $\mathfrak{g}$.

By using Lemma 4.5, we can rewrite Cp_2, Cp_3 as we wished. Now, since for all $F(a_t) \in C^{\infty}_{\eta,\mathbf{1}}(L)|_A$ is annihilated by the action of $U(\mathfrak{g})\mathfrak{k}$ and $\mathrm{Ad}(a_t^{-1})Y_2U(\mathfrak{g})\mathrm{Ad}(a_t^{-1})Y_3U(\mathfrak{g})$, and the actions of $\mathrm{Ad}(a_t^{-1})Y_1$ and $\mathrm{Ad}(a_t^{-1})Y_4$ on F are the same (the multiplication by s), we may regard Cp_2, Cp_3 as the elements in $U(\mathfrak{g})(\mathrm{mod}\ \mathfrak{P})$, where $\mathfrak{P}$ is the subalgebra of $U(\mathfrak{g})$ defined by

$$\mathfrak{P} = \left(\mathrm{Ad}\left(a_t^{-1}\right)Y_1 - \mathrm{Ad}\left(a_t^{-1}\right)Y_4\right)U(\mathfrak{g}) + \mathrm{Ad}\left(a_t^{-1}\right)Y_2U(\mathfrak{g}) + \mathrm{Ad}\left(a_t^{-1}\right)Y_3U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}. \tag{4.12}$$

Lemma 4.6. *One has the congruences*

$$Cp_2 \equiv -\frac{1}{4}X_{13}^2 - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)X_{13} + \left(\frac{1}{4}\tanh^2(2t) - \frac{1}{3}\right)\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_1\right)^2 - 1\ (\mathrm{mod}\ \mathfrak{P}),$$

$$Cp_3 \equiv -\frac{1}{12}\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_1\right)X_{13}^2 - \frac{1}{6}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_1\right)X_{13} + \left(-\frac{1}{27} + \frac{1}{12}\tanh^2(2t)\right)\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_1\right)^3 - \frac{1}{3}\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_1\right)\ (\mathrm{mod}\ \mathfrak{P}). \tag{4.13}$$

By combining Proposition 4.2, and Lemmas 4.3 and 4.6, we have the following two differential equations.

Theorem 4.7. *The Shintani function* $F(a_t) \in C^{\infty}_{\eta,\mathbf{1}}(L)_{\pi_{\sigma_0,\nu}}|_A$ *satisfies the following equations:*

(1) *the differential equation obtained from the action of* Cp_2*:*

$$-\frac{1}{4}\frac{d^2F}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\frac{dF}{dt}(a_t) + \left\{\left(-\frac{1}{3} + \frac{1}{4}\tanh^2(2t)\right)s^2 - 1\right\}F(a_t) = \lambda_2 F(a_t); \tag{4.14}$$

(2) *the differential equation obtained from the action of* Cp_3*:*

$$-\frac{1}{12}s\frac{d^2F}{dt^2}(a_t) - \frac{1}{6}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)s\frac{dF}{dt}(a_t) + \left\{\left(-\frac{2}{27} + \frac{1}{12}\tanh^2(2t)\right)s^3 - \frac{1}{3}s\right\}F(a_t) = \lambda_3 F(a_t). \tag{4.15}$$

Here,

$$\lambda_2 = -\frac{1}{3}\left(\nu_1^2 - \nu_1\nu_2 + \nu_2^2\right),$$
$$\lambda_3 = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2) \tag{4.16}$$

are the eigenvalues of the Capelli elements on principal series representations.

Equations (4.15)–(4.14) × (1/3)s *give*

$$\left\{\frac{1}{27}s^3 + \frac{1}{3}\lambda_2 s - \lambda_3\right\}F(a_t) = 0. \tag{4.17}$$

Therefore, if $F(a_t)$ is not identically zero, we have

$$s^3 - 3\left(\nu_1^2 - \nu_1\nu_2 + \nu_2^2\right)s + (2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2) = 0. \tag{4.18}$$

By solving this equation, we have

$$s = 2\nu_1 - \nu_2,\ 2\nu_2 - \nu_1,\ -\nu_1 - \nu_2. \tag{4.19}$$

Therefore, one of the necessary conditions of the existence of nontrivial Shintani functions is that the parameter s is one of the above three values. Now, we assume that s satisfies this condition. We put $x = \tanh(2t)$, $\widetilde{F}(x) := F(a_t)$ in (4.14). Then, we have

$$-4x(1-x)^2\frac{d^2\widetilde{F}}{dx^2}(x) - 4(1-x)^2\frac{d\widetilde{F}}{dx}(x) + \left\{\left(-\frac{1}{3} + \frac{1}{4}x\right)s^2 - 1 - \lambda_2\right\}\widetilde{F}(x) = 0. \tag{4.20}$$

Next, we put $\widetilde{F}(x) = (1-x)^{\mu}G_0(x)$ $(\mu \in \mathbf{C})$. Then, G_0 satisfies

$$\begin{aligned} &-4x(1-x)^{\mu+2}\frac{d^2G_0}{dx^2}(x) + (-4 + (8\mu+4)x)(1-x)^{\mu+1}\frac{dG_0}{dx}(x) \\ &\quad + \left\{-4\mu(\mu-1)x + 4\mu - 4\mu x + \left(-\frac{1}{3} + \frac{1}{4}x\right)s^2 - 1 - \lambda_2\right\}(1-x)^{\mu}G_0(x) = 0. \end{aligned} \tag{4.21}$$

We want to divide the left-hand side of (4.21) by $(1-x)^{\mu+1}$. To do this, we take $\mu \in \mathbf{C}$ so that μ satisfies

$$-4\mu^2 + 4\mu - \frac{1}{12}s^2 - 1 - \lambda_2 = 0. \tag{4.22}$$

The value of $\mu \in \mathbf{C}$ is as follows.

(1) If $s = 2\nu_1 - \nu_2$, $\mu = (2 \pm \nu_2)/4$. So we take $\mu = (2 + \nu_2)/4$.

(2) If $s = 2\nu_2 - \nu_1$, $\mu = (2 \pm \nu_1)/4$. So we take $\mu = (2 + \nu_1)/4$.

(3) If $s = -\nu_1 - \nu_2$, $\mu = \{2 \pm (\nu_1 - \nu_2)\}/4$. So we take $\mu = (2 + \nu_1 - \nu_2)/4$.

For this μ, the left-hand side of (4.21) is divided by $(1-x)^{\mu+1}$, and the equation becomes

$$x(x-1)\frac{d^2G_0}{dx^2}(x)+\left(-1+(1+2\mu)x\right)\frac{dG_0}{dx}+\left(\mu^2-\frac{1}{16}s^2\right)G_0(x)=0. \tag{4.23}$$

This is the Gaussian hypergeometric differential equation. Note that the Shintani function $F(a_t)$ on A is regular at the origin ($\Leftrightarrow x=0$). Therefore, $G_0(x)=(1-x)^{-\mu}\widetilde{F}(x)$ is also regular at $x=0$. Equation (4.23) has just one solution which is regular around $x=0$ (up to constant multiples), and it is given by

$$G_0(x)={}_2F_1(\alpha,\beta;1;x), \tag{4.24}$$

where ${}_2F_1$ is the Gaussian hypergeometric function and $\alpha,\beta\in\mathbf{C}$ are defined by

$$\begin{aligned}1+\alpha+\beta&=1+2\mu,\\ \alpha\beta&=\mu^2-\frac{1}{16}s^2.\end{aligned} \tag{4.25}$$

Explicitly, by solving these equations, α,β are given as follows.

(1) In case of $s=2\nu_1-\nu_2$, $(\alpha,\beta)=((\nu_1+1)/2,(-\nu_1+\nu_2+1)/2)$.

(2) In case of $s=2\nu_2-\nu_1$, $(\alpha,\beta)=((\nu_2+1)/2,(-\nu_2+\nu_1+1)/2)$.

(3) In case of $s=-\nu_1-\nu_2$, $(\alpha,\beta)=((\nu_1+1)/2,(-\nu_2+1)/2)$.

Finally, we consider the three conditions in (3.8). Condition (1) is equivalent to $(-1)^k F(a_t)=F(a_t)$. Condition (2) always holds. Condition (3) is equivalent to $(-1)^kF(1)=F(1)$, which holds if condition (1) is satisfied. Summing up, we have the following theorem.

Theorem 4.8. *Let $\eta=\eta_{s,k}$ be the unitary character of H defined by* (3.1)*. Then, the necessary condition of the existence of the nontrivial elements in $C^\infty_{\eta,\mathbf{1}}(L)_{\pi_{\sigma_0,\nu}}$ is that*

$$k=0,\qquad s=2\nu_1-\nu_2,\ 2\nu_2-\nu_1,\ -\nu_1-\nu_2. \tag{4.26}$$

Suppose that this condition is satisfied and nontrivial Shintani functions exist. If one puts $x=\tanh^2(2t)$, $F(a_t)=\widetilde{F}(x)\in C^\infty_{\eta,\mathbf{1}}(L)_{\pi_{\sigma_0,\nu}}|_A$ is given as follows (up to constant multiples).

(1) *In case of $s=2\nu_1-\nu_2$, one has*

$$\widetilde{F}(x)=(1-x)^{(2+\nu_2)/4}{}_2F_1\left(\frac{\nu_1+1}{2},\frac{-\nu_1+\nu_2+1}{2};1;x\right). \tag{4.27}$$

(2) *In case of $s=2\nu_2-\nu_1$, one has*

$$\widetilde{F}(x)=(1-x)^{(2+\nu_1)/4}{}_2F_1\left(\frac{\nu_2+1}{2},\frac{-\nu_2+\nu_1+1}{2};1;x\right). \tag{4.28}$$

(3) *In case of* $s = -\nu_1 - \nu_2$, *one has*

$$\widetilde{F}(x) = (1-x)^{(2+\nu_1-\nu_2)/4}\,{}_2F_1\left(\frac{\nu_1+1}{2}, \frac{-\nu_2+1}{2}; 1; x\right). \tag{4.29}$$

Especially, one has

$$\dim C^{\infty}_{\eta,\mathbf{1}}(L)_{\pi_{\sigma_0,\nu}} \leq 1. \tag{4.30}$$

5. Shintani Functions Attached to the Nonspherical Principal Series Representations

In this section, as a character of M, we take a nontrivial character $\sigma = \sigma_i$ ($i = 1,2,3$). Then, the minimal K-type of $\pi_{\sigma_i,\nu}$ is the three-dimensional representation of K which is isomorphic to the tautological representation $\tau_2 : K = SO(3,\mathbf{R}) \hookrightarrow GL(3,\mathbf{R})$ which occurs of multiplicity one in $\pi_{\sigma_i,\nu}|_K$. We take τ_2^* instead of τ_2 as a minimal K-type of $\pi_{\sigma_i,\nu}$. The representation space of τ_2 is denoted by V_{τ_2} $(= \mathbf{R}^3)$, and we take $s_1 = {}^t(1,0,0)$, $s_2 = {}^t(0,1,0)$, $s_3 = {}^t(0,0,1)$ as a basis of V_{τ_2}. Let $\Psi \in C^{\infty}_{\eta,\tau_2}(L)_{\pi_{\sigma_i,\nu}}$ be the Shintani function. Then, Ψ is expressed by

$$\Psi(g) = F_0(g)s_1 + G_0(g)s_2 + H_0(g)s_3. \tag{5.1}$$

Ψ is characterized by its restriction to A. To investigate $\Psi|_A$, we construct two kinds of differential equations. One is the Casimir equation of degree two and the other is the gradient equation (or the Dirac-Schmidt equation).

5.1. The Casimir Equation

Firstly, we construct the Casimir equation of degree two. Since the Capelli element Cp_2 acts on the representation space of the principal series representation $\pi_{\sigma_i,\nu}$ as a scalar operator (λ_2-multiple) and the space of Shintani functions $C^{\infty}_{\eta,\tau_2}(L)_{\pi_{\sigma_i,\nu}}$ is the image of $(\mathfrak{g}_{\mathbf{C}}, K)$-homomorphism of $\pi_{\sigma_i,\nu}$, Cp_2 acts on this space as the same scalar operator. Since for all $\Psi(a_t) \in C^{\infty}_{\eta,\tau_2}(L)|_A$ is annihilated by the action of $\mathrm{Ad}(a_t^{-1})Y_2U(\mathfrak{g})$, $\mathrm{Ad}(a_t^{-1})Y_3U(\mathfrak{g})$ and the actions of $\mathrm{Ad}(a_t^{-1})Y_1$ and $\mathrm{Ad}(a_t^{-1})Y_4$ on F are the same (the multiplication by s), we may regard Cp_2 as the element in $U(\mathfrak{g})(\bmod\ \mathfrak{P}')$, where $\mathfrak{P}'$ is a subalgebra of $U(\mathfrak{g})$ defined by

$$\mathfrak{P}' = \left(\mathrm{Ad}\left(a_t^{-1}\right)Y_1 - \mathrm{Ad}\left(a_t^{-1}\right)Y_4\right)U(\mathfrak{g}) + \mathrm{Ad}\left(a_t^{-1}\right)Y_2U(\mathfrak{g}) + \mathrm{Ad}\left(a_t^{-1}\right)Y_3U(\mathfrak{g}). \tag{5.2}$$

By using Lemmas 4.4 and 4.5, we can rewrite Cp_2 in Proposition 4.1 as follows.

Lemma 5.1. *One has*

$$Cp_2 \equiv -\frac{1}{4}X_{13}^2 - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)X_{13} + \left(\frac{1}{4}\tanh^2(2t) - \frac{1}{3}\right)\left(\mathrm{Ad}\left(a_t^{-1}\right)Y_1\right)^2 - 1$$
$$+ \frac{\tanh(2t)}{2\cosh(2t)}\mathrm{Ad}\left(a_t^{-1}\right)Y_1K_{13} + \left(-\frac{1}{4}\tanh^2(2t) + \frac{1}{4}\right)K_{13}^2$$

$$+\left(-\frac{1}{4}\tanh^2 t+\frac{1}{4}\right)K_{12}^2+\left(\frac{1}{4}-\frac{1}{4}\tanh^2 t\right)K_{23}^2$$
$$\cdot\left(\operatorname{mod}\left(\operatorname{Ad}\left(a_t^{-1}\right)Y_1-\operatorname{Ad}\left(a_t^{-1}\right)Y_4\right)U(\mathfrak{g})+\operatorname{Ad}\left(a_t^{-1}\right)Y_2U(\mathfrak{g})+\operatorname{Ad}\left(a_t^{-1}\right)Y_3U(\mathfrak{g})\right). \tag{5.3}$$

By using this lemma, the action of Cp_2 on $\Psi(a_t) = F_0(a_t)s_1 + G_0(a_t)s_2 + H_0(a_t)s_3 \in C^\infty_{\eta,\tau_2}(L)|_A$ can be computed easily. We have

$$Cp_2\Psi(a_t) = F'(a_t)s_1 + G'(a_t)s_2 + H'(a_t)s_3, \tag{5.4}$$

where

$$F'(a_t) = -\frac{1}{4}\frac{d^2F_0}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t)+\frac{1}{\tanh(2t)}\right)\frac{dF_0}{dt}(a_t)$$
$$+\left\{\left(\frac{1}{4}\tanh^2(2t)-\frac{1}{3}\right)s^2+\frac{1}{4}\tanh^2(2t)+\frac{1}{4\tanh^2 t}-\frac{3}{2}\right\}F_0(a_t)-\frac{\tanh(2t)}{2\cosh(2t)}sH_0(a_t),$$
$$G'(a_t) = -\frac{1}{4}\frac{d^2G_0}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t)+\frac{1}{\tanh(2t)}\right)\frac{dG_0}{dt}(a_t)$$
$$+\left\{\left(\frac{1}{4}\tanh^2 t-\frac{1}{3}\right)s^2+\frac{1}{4}\tanh^2 t+\frac{1}{4\tanh^2 t}-\frac{3}{2}\right\}G_0(a_t),$$
$$H'(a_t) = -\frac{1}{4}\frac{d^2H_0}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t)+\frac{1}{\tanh(2t)}\right)\frac{dH_0}{dt}(a_t)$$
$$+\left\{\left(\frac{1}{4}\tanh^2(2t)-\frac{1}{3}\right)s^2+\frac{1}{4}\tanh^2(2t)+\frac{1}{4}\tanh^2 t-\frac{3}{2}\right\}H_0(a_t)+\frac{\tanh(2t)}{2\cosh(2t)}sF_0(a_t). \tag{5.5}$$

Since $\Psi(a_t)$ satisfies $Cp_2\Psi(a_t) = \lambda_2\Psi(a_t)$, we have the following three differential equations.

Theorem 5.2. *For $\Psi(a_t) = F_0(a_t)s_1 + G_0(a_t)s_2 + H_0(a_t)s_3 \in C^\infty_{\eta,\tau_2}(L)_{\pi_{\sigma_i,\nu}}|_A$, the functions F_0, G_0, H_0 satisfy the following equations:*

$$-\frac{1}{4}\frac{d^2F_0}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t)+\frac{1}{\tanh(2t)}\right)\frac{dF_0}{dt}(a_t)$$
$$+\left\{\left(\frac{1}{4}\tanh^2(2t)-\frac{1}{3}\right)s^2+\frac{1}{4}\tanh^2(2t)+\frac{1}{4\tanh^2 t}-\frac{3}{2}\right\}F_0(a_t) \tag{5.6}$$
$$-\frac{\tanh(2t)}{2\cosh(2t)}sH_0(a_t) = \lambda_2F_0(a_t),$$

$$
\begin{aligned}
&-\frac{1}{4}\frac{d^2G_0}{dt^2}(a_t)-\frac{1}{2}\left(\tanh(2t)+\frac{1}{\tanh(2t)}\right)\frac{dG_0}{dt}(a_t)\\
&\quad+\left\{\left(\frac{1}{4}\tanh^2 t-\frac{1}{3}\right)s^2+\frac{1}{4}\tanh^2 t+\frac{1}{4\tanh^2 t}-\frac{3}{2}\right\}G_0(a_t)=\lambda_2 G_0(a_t),
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
&-\frac{1}{4}\frac{d^2H_0}{dt^2}(a_t)-\frac{1}{2}\left(\tanh(2t)+\frac{1}{\tanh(2t)}\right)\frac{dH_0}{dt}(a_t)\\
&\quad+\left\{\left(\frac{1}{4}\tanh^2(2t)-\frac{1}{3}\right)s^2+\frac{1}{4}\tanh^2(2t)+\frac{1}{4}\tanh^2 t-\frac{3}{2}\right\}H_0(a_t)\\
&\quad+\frac{\tanh(2t)}{2\cosh(2t)}sF_0(a_t)=\lambda_2 H_0(a_t).
\end{aligned}
\tag{5.8}
$$

5.2. The Gradient Equation

For the spherical function $\Psi(g)\in C^{\infty}_{\eta,\tau_2}(L)$, we define the right gradient operator ∇^R as follows.

Definition 5.3. For the orthonormal basis $\{X_i\}_{i=1}^5$ of $\mathfrak{p}$, the right gradient operator ∇^R is defined by

$$
\nabla^R\Psi(g):=\sum_{i=1}^{5}R(X_i)\Psi\otimes X_i^*.
\tag{5.9}
$$

Here, X_i^* is the dual basis of X_i with respect to the inner product $(X,Y)\in\mathfrak{p}\times\mathfrak{p}\to\mathrm{Tr}(XY)\in\mathbf{C}$.

The set $\{H_{12},H_{23},X_{12},X_{23},X_{13}\}$ becomes the orthonormal basis of $\mathfrak{p}$, and $\{(1/3)(2H_{12}+H_{23}),(1/3)(H_{12}+2H_{23}),(1/2)X_{12},(1/2)X_{23},(1/2)X_{13}\}$ is its dual basis. Therefore, the gradient operator ∇^R is explicitly given by

$$
\nabla^R\Psi(g)=\frac{1}{3}R(H_{12})\Psi\otimes(2H_{12}+H_{23})+\frac{1}{3}R(H_{23})\Psi\otimes(H_{12}+2H_{23})+\frac{1}{2}\sum_{i<j}R(X_{ij})\Psi\otimes X_{ij}.
\tag{5.10}
$$

We rewrite this by using the basis of $\mathfrak{p}_{\mathbf{C}}$.

Claim 1. We define five elements w_i $(0\le i\le 4)$ *in* $\mathfrak{p}_{\mathbf{C}}$ *by*

$$
\begin{aligned}
w_0&:=-2\left(H_{23}-\sqrt{-1}X_{23}\right),\qquad w_4:=-2\left(H_{23}+\sqrt{-1}X_{23}\right),\\
w_2&:=\frac{2}{3}(2H_{12}+H_{23}),\\
w_1&:=X_{13}+\sqrt{-1}X_{12},\qquad w_3:=-X_{13}+\sqrt{-1}X_{12}.
\end{aligned}
\tag{5.11}
$$

Then, $\{w_i\mid 0\le i\le 4\}$ becomes the basis of $\mathfrak{p}_{\mathbf{C}}$.

Table 2

	w_0	w_1	w_2	w_3	w_4
s_1	0	$-\frac{1}{4}(s_3+\sqrt{-1}s_2)$	$-\frac{1}{3}s_1$	$\frac{1}{4}(s_3-\sqrt{-1}s_2)$	0
s_2	$\frac{1}{2}(s_2-\sqrt{-1}s_3)$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{6}s_2$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{2}(s_2+\sqrt{-1}s_3)$
s_3	$-\frac{1}{2}(s_3+\sqrt{-1}s_2)$	$-\frac{1}{4}s_1$	$\frac{1}{6}s_3$	$\frac{1}{4}s_1$	$\frac{1}{2}(-s_3+\sqrt{-1}s_2)$

With this basis, the gradient operator ∇^R is rewritten as

$$\begin{aligned}\nabla^R\Psi &= \frac{1}{16}R(w_4)\Psi\otimes w_0+\frac{1}{16}R(w_0)\Psi\otimes w_4-\frac{1}{4}R(w_3)\Psi\otimes w_1\\ &\quad -\frac{1}{4}R(w_1)\Psi\otimes w_3+\frac{3}{8}R(w_2)\Psi\otimes w_2\\ &=\frac{1}{4}\left\{\frac{1}{4}R(w_4)\Psi\otimes w_0+\frac{1}{4}R(w_0)\Psi\otimes w_4-R(w_3)\Psi\otimes w_1\right.\\ &\quad\left. -R(w_1)\Psi\otimes w_3+\frac{3}{2}R(w_2)\Psi\otimes w_2\right\}.\end{aligned}\tag{5.12}$$

The Lie algebra $\mathfrak{p}_{\mathbf{C}}$ becomes the representation space of the adjoint action of K. We denote this representation by (τ_4, W_4). By the Clebsch-Gordan theorem, $\tau_2\otimes\tau_4$ has the irreducible decomposition

$$\tau_2\otimes\tau_4\cong\tau_2\oplus\tau_4\oplus\tau_6.\tag{5.13}$$

Here, each τ_n is the $(n+1)$-dimensional irreducible representation of K. In this decomposition, the projector of K-modules

$$pr_2:\tau_2\otimes\tau_4\twoheadrightarrow\tau_2,\qquad s_i\otimes w_j\longmapsto pr_2\left(s_i\otimes w_j\right),\tag{5.14}$$

is described as in Table 2.

$\nabla^R\Psi$ is a $\tau_2\otimes(\tau_2\otimes\mathfrak{p}_{\mathbf{C}})$-valued function. Then, by mapping $s_i\otimes w_k$ to $pr_2(s_i\otimes w_k)$, we have a K-homomorphism

$$\widetilde{pr_2}\circ\nabla^R: C^{\infty}_{\eta,\tau_2}(L)\longrightarrow C^{\infty}_{\eta,\tau_2}(L).\tag{5.15}$$

Since the minimal K-type τ_2^* occurs of multiplicity one, $\widetilde{pr_2}\circ\nabla^R$ is a map of constant multiple. To compute the action of the gradient operator $\widetilde{pr_2}\circ\nabla^R$ on the space of the Shintani functions $C^{\infty}_{\eta,\tau_2}(L)_{\pi_{\sigma_i,\nu}}$, we have to decompose w_i $(i=0,1,2,3,4)$ along the decomposition $\mathfrak{g}_{\mathbf{C}}=\mathrm{Ad}(a_t^{-1})\mathfrak{h}_{\mathbf{C}}\oplus\mathfrak{a}_{\mathbf{C}}\oplus\mathfrak{k}_{\mathbf{C}}$.

Lemma 5.4. *One has*

$$\begin{gathered}
w_0 = \frac{2\sinh^2 t}{\cosh(2t)}\operatorname{Ad}\left(a_t^{-1}\right)Y_1 - 2\operatorname{Ad}\left(a_t^{-1}\right)Y_4 + \tanh(2t)K_{13} + \frac{2\sqrt{-1}}{\sinh t}\operatorname{Ad}\left(a_t^{-1}\right)Y_3 + \frac{2\sqrt{-1}}{\tanh t}K_{12}, \\
w_4 = \frac{2\sinh^2 t}{\cosh(2t)}\operatorname{Ad}\left(a_t^{-1}\right)Y_1 - 2\operatorname{Ad}\left(a_t^{-1}\right)Y_4 + \tanh(2t)K_{13} - \frac{2\sqrt{-1}}{\sinh t}\operatorname{Ad}\left(a_t^{-1}\right)Y_3 - \frac{2\sqrt{-1}}{\tanh t}K_{12}, \\
w_2 = \frac{2\left(\cosh^2 t + 1\right)}{3\cosh(2t)}\operatorname{Ad}\left(a_t^{-1}\right)Y_1 - \frac{2}{3}\operatorname{Ad}\left(a_t^{-1}\right)Y_4 - \tanh(2t)K_{13}, \\
w_1 = X_{13} + \frac{\sqrt{-1}}{\cosh t}\operatorname{Ad}\left(a_t^{-1}\right)Y_2 - \sqrt{-1}\tanh t K_{23}, \\
w_3 = -X_{13} + \frac{\sqrt{-1}}{\cosh t}\operatorname{Ad}\left(a_t^{-1}\right)Y_2 - \sqrt{-1}\tanh t K_{23}.
\end{gathered} \tag{5.16}$$

By using Lemmas 3.4 and 4.3 and the table of projections, we can compute the action of the gradient operator. For $\Psi(a_t) = F_0(a_t)s_1 + G_0(a_t)s_2 + H_0(a_t)s_3 \in C^\infty_{\eta,\tau_2}(L)|_A$, we have

$$\widetilde{pr_2}\circ\nabla^R\Psi(a_t) = F''(a_t)s_1 + G''(a_t)s_2 + H''(a_t)s_3, \tag{5.17}$$

where

$$\begin{gathered}
F''(a_t) = -\left(\frac{1}{2\cosh(2t)} - \frac{1}{6}\right)sF_0(a_t) - \frac{1}{2}\frac{dH_0}{dt}(a_t) - \frac{1}{2}(\tanh(2t) + \tanh t)H_0(a_t), \\
G''(a_t) = -\frac{1}{3}sG_0(a_t), \\
H''(a_t) = \left(\frac{1}{2\cosh(2t)} + \frac{1}{6}\right)sH_0(a_t) - \frac{1}{2}\frac{dF_0}{dt}(a_t) - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh t}\right)F_0(a_t).
\end{gathered} \tag{5.18}$$

On the other hand, the eigenvalue of the gradient operator on the spherical functions of the principal series representation $\pi_{\sigma_i,\nu}$ depends on the choice of σ_i, denoted by λ_{σ_i} $(i = 1,2,3)$. These values are computed in [11] and they are as follows:

$$\lambda_{\sigma_1} = -\frac{1}{3}(2\nu_1 - \nu_2), \qquad \lambda_{\sigma_2} = -\frac{1}{3}(2\nu_2 - \nu_1), \qquad \lambda_{\sigma_3} = \frac{1}{3}(\nu_1 + \nu_2). \tag{5.19}$$

Therefore, since $\Psi(a_t) = F_0(a_t)s_1 + G_0(a_t)s_2 + H_0(a_t)s_3 \in C^\infty_{\eta,\tau_2}(L)_{\pi_{\sigma_i,\nu}}|_A$ satisfies $\widetilde{pr_2}\circ\nabla^R\Psi(a_t) = \lambda_{\sigma_i}\Psi(a_t)$, we have the following three differential equations.

Theorem 5.5. *For* $\Psi(a_t) = F_0(a_t)s_1 + G_0(a_t)s_2 + H_0(a_t)s_3 \in C^\infty_{\eta,\tau_2}(L)_{\pi_{\sigma_i,\nu}}|_A$, *one has*

$$-\left(\frac{1}{2\cosh(2t)} - \frac{1}{6}\right)sF_0(a_t) - \frac{1}{2}\frac{dH_0}{dt}(a_t) - \frac{1}{2}(\tanh(2t) + \tanh t)H_0(a_t) = \lambda_{\sigma_i}F_0(a_t), \tag{5.20}$$

$$-\frac{1}{3}sG_0(a_t) = \lambda_{\sigma_i}G_0(a_t), \tag{5.21}$$

$$\left(\frac{1}{2\cosh(2t)} + \frac{1}{6}\right)sH_0(a_t) - \frac{1}{2}\frac{dF_0}{dt}(a_t) - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh t}\right)F_0(a_t) = \lambda_{\sigma_i}H_0(a_t). \tag{5.22}$$

We consider the case of $\sigma = \sigma_1$. We have $\lambda_{\sigma_1} = -(1/3)(2\nu_1 - \nu_2)$. By (5.21), if $s \neq -3\lambda_{\sigma_1} = 2\nu_1 - \nu_2$, we have $G_0(a_t) \equiv 0$. Suppose that $s = 2\nu_1 - \nu_2$. We put $x = \tanh^2(2t)$, $G_0(a_t) = G_1(x)$ in (5.21). Then, the equation becomes

$$-4x^2(1-x)^2\frac{d^2G_1}{dx^2}(x) - 4x(1-x)^2\frac{dG_1}{\mathrm{d}x}(x) + \left\{\left(\frac{1}{4}x^2 - \frac{1}{3}x\right)s^2 - (2+\lambda_2)x + 1\right\}G_1(x) = 0. \tag{5.23}$$

We put $G_1(x) = x^{\alpha}(1-x)^{\beta}\widetilde{G}_1(x)$ $(\alpha, \beta \in \mathbf{C})$. Then, the equation becomes

$$\begin{aligned}&-4x^2(1-x)^2\frac{d^2\widetilde{G}_1}{dx^2}(x) + 4x(1-x)\{-2\alpha - 1 + (2\alpha + 2\beta + 1)x\}\frac{d\widetilde{G}_1}{dx}(x)\\ &\quad + \Big\{-4\alpha(\alpha-1)(1-x)^2 - 4\beta(\beta-1)x^2 + 8\alpha\beta x(1-x) - 4\alpha(1-x)^2 + 4\beta x(1-x)\\ &\qquad + \left(\frac{1}{4}x^2 - \frac{1}{3}x\right)s^2 - (2+\lambda_2)x + 1\Big\}\widetilde{G}_1(x) = 0.\end{aligned} \tag{5.24}$$

We want to divide the left-hand side of (5.24) by $x(1-x)$. For this purpose, $\alpha, \beta \in \mathbf{C}$ must satisfy

$$\begin{aligned}&-4\alpha^2 + 1 = 0,\\ &-4\beta(\beta-1) - \frac{1}{12}s^2 - \lambda_2 - 1 = 0.\end{aligned} \tag{5.25}$$

The solutions are $\alpha = \pm 1/2$, $\beta = (2 \pm \nu_2)/4$. We choose $\alpha = 1/2$, $\beta = (2+\nu_2)/4$. Then the left-hand side of (5.24) can be divided by $x(1-x)$, and the equation becomes

$$x(x-1)\frac{d^2\widetilde{G}_1}{dx^2}(x) + \left\{-2 + \left(3 + \frac{\nu_2}{2}\right)x\right\}\frac{d\widetilde{G}_1}{dx}(x) + \frac{1}{4}\left(-\nu_1^2 + \nu_1\nu_2 + 2\nu_2 + 4\right)\widetilde{G}_1(x) = 0. \tag{5.26}$$

This is a Gaussian hypergeometric differential equation, and its regular solution is given by

$$\widetilde{G}_1(x) = {}_2F_1\left(\frac{1}{2}\nu_1 + 1, \frac{1}{2}\nu_2 - \frac{1}{2}\nu_1 + 1; 2; x\right) \tag{5.27}$$

up to constant multiples. (The other solutions are not regular around $x = 0$, since they contain $\log x$.) Therefore, we have

$$G_0(a_t) = G_0(x) = Cx^{1/2}(1-x)^{(2+\nu_2)/4}\,{}_2F_1\left(\frac{1}{2}\nu_1 + 1, \frac{1}{2}\nu_2 - \frac{1}{2}\nu_1 + 1; 2; x\right), \tag{5.28}$$

where C is a constant number and $x = \tanh^2(2t)$. Next, we consider the equations satisfied by F_0 and H_0. From (5.20) and (5.22), we have

$$\begin{aligned}\frac{dF_0}{dt}(a_t) &= \left(\frac{1}{\cosh(2t)} + \frac{1}{3}\right)sH_0(a_t) - \left(\tanh(2t) + \frac{1}{\tanh t}\right)F_0(a_t) - 2\lambda_{\sigma_1}H_0(a_t),\\ \frac{dH_0}{dt}(a_t) &= -\left(\frac{1}{\cosh(2t)} - \frac{1}{3}\right)sF_0(a_t) - (\tanh(2t) + \tanh t)H_0(a_t) - 2\lambda_{\sigma_1}F_0(a_t).\end{aligned} \tag{5.29}$$

By differentiating both sides of (5.29) by t, we have

$$\begin{aligned}\frac{d^2F_0}{dt^2}(a_t) = &\left\{3\tanh^2(2t) + \frac{2}{\tan\mathrm{h}^2 t} + \frac{2\tanh(2t)}{\tan\mathrm{h}t} - 3 - \frac{4}{3}s\lambda_{\sigma_1}\right.\\ &\left.- \left(\frac{1}{\cosh^2(2t)} - \frac{1}{9}\right)s^2 + 4\lambda_{\sigma_1}^2\right\}F_0(a_t)\\ &+\left\{-\left(\frac{4\tanh(2t)}{\cosh(2t)} + \frac{2}{3}\tanh(2t) + \frac{2}{3\tanh(2t)} + \frac{2}{\sinh(2t)}\right)s\right.\\ &\left.+ 4\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\lambda_{\sigma_1}\right\}H_0(a_t),\\ \frac{d^2H_0}{dt^2}(a_t) = &\left\{3\tanh^2(2t) + 2\tanh^2 t + 2\tanh t\tanh(2t) - 3\right.\\ &\left.- \frac{4}{3}s\lambda_{\sigma_1} - \left(\frac{1}{\cosh^2(2t)} - \frac{1}{9}\right)s^2 + 4\lambda_{\sigma_1}^2\right\}H_0(a_t)\\ &+\left\{\left(\frac{4\tanh(2t)}{\cosh(2t)} - \frac{2}{3}\tanh(2t) - \frac{2}{3\tanh(2t)} + \frac{2}{\sinh(2t)}\right)s\right.\\ &\left.+ 4\left(\tan\mathrm{h}(2t) + \frac{1}{\tanh(2t)}\right)\lambda_{\sigma_1}\right\}F_0(a_t).\end{aligned} \tag{5.30}$$

By inserting (5.29) and (5.30) into the Casimir equation (5.6), (5.8) to eliminate the differential terms, we have

$$\begin{aligned}\left(\frac{1}{3}s\lambda_{\sigma_1} - \lambda_{\sigma_1}^2 - \frac{1}{9}s^2 - \lambda_2\right)F_0(a_t) = 0,\\ \left(\frac{1}{3}s\lambda_{\sigma_1} - \lambda_{\sigma_1}^2 - \frac{1}{9}s^2 - \lambda_2\right)H_0(a_t) = 0.\end{aligned} \tag{5.31}$$

Therefore, if the parameter $s \in \mathbf{C}$ satisfies $(1/3)s\lambda_{\sigma_1} - \lambda_{\sigma_1}^2 - (1/9)s^2 - \lambda_2 \neq 0$, then $F_0(a_t)$, $H_0(a_t) \equiv 0$. Suppose that $(1/3)s\lambda_{\sigma_1} - \lambda_{\sigma_1}^2 - (1/9)s^2 - \lambda_2 = 0$. Since $\lambda_{\sigma_1} = -(1/3)(2\nu_1 - \nu_2)$,

$\lambda_2 = -(1/3)(\nu_1^2 - \nu_1\nu_2 + \nu_2^2)$, we have

$$s^2 + (2\nu_1 - \nu_2)s + \left(\nu_1^2 - \nu_1\nu_2 - 2\nu_2^2\right) = 0. \tag{5.32}$$

Therefore, we have

$$s = -\nu_1 - \nu_2,\ -\nu_1 + 2\nu_2. \tag{5.33}$$

That is, (5.33) is the necessary condition of the existence of nontrivial $F_0(a_t), H_0(a_t)$. We put

$$\begin{aligned} F_0(a_t) &= \cosh^{-1/2}(2t)(\sinh t)^{-1}\widetilde{F}(t), \\ H_0(a_t) &= \cosh^{-1/2}(2t)(\cosh t)^{-1}\widetilde{H}(t) \end{aligned} \tag{5.34}$$

and insert these into (5.20), (5.22). Then we have

$$\begin{aligned} -\left(\frac{1}{2\cosh(2t)} - \frac{1}{6}\right)s\frac{1}{\tanh t}\widetilde{F}(t) - \frac{1}{2}\frac{d\widetilde{H}}{dt}(t) &= \lambda_{\sigma_1}\frac{1}{\tanh t}\widetilde{F}(t), \\ \left(\frac{1}{2\cosh(2t)} + \frac{1}{6}\right)s\tanh t\widetilde{H}(t) - \frac{1}{2}\frac{d\widetilde{F}}{dt}(t) &= \lambda_{\sigma_1}\tanh t\widetilde{H}(t). \end{aligned} \tag{5.35}$$

Next, we put $\tanh^2 t = u$, $\widetilde{F}(t) = \widetilde{F}_1(u)$, $\widetilde{H}(t) = \widetilde{H}_1(u)$. Then, the above equations become

$$-\left(\frac{1-u}{2(1+u)} - \frac{1}{6}\right)s\widetilde{F}_1(u) - u(1-u)\frac{d\widetilde{H}_1}{du}(u) = \lambda_{\sigma_1}\widetilde{F}_1(u), \tag{5.36}$$

$$-(1-u)\frac{d\widetilde{F}_1}{du}(u) + \left(\frac{1-u}{2(1+u)} + \frac{1}{6}\right)s\widetilde{H}_1(u) = \lambda_{\sigma_1}\widetilde{H}_1(u). \tag{5.37}$$

For a while, we consider the case of $s = 0$, that is, $\eta = \eta_{0,k}$ is a signature $\mathrm{sgn}_{(k)}$ of H, where

$$\mathrm{sgn}_{(k)}\left(\begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_1 \end{pmatrix}\right) = \det(H_1)^k|\det(H_1)|^{-k}, \qquad H_1 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}. \tag{5.38}$$

Then, by combining (5.36), (5.37), we have

$$u(1-u)^2\lambda_{\sigma_1}\frac{d^2\widetilde{F}_1}{du^2}(u) - u(1-u)\lambda_{\sigma_1}\frac{d\widetilde{F}_1}{du}(u) - \lambda_{\sigma_1}^3\widetilde{F}_1(u) = 0. \tag{5.39}$$

Suppose that $\lambda_{\sigma_1} \neq 0$. Then, the equation becomes

$$u(1-u)^2\frac{d^2\widetilde{F}_1}{du^2}(u) - u(1-u)\frac{d\widetilde{F}_1}{du}(u) - \lambda_{\sigma_1}^2\widetilde{F}_1(u) = 0. \tag{5.40}$$

We put $\widetilde{F}_1(u) = (1-u)^{\lambda_{\sigma_1}} F_2(u)$. Then, $F_2(u)$ satisfies

$$u(1-u)\frac{d^2F_2}{du^2}(u) - (2\lambda_{\sigma_1}+1)u\frac{dF_2}{du}(u) - \lambda_{\sigma_1}^2 F_2(u) = 0. \tag{5.41}$$

Equation (5.41) is the Gaussian hypergeometric differential equation. Now, since $F_2(u) = (1-u)^{-\lambda_{\sigma_1}}\cosh^{1/2}(2t)\sinh t F_0(a_t)$ and $F_0(a_t)$ is regular at $a_t = 1$ ($\Leftrightarrow t = 0 \Leftrightarrow u = 0$), $F_2(u)$ must be regular at $u = 0$. The regular solution of (5.41) is given by

$$F_2(u) = u\,{}_2F_1(\lambda_{\sigma_1}+1, \lambda_{\sigma_1}+1; 2; u) \tag{5.42}$$

up to constant multiples. Therefore, we have

$$\widetilde{F}_1(u) = C\cdot u(1-u)^{\lambda_{\sigma_1}}\,{}_2F_1(\lambda_{\sigma_1}+1, \lambda_{\sigma_1}+1; 2; u) \quad (C : \text{constant}). \tag{5.43}$$

Similarly, we have

$$\widetilde{H}_1(u) = C'\cdot(1-u)^{\lambda_{\sigma_1}}\,{}_2F_1(\lambda_{\sigma_1}+1, \lambda_{\sigma_1}; 1; u) \quad (C' : \text{constant}). \tag{5.44}$$

We want to find the relation between C and C'. By expanding $\widetilde{F}_1(u)$ and $\widetilde{H}_1(u)$ around $u = 0$, we have

$$\begin{gathered}\widetilde{F}_1(u) = C\cdot\left(u + \frac{\lambda_{\sigma_1^2}+1}{2}u^2 + u^3P_1(u)\right),\\ \widetilde{H}_1(u) = C'\cdot\left(1+\lambda_{\sigma_1}^2 u + u^2P_2(u)\right),\end{gathered} \tag{5.45}$$

where $P_1(u)$ and $P_2(u)$ are the analytic functions around $u = 0$. By inserting these into (5.36), we have

$$-3u\left(1-u^2\right)C'\left(\lambda_{\sigma_1}^2 + uP_3(u)\right) = 3(1+u)\lambda_{\sigma_1}C\left(u+u^2P_4(u)\right), \tag{5.46}$$

where $P_3(u)$ and $P_4(u)$ are also the analytic functions around $u = 0$. By comparing the coefficients of u of both sides, we have

$$C = -\lambda_{\sigma_1}C'. \tag{5.47}$$

Summing up, $F_0(a_t)$ and $H_0(a_t)$ are given by

$$\begin{gathered}F_0(a_t) = -C'\lambda_{\sigma_1}\cosh^{-1/2}(2t)(\sinh t)^{-1}u(1-u)^{\lambda_{\sigma_1}}\,{}_2F_1(\lambda_{\sigma_1}+1, \lambda_{\sigma_1}+1; 2; u),\\ H_0(a_t) = C'\cosh^{-1/2}(2t)(\cosh t)^{-1}(1-u)^{\lambda_{\sigma_1}}\,{}_2F_1(\lambda_{\sigma_1}+1, \lambda_{\sigma_1}; 1; u).\end{gathered} \tag{5.48}$$

(C': some constant, $u = \tanh^2 t$). In our computation, we assumed that $\lambda_{\sigma_1} \neq 0$, but the result above holds without this assumption.

Next, we consider conditions (1), (2), and (3) in (3.8). For $\Psi = {}^t(F_0, G_0, H_0) \in C^\infty_{\mathrm{sgn}_{(k)},\tau_2}(L)$, condition (1) is equivalent to

$$(-1)^k \cdot {}^t(-F_0(a_t), G_0(a_t), -H_0(a_t)) = {}^t(F_0(a_t), G_0(a_t), H_0(a_t)). \tag{5.49}$$

This is equivalent to

$$\begin{gathered} k = 0 \Longrightarrow F_0 \equiv 0, \qquad H_0 \equiv 0, \\ k = 1 \Longrightarrow G_0 \equiv 0. \end{gathered} \tag{5.50}$$

Condition (2) is equivalent to

$${}^t(-F_0(a_t), -G_0(a_t), H_0(a_t)) = {}^t(F_0(a_t), G_0(a_t), H_0(a_t)). \tag{5.51}$$

The solutions we have always satisfy this condition. Condition (3) is equivalent to

$$\begin{gathered} {}^t(\cos\theta F_0(1) + \sin\theta G_0(1), -\sin\theta F_0(1) + \cos\theta G_0(1), H_0(1)) = {}^t(F_0(1), G_0(1), H_0(1)), \\ {}^t(\cos\theta F_0(1) + \sin\theta G_0(1), \sin\theta F_0(1) - \cos\theta G_0(1), -H_0(1)) = (-1)^k \cdot {}^t(F_0(1), G_0(1), H_0(1)). \end{gathered} \tag{5.52}$$

(for all $\theta \in \mathbf{R}$). Since $F_0(1) = G_0(1) = 0$, (5.52) are equivalent to

$$k = 0 \Longrightarrow H_0 \equiv 0. \tag{5.53}$$

But this condition holds if condition (5.50) is satisfied. We have obtained a result about the Shintani functions attached to the nonspherical principal series representation $\pi_{\sigma_1,\nu}$. Note that since the transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ does not change the eigenvalue of Casimir operator λ_2 and changes the eigenvalue of gradient operator λ_{σ_1} to λ_{σ_2}, this transform gives the result in case of $\sigma = \sigma_2$. Similarly, the transform $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ gives the result in case of $\sigma = \sigma_3$. Summing up these results, we have the following theorem.

Theorem 5.6. *Let $\eta = \mathrm{sgn}_{(k)}$ ($k \in \{0,1\}$) be a signature of H defined by* (5.38) *and τ_2 a three-dimensional tautological representation of K, and let $\Psi = {}^t(F_0, G_0, H_0) \in C^\infty_{\mathrm{sgn}_{(k)},\tau_2}(L)_{\pi_{\sigma_1,\nu}}$ be a Shintani function corresponding to the nonspherical principal series representation $\pi_{\sigma_1,\nu}$ of G. Then, the restriction of Ψ to A is given as follows.*

(1) *In case of $k = 0$,*

(a) *if $2\nu_1 - \nu_2 \neq 0$, $\Psi \equiv 0$;*

(b) *if* $2\nu_1 - \nu_2 = 0$, *one has*

$$\begin{pmatrix} F_0(a_t) \\ G_0(a_t) \\ H_0(a_t) \end{pmatrix} = C \cdot \begin{pmatrix} 0 \\ x^{1/2}(1-x)^{(2+\nu_2)/4}\, {}_2F_1\left(\frac{1}{2}\nu_1 + 1, \frac{1}{2}\nu_2 - \frac{1}{2}\nu_1 + 1, ; 2; x\right) \\ 0 \end{pmatrix}. \tag{5.54}$$

$$\left(x = \tanh^2(2t),\ C : \textit{some constant}\right).$$

(2) *In case of* $k = 1$,

(a) *if* $(-\nu_1 - \nu_2)(-\nu_1 + 2\nu_2) \neq 0$, $\Psi \equiv 0$;

(b) *if* $(-\nu_1 - \nu_2)(-\nu_1 + 2\nu_2) = 0$, *one has*

$$\begin{pmatrix} F_0(a_t) \\ G_0(a_t) \\ H_0(a_t) \end{pmatrix} = C \cdot \begin{pmatrix} -\lambda_{\sigma_1} \cosh^{-1/2}(2t)(\sinh t)^{-1} u(1-u)^{\lambda_{\sigma_1}}\, {}_2F_1(\lambda_{\sigma_1} + 1, \lambda_{\sigma_1} + 1; 2; u) \\ 0 \\ \cosh^{-1/2}(2t)(\cosh t)^{-1}(1-u)^{\lambda_{\sigma_1}}\, {}_2F_1(\lambda_{\sigma_1} + 1, \lambda_{\sigma_1}; 1; u) \end{pmatrix}. \tag{5.55}$$

$$\left(u = \tanh^2 t,\ C : \textit{some constant}\right).$$

Especially, in any case, one has

$$\dim C^{\infty}_{\mathrm{sgn}_{(k)}, \tau_2}(L)_{\pi_{\sigma_1,\nu}} \leq 1. \tag{5.56}$$

The transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ *gives the result in case of* $\sigma = \sigma_2$ *and the transform* $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ *gives the result in case of* $\sigma = \sigma_3$.

Next, we compute the Shintani functions $\Psi \in C^{\infty}_{\eta,\tau_2}(L)_{\pi_{\sigma_1,\nu}}$ for general unitary character $\eta = \eta_{s,k}$ under the assumption that $1, \nu_1, \nu_2$ are linearly independent over $\mathbf{Q}$. We have already known that the necessary condition of the existence of non zero Ψ is that the parameter s is one of $2\nu_1 - \nu_2, 2\nu_2 - \nu_1$, or $-\nu_1 - \nu_2$, and we have already solved the differential equations in case of $s = 2\nu_1 - \nu_2$. Hereafter, we suppose that the parameter s is either $2\nu_2 - \nu_1$ or $-\nu_1 - \nu_2$. We put $w = (1-u)/(1+u)$, $\widetilde{F}_2(w) = \widetilde{F}_1(u)$, $\widetilde{H}_2(w) = \widetilde{H}_1(u)$ in (5.36), (5.37). Then we have

$$\begin{aligned} -\left(\frac{w}{2} - \frac{1}{6}\right)s\widetilde{F}_2(w) + w(1-w)\frac{d\widetilde{H}_2}{dw}(w) &= \lambda_{\sigma_1}\widetilde{F}_2(w), \\ w(1+w)\frac{d\widetilde{F}_2}{dw}(w) + \left(\frac{w}{2} + \frac{1}{6}\right)s\widetilde{H}_2(w) &= \lambda_{\sigma_1}\widetilde{H}_2(w). \end{aligned} \tag{5.57}$$

We put

$$\widetilde{F}_2(w) = \sum_{n=0}^{\infty} a_n w^{n+\alpha}, \qquad \widetilde{H}_2(w) = \sum_{n=0}^{\infty} c_n w^{n+\gamma} \tag{5.58}$$

$(\alpha, \gamma \in \mathbf{C}, a_0, c_0 \neq 0)$ in (5.57) and compute the power series solutions. By inserting these series into (5.57), we have

$$\frac{s}{6}a_0 w^{\alpha} + w^{\alpha+1}Q_1(w) + \gamma c_0 w^{\gamma} + w^{\gamma+1}Q_2(w) = \lambda_{\sigma_1} a_0 w^{\alpha} + w^{\alpha+1}Q_3(w),$$
$$\alpha a_0 w^{\alpha} + w^{\alpha+1}Q_4(w) + \frac{s}{6}c_0 w^{\gamma} + w^{\gamma+1}Q_5(w) = \lambda_{\sigma_1} c_0 w^{\gamma} + w^{\gamma+1}Q_6(w), \tag{5.59}$$

where $Q_i(w)$ $(i = 1, \ldots, 6)$ are the analytic functions around $w = 0$. By comparing the lowest terms in power series, easily we have $\alpha = \gamma$ (in this argument, we use the fact that $1, \nu_1, \nu_2$ are linearly independent over $\mathbf{Q}$ carefully). Therefore, from (5.59), we have

$$\left(\frac{s}{6} - \lambda_{\sigma_1}\right) + \gamma c_0 = 0,$$
$$\alpha a_0 + \left(\frac{s}{6} - \lambda_{\sigma_1}\right) c_0 = 0. \tag{5.60}$$

Since $(a_0, c_0) \neq (0, 0)$, we have

$$\left(\frac{s}{6} - \lambda_{\sigma_1}\right)^2 - \alpha\gamma = 0. \tag{5.61}$$

By combining this and $\alpha = \gamma$, we have

$$\alpha = \gamma = \pm\left(\frac{s}{6} - \lambda_{\sigma_1}\right). \tag{5.62}$$

Hereafter, we put $A = A(\nu_1, \nu_2) = (s/6) - \lambda_{\sigma_1}$. Then, $\widetilde{F}_2(w)$ (resp., $\widetilde{H}_2(w)$) are expressed by the linear combination of some power series $\sum_{n=0}^{\infty} a_n w^{n+A}$, $\sum_{n=0}^{\infty} a'_n w^{n-A}$ (resp., $\sum_{n=0}^{\infty} c_n w^{n+A}$, $\sum_{n=0}^{\infty} c'_n w^{n-A}$). That is, there exist common constants C_+, C_- such that

$$\widetilde{F}_2(w) = C_+ \sum_{n=0}^{\infty} a_n w^{n+A} + C_- \sum_{n=0}^{\infty} a'_n w^{n-A},$$
$$\widetilde{H}_2(w) = C_+ \sum_{n=0}^{\infty} c_n w^{n+A} + C_- \sum_{n=0}^{\infty} c'_n w^{n-A}. \tag{5.63}$$

By inserting

$$\widetilde{F}_2(w) = \sum_{n=0}^{\infty} a_n w^{n+A}, \qquad \widetilde{H}_2(w) = \sum_{n=0}^{\infty} c_n w^{n+A} \tag{5.64}$$

into (5.57) and picking up the coefficients of w^{n+A}, we have the following recurrence relations:

$$-\frac{s}{2}a_{n-1} + Aa_n - (n+A-1)c_{n-1} + (n+A)c_n = 0, \tag{5.65}$$

$$(n+A-1)a_{n-1} + (n+A)a_n + \frac{s}{2}c_{n-1} + Ac_n = 0, \tag{5.66}$$

for all $n \geq 0$. Here, we assume that $a_l = c_l = 0$ if $l < 0$. From (5.65), (5.66), easily we have $c_n = (-1)^{n+1}a_n$ for all $n \geq 0$ by induction. Therefore, by inserting $c_n = (-1)^{n+1}a_n,\ c_{n-1} = (-1)^n a_{n-1}$ into (5.65), we have

$$\left(A + (n+A)(-1)^{n+1}\right)a_n = \left(\frac{s}{2} + (n+A-1)(-1)^n\right)a_{n-1}. \tag{5.67}$$

Thus, we have

$$a_n = \left\{\prod_{k=1}^{n}\frac{s/2 + (n+A-1)(-1)^k}{A + (k+A)(-1)^{k+1}}\right\}a_0, \tag{5.68}$$

$$c_n = (-1)^{n+1}\left\{\prod_{k=1}^{n}\frac{s/2 + (n+A-1)(-1)^k}{A + (k+A)(-1)^{k+1}}\right\}a_0. \tag{5.69}$$

Similarly, if the characteristic roots are $\alpha = \gamma = -A$, by inserting

$$\widetilde{F}_2(w) = \sum_{n=0}^{\infty}a'_n w^{n-A}, \qquad \widetilde{H}_2(w) = \sum_{n=0}^{\infty}c'_n w^{n-A} \tag{5.70}$$

into (5.57), we have

$$\begin{aligned} a'_n &= \left\{\prod_{k=1}^{n}\frac{s/2 + (n-A-1)(-1)^k}{A + (k-A)(-1)^{k+1}}\right\}a_0, \\ c'_n &= (-1)^n\left\{\prod_{k=1}^{n}\frac{s/2 + (n-A-1)(-1)^k}{A + (k-A)(-1)^{k+1}}\right\}a_0. \end{aligned} \tag{5.71}$$

(for all $n \geq 1$). From (5.68), a_{2n} and a_{2n-1} are expressed by

$$\begin{aligned} a_{2n} &= \frac{(s/4 + A/2 + 1/2)_n(-s/4 + A/2)_n}{n!(A+1/2)_n}a_0, \\ a_{2n-1} &= -\frac{(s/4 + A/2 + 1/2)_{n-1}(-s/4 + A/2)_n}{(n-1)!(A+1/2)_n}a_0. \end{aligned} \tag{5.72}$$

Here, for $\alpha \in \mathbf{C}, n \in \mathbf{Z}_{>0}$, we define $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$ and $(\alpha)_0 := 1$. Therefore, if we normalize $a_0 = 1$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{2n} w^{2n} &= \sum_{n=0}^{\infty} \frac{(s/4+A/2+1/2)_n(-s/4+A/2)_n}{n!(A+1/2)_n} w^{2n} \\
&= {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right), \\
\sum_{n=1}^{\infty} a_{2n-1} w^{2n-1} &= -\frac{-s/4+A/2}{A+1/2} \sum_{n=1}^{\infty} \frac{(s/4+A/2+1/2)_{n-1}(-s/4+A/2+1)_{n-1}}{(n-1)!(A+3/2)_{n-1}} w^{2n-2} \cdot w \\
&= \frac{s/4-A/2}{A+1/2} w \sum_{n=0}^{\infty} \frac{(s/4+A/2+1/2)_n(-s/4+A/2+1)_n}{n!(A+3/2)_n} w^{2n} \\
&= \frac{s/4-A/2}{A+1/2} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right).
\end{aligned} \tag{5.73}$$

Thus, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n w^{n+A} = w^A \Bigg\{ &{}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \\
&+ \frac{s/4-A/2}{A+1/2} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right) \Bigg\}.
\end{aligned} \tag{5.74}$$

And since $c_n = (-1)^{n+1} a_n$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n w^{n+A} = w^A \Bigg\{ &- {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \\
&+ \frac{s/4-A/2}{A+1/2} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right) \Bigg\}.
\end{aligned} \tag{5.75}$$

Similarly, by using

$$\begin{aligned}
a'_{2n} &= \frac{(-s/4-A/2+1/2)_n(s/4-A/2)_n}{n!(-A+1/2)_n} a'_0, \\
a'_{2n-1} &= -\frac{(-s/4-A/2+1/2)_{n-1}(s/4-A/2)_n}{(n-1)!(-A+1/2)_n} a'_0,
\end{aligned} \tag{5.76}$$

and $c'_n = (-1)^n a'_n$, if we normalize $a'_0 = 1$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a'_n w^{n-A} = w^{-A} \Bigg\{ &{}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \\
&- \frac{s/4-A/2}{-A+1/2} w\, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right) \Bigg\},
\end{aligned}$$

$$\sum_{n=0}^{\infty} c'_n w^{n-A} = w^{-A}\left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \right. \\ \left. + \frac{s/4-A/2}{-A+1/2} w\, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\}. \tag{5.77}$$

Therefore, $\widetilde{F}_2(w)$ and $\widetilde{H}_2(w)$ are expressed as follows:

$$\begin{aligned}
\widetilde{F}_2(w) = {} & C_+ w^A \left\{ {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \right. \\
& \left. + \frac{s/4-A/2}{A+1/2} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right)\right\} \\
& + C_- w^{-A} \left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \right. \\
& \left. - \frac{s/4-A/2}{-A+1/2} w\, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\}, \\
\widetilde{H}_2(w) = {} & C_+ w^A \left\{ -{}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \right. \\
& \left. + \frac{s/4-A/2}{A+1/2} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right)\right\} \\
& + C_- w^{-A} \left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \right. \\
& \left. + \frac{s/4-A/2}{-A+1/2} w\, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\}.
\end{aligned} \tag{5.78}$$

We want the relation between C_+ and C_-. We can find the relation by using the regularity of the Shintani function and the asymptotic formula of Gaussian hypergeometric function ${}_2F_1$. Now, since $F_0(a_t)$ and $H_0(a_t)$ are regular around $t = 0$ ($\Leftrightarrow w = 1$) and since

$$\begin{aligned}
\widetilde{F}(t) &= \cosh^{1/2}(2t) \sinh t F_0(a_t), \\
\widetilde{H}(t) &= \cosh^{1/2}(2t) \cosh t H_0(a_t),
\end{aligned} \tag{5.79}$$

$\widetilde{F}_2(w) = \widetilde{F}(t)$ and $\widetilde{H}_2(w) = \widetilde{H}(t)$ must be regular around $w = 1$ and $\widetilde{F}_2(w)$ must satisfy $\widetilde{F}_2(1) = 0$. Since all hypergeometric functions appearing in the right-hand sides of (5.78) are in the form of ${}_2F_1(a, b; a+b; w^2)$, to investigate the behavior of the right-hand sides of (5.78) around $w = 1$, we use the following asymptotic formula.

Formula 1

We have

$$ {}_2F_1(a,b;a+b;z) = -\frac{\Gamma(a+b)\left(\log(1-z)+\psi(a)+\psi(b)+2\gamma\right)}{\Gamma(a)\Gamma(b)}(1+O(z-1)) \quad (z \longrightarrow 1). \tag{5.80} $$

Here, γ is the Euler constant and ψ is defined by

$$ \psi(z) = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+z-1}\right)-\gamma. \tag{5.81} $$

We apply this formula to the right-hand sides of (5.78). Firstly, the coefficient of $\log(1-w^2)$ of $\widetilde{F}_2(w)$ equals

$$ \begin{aligned} &C_+\left\{-\frac{\Gamma(A+1/2)}{\Gamma(s/4+A/2+1/2)\Gamma(-s/4+A/2)}\right. \\ &\qquad \left.-\frac{s/4-A/2}{A+1/2}\frac{\Gamma(A+3/2)}{\Gamma(s/4+A/2+1/2)\Gamma(-s/4+A/2+1)}\right\} \\ &\quad +C_-\left\{-\frac{\Gamma(-A+1/2)}{\Gamma(-s/4-A/2+1/2)\Gamma(s/4-A/2)}\right. \\ &\qquad \left.+\frac{s/4-A/2}{-A+1/2}\frac{\Gamma(-A+3/2)}{\Gamma(-s/4-A/2+1/2)\Gamma(s/4-A/2+1)}\right\} \\ &= C_+\cdot 0 + C_-\cdot 0 = 0. \end{aligned} \tag{5.82} $$

Therefore, $\widetilde{F}_2(w)$ is regular around $w = 1$ regardless of the values of C_+, C_-. Next, the coefficient of $\log(1-w^2)$ of $\widetilde{H}_2(w)$ equals

$$ \begin{aligned} &C_+\left\{\frac{\Gamma(A+1/2)}{\Gamma(s/4+A/2+1/2)\Gamma(-s/4+A/2)}\right. \\ &\qquad \left.-\frac{s/4-A/2}{A+1/2}\frac{\Gamma(A+3/2)}{\Gamma(s/4+A/2+1/2)\Gamma(-s/4+A/2+1)}\right\} \\ &\quad +C_-\left\{-\frac{\Gamma(-A+1/2)}{\Gamma(-s/4-A/2+1/2)\Gamma(s/4-A/2)}\right. \\ &\qquad \left.-\frac{s/4-A/2}{-A+1/2}\frac{\Gamma(-A+3/2)}{\Gamma(-s/4-A/2+1/2)\Gamma(s/4-A/2+1)}\right\} \\ &= \frac{2\Gamma(A+1/2)}{\Gamma(s/4+A/2+1/2)\Gamma(-s/4+A/2)}C_+ \\ &\quad -\frac{2\Gamma(-A+1/2)}{\Gamma(-s/4-A/2+1/2)\Gamma(s/4-A/2)}C_-. \end{aligned} \tag{5.83} $$

Since this coefficient must be 0, we have

$$C_+ = \frac{\Gamma(-A+1/2)}{\Gamma(-s/4-A/2+1/2)\Gamma(s/4-A/2)}, \quad C_- = \frac{\Gamma(A+1/2)}{\Gamma(s/4+A/2+1/2)\Gamma(-s/4+A/2)} \tag{5.84}$$

up to (the same) constant multiples. Note that we can easily verify that $\widetilde{F}_2(1) = 0$ for these C_+, C_- by using formulas $\Gamma(z+1) = z\Gamma(z)$, $\psi(z+1) = \psi(z) + 1/z$. Now, we have completely determined $\widetilde{F}_2(w), \widetilde{H}_2(w)$. We have

$$\begin{aligned}
\widetilde{F}_2(w) = {} & \frac{\Gamma(-A+1/2)}{\Gamma(-s/4-A/2+1/2)\Gamma(s/4-A/2)} w^A \\
& \times \left\{ {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \right. \\
& \quad \left. + \frac{s/4-A/2}{A+1/2} w \, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right)\right\} \\
& + \frac{\Gamma(A+1/2)}{\Gamma(s/4+A/2+1/2)\Gamma(-s/4+A/2)} w^{-A} \\
& \times \left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \right. \\
& \quad \left. - \frac{s/4-A/2}{-A+1/2} w \, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\}, \\
\widetilde{H}_2(w) = {} & \frac{\Gamma(-A+1/2)}{\Gamma(-s/4-A/2+1/2)\Gamma(s/4-A/2)} w^A \\
& \times \left\{ -{}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \right. \\
& \quad \left. + \frac{s/4-A/2}{A+1/2} w \, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right)\right\} \\
& + \frac{\Gamma(A+1/2)}{\Gamma(s/4+A/2+1/2)\Gamma(-s/4+A/2)} w^{-A} \\
& \times \left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \right. \\
& \quad \left. + \frac{s/4-A/2}{-A+1/2} w \, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\}
\end{aligned} \tag{5.85}$$

up to (the same) constant multiples. Here, $A = A(\nu_1, \nu_2) = s/6 - \lambda_{\sigma_1}$, $\lambda_{\sigma_1} = -(1/3)(2\nu_1 - \nu_2)$, $w = (1-u)/(1+u) = (1-\tanh^2 t)/(1+\tanh^2 t)$, $\widetilde{F}_2(w) = \widetilde{F}(t) = \cosh^{1/2}(2t) \sinh t F_0(a_t)$, $\widetilde{H}_2(w) = \widetilde{H}(t) = \cosh^{1/2}(2t) \cosh t H_0(a_t)$. By the same argument we have done when η is the signature, we can easily verify that conditions (1), (2), and (3) in (3.8) are equivalent to condition (5.50). We have already computed $F_0(a_t)$, $G_0(a_t)$, and $H_0(a_t)$ in any case. Summing up, we obtain the following theorem.

Theorem 5.7. *Assume that* $1, \nu_1, \nu_2$ *are linearly independent over* $\mathbf{Q}$. *Let* $\eta = \eta_{s,k}$ *be a unitary character of* H *defined by* (3.1) *and* $\sigma = \sigma_1$ *the character of* M. *Then, the necessary condition of the existence of the nontrivial Shintani functions attached to the nonspherical principal series representation* $\pi_{\sigma_1,\nu}$ *is that*

$$k = 0, \qquad s = 2\nu_1 - \nu_2 \tag{5.86}$$

or

$$k = 1, \qquad s = -\nu_1 - \nu_2 \quad \text{or} \quad -\nu_1 + 2\nu_2. \tag{5.87}$$

That is, if the condition above is not satisfied, one has

$$C^{\infty}_{\eta,\tau_2}(L)_{\pi_{\sigma_1,\nu}} = \{0\}. \tag{5.88}$$

Let $\Psi(a_t) = {}^t(F_0(a_t), G_0(a_t), H_0(a_t)) \in C^{\infty}_{\eta,\tau_2}(L)_{\pi_{\sigma_1,\nu}}|_A$ *and suppose that the condition above is satisfied. Then,*

(1) *if* $k = 0$ *and* $s = 2\nu_1 - \nu_2$, *one has*

$$\begin{pmatrix} F_0(a_t) \\ G_0(a_t) \\ H_0(a_t) \end{pmatrix} = C \cdot \begin{pmatrix} 0 \\ x^{1/2}(1-x)^{(2+\nu_2)/4}\,{}_2F_1\left(\frac{1}{2}\nu_1 + 1, \frac{1}{2}\nu_2 - \frac{1}{2}\nu_1 + 1, ; 2; x\right) \\ 0 \end{pmatrix}. \tag{5.89}$$

$$\left(x = \tan h^2(2t),\ C : \text{someconstant}\right);$$

(2) *if* $k = 1$ *and* $s = -\nu_1 - \nu_2$ *or* $-\nu_1 + 2\nu_2$, *one has*

$$\begin{pmatrix} F_0(a_t) \\ G_0(a_t) \\ H_0(a_t) \end{pmatrix} = C \cdot \begin{pmatrix} \cosh^{-1/2}(2t)\sinh^{-1} t\widetilde{F}_2(w) \\ 0 \\ \cosh^{-1/2}(2t)\cosh^{-1} t\widetilde{H}_2(w) \end{pmatrix}, \tag{5.90}$$

where C *is some constant and* $\widetilde{F}_2(w)$ *and* $\widetilde{H}_2(w)$ *are the functions given by* (5.85). *Especially, in any case, one has*

$$\dim C^{\infty}_{\eta,\tau_2}(L)_{\pi_{\sigma_1,\nu}} \leq 1. \tag{5.91}$$

The transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ *gives the result in case of* $\sigma = \sigma_2$, *and the transform* $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ *gives the result in case of* $\sigma = \sigma_3$.

Remark 5.8. By using the relation $1 - x = w^2$ and the formulas of the hypergeometric function

$$\begin{aligned}
\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2F_1(\alpha,\beta;\gamma;z) &= \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)}(1-z)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z}\right) \\
&\quad + \frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)}(1-z)^{-\beta}\,{}_2F_1\left(\beta,\gamma-\alpha;\beta-\alpha+1;\frac{1}{1-z}\right), \\
{}_2F_1(\alpha,\beta,\gamma;z) &= (1-z)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right) \\
&= (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z),
\end{aligned} \tag{5.92}$$

we can rewrite $\widetilde{F}_2(w)$, $\widetilde{H}_2(w)$ in Theorem 5.7 as functions in $x = \tanh^2(2t)$. We put $\widetilde{F}_2(w) = F_3(x)$, $\widetilde{H}_2(w) = H_3(x)$. Then, we have

$$\begin{aligned}
F_3(x) &= \left(-\frac{s}{4}+\frac{A}{2}\right)\left\{-(1-x)^{A/2}\,{}_2F_1\left(\frac{s}{4}+\frac{A+1}{2},-\frac{s}{4}+\frac{A}{2};1;x\right)\right. \\
&\qquad \left. +(1-x)^{(A+1)/2}\,{}_2F_1\left(\frac{s}{4}+\frac{A+1}{2},-\frac{s}{4}+\frac{A}{2}+1;1;x\right)\right\}, \\
H_3(x) &= \left(-\frac{s}{4}+\frac{A}{2}\right)\left\{(1-x)^{A/2}\,{}_2F_1\left(\frac{s}{4}+\frac{A+1}{2},-\frac{s}{4}+\frac{A}{2};1;x\right)\right. \\
&\qquad \left. +(1-x)^{(A+1)/2}\,{}_2F_1\left(\frac{s}{4}+\frac{A+1}{2},-\frac{s}{4}+\frac{A}{2}+1;1;x\right)\right\}.
\end{aligned} \tag{5.93}$$

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