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Research Article

Folding List of Graphs Obtained from a Given Graph

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In this paper, we examine the relation between graph folding of a given graph and foldings of new graphs obtained from this graph by some techniques like dual, gear, subdivision, web, crown, simplex, crossed prism, and clique-sum graphs. In each case, we obtained the necessary and sufficient conditions, if exist, for these new graphs to be folded.

1. Introduction

Let $G = (V, E)$ be a graph, where V is the set of its vertices and E is the set of its edges. By a graph, we mean a simple and finite connected graph; that is, a graph without multiple edges or loops. Let G be a graph, then

(1) The dual graph G^* of a graph G is obtained by placing a vertex in every face of G and an edge joining every two vertices in neighbouring faces [1].

(2) A gear graph, denoted G_n, is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph W_n. Thus, G_n has $2n + 1$ vertices and $3n$ edges [2].

(3) If the edge e joins vertices v and w, then the subdivision of e replaces e by a new vertex u and two new edges vu and uw [3]. A subdivision of a graph G is a graph obtained from G by applying a finite number of subdivisions of edges in succession.

(4) The web graph $W_{n,r}$ is a graph consisting of r concentric copies of the cycle graphs C_n, with corresponding vertices connected by edges [4].

(5) Crown graph on $2n$ vertices is an undirected graph with two sets of vertices $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ and with an edge from u_i to v_j whenever $i \neq j$ [5]. The crown graph can be viewed as a complete bipartite graph from which edges $u_i, v_j, i = 1, \ldots, n$, have been removed.

(6) A simplex graph $\kappa(G)$ of an undirected graph G is itself a graph, with a vertex for each clique in G. Two vertices of $\kappa(G)$ are joined by an edge whenever the corresponding two cliques differ in the presence or absence of a single vertex. The single vertices are called the zero vertices [6].

(7) A crossed prism graph for positive even n is a graph obtained by taking two disjoint cycle graphs C_n and adding edges (v_k, v_{2k+1}) and (v_{k+1}, v_{2k}) for $k = 1, 3, \ldots, (n-1)$ [7]. We will denote this graph by CP_n.

(8) If two graphs G and H each contain cliques of equal size, the clique-sum of G and H is formed from their disjoint union by identifying pairs of vertices in these two cliques to form a single shared clique, without deleting any of the clique edges [8].

(9) Let G_1 and G_2 be two simple graphs and $f: G_1 \longrightarrow G_2$ a continuous map. Then, f is called a graph map, if

(i) For each vertex $v \epsilon V(G_1)$, $f(v)$ is a vertex in $V(G_2)$.

(ii) For each edge $e \varepsilon E(G_1)$, $f(e)$ is either a vertex or an edge of the graph G_2, i.e., $f(e) \varepsilon V(G_2)$ or $f(e) \varepsilon E(G_2)$ [9].

(10) A graph map $f\colon G_1 \longrightarrow G_2$ is called a graph folding if and only if f maps vertices to vertices and edges to edges [10].

(11) A graph G is planar if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are both incident. Any such drawing is called a plane drawing of G. Any plane drawing of G divides the plane into regions called faces.

If the edges and vertices of a face σ_i of G_1 are mapped to the edges and vertices of a face σ_j of G_2, then we write $f(\sigma_i) = \sigma_j$.

2. Graph Folding of the Dual Graph

Theorem 1. *Let G_1 and G_2 be graphs and $f\colon G_1 \longrightarrow G_2$ a graph folding. Consider the graph map $g\colon G_1^* \longrightarrow G_2^*$ defined by*

(i) For all $v_i^* \in V(G_1^*)$, $g(v_i^*) = v_i^*$ if and only if $f(\sigma_i) = \sigma_i$, where σ_i is a face of G_1

(ii) If $f(\sigma_i) = \sigma_j$ and $f(\sigma_j) = \sigma_j$ where σ_i and σ_j are the neighbouring faces, then $g\{v_i^*, v_j^*\} = g\{v_k^*, v_j^*\}$ where v_k^* is the vertex of the face σ_k which is neighbouring to σ_j but not neighbouring to σ_i

(iii) If $f(\sigma_i) = \sigma_j$ and $f(\sigma_k) = \sigma_l$, then $g(v_i^*) = (v_l^*)$ and $g(v_k^*) = (v_j^*)$ such that each of σ_i, σ_k and σ_j, σ_l are neighbouring faces

Proof. Let $f\colon G_1 \longrightarrow G_2$ be a graph folding. Suppose that σ_i, σ_j, σ_k, and σ_l are the faces of the graph G_1 such that $f(\sigma_i) = \sigma_j$ and $f(\sigma_j) = \sigma_j$ where σ_i and σ_j are the neighbouring faces. If σ_k is neighbouring to σ_j and not neighbouring to σ_i, then there are no edges joining v_i^* and v_k^* in G_1^*, but each of $\{v_i^*, v_j^*\}$ and $\{v_k^*, v_j^*\}$ is an edge of G_1^*. Thus, by the given definition, g maps edges to edges. Now, let $f(\sigma_i) = \sigma_j$ and $f(\sigma_k) = \sigma_l$ Then, by the given definition of g, it maps the vertex v_i^* to v_l^* and the vertex v_k^* to v_j^*. Now, since each of the faces σ_i, σ_k and σ_j, σ_l are neighbouring, then each of $\{v_i^*, v_k^*\}$ and $\{v_l^*, v_j^*\}$ is an edge of G_1^*, i.e., the map g maps edges to edges. And, consequently g is a graph folding of the dual graph of G_1. □

Example 1. Consider the graphs G_1 and G_2 shown in Figure 1(a). Let $f\colon G_1 \longrightarrow G_2$ be a graph folding defined by $f(v_6) = (v_1)$ and $f\{(v_6, v_2), (v_6, v_3), (v_6, v_4), (v_6, v_5)\} = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5)\}$, i.e., $f\{\sigma_5, \sigma_6, \sigma_7, \sigma_8\} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. The graph map $g\colon G_1* \longrightarrow G_2*$ defined $g\{v_5^*, v_6^*, v_7^*, v_8^*\} = \{v_3^*, v_4^*, v_1^*, v_2^*\}$ and $\{(v_5^*, v_1^*), (v_5^*, v_6^*), (v_5^*, v_7^*), (v_6^*, v_2^*), (v_6^*, v_8^*), (v_7^*, v_3^*), (v_7^*, v_8^*), (v_8^*, v_4^*)\} = \{(v_3^*, v_1^*), (v_3^*, v_4^*), (v_3^*, v_1^*), (v_4^*, v_2^*), (v_4^*, v_2^*), (v_1^*, v_3^*), (v_1^*, v_2^*)\}$ is a graph folding, see Figure 1(b). The omitted vertices and edges, or faces, are mapped to themselves through this paper.

3. Graph Folding of the Gear and the Subdivision Graphs

It should be noted that any graph folding of the wheel graph W_n maps the hup into itself.

Theorem 2. *Let W_n be a wheel graph and G_n be the corresponding gear graph. Let $f\colon W_n \longrightarrow W_n$ be a graph folding. Then, the graph map $g\colon G_n \longrightarrow G_n$ is defined by*

(i) $g\{v_i, v_r\} = \{v_k, v_s\}$ and $g\{v_r, v_j\} = \{v_s, v_l\}$ if and only if $f\{v_i, v_j\} = \{v_k, v_l\}$, where v_r and v_s are the extra vertices inserting between the adjacent vertices v_i, v_j and v_k, v_l, respectively.

(ii) For the hub v, $g(v) = v$.

Proof. Let $f\colon W_n \longrightarrow W_n$ be a graph folding, and consider the edges $\{v_i, v_j\}$, $\{v_k, v_l\} \in E(W_n)$ such that $f\{v_i, v_j\} = \{v_k, v_l\}$, i.e., $f\{v_i\} = \{v_k\}$ and $f\{v_j\} = \{v_l\}$. Now, let v_r and v_s be the new vertices inserted between the vertices of the edges $\{v_i, v_j\}$ and $\{v_k, v_l\}$, respectively. Then, we have four new edges $\{v_i, v_r\}$, $\{v_r, v_j\}$, $\{v_k, v_s\}$, and $\{v_s, v_l\} \in E(G_n)$, but $g\{v_i, v_r\} = \{v_k, v_s\}$ and $g\{v_r, v_j\} = \{v_s, v_l\}$, i.e., the map g maps edges of G_n to other edges of G_n. Also, for all $v_i, v_k \in V(W_n)$ if $f(v_i) = v_k$, then g maps the edge $\{v_i, v\}$ to the edge $\{v_k, v\}$ where v is the hub, and consequently g is a graph folding of the gear graph G_n. □

Example 2. Consider the wheel graph W_7 and the corresponding gear graph G_7. Let $f\colon W_7 \longrightarrow W_7$ be a graph folding defined by $f\{v_2, v_3\} = \{v_6, v_5\}$ and $f\{(v_2, v_1), (v_2, v_3), (v_2, v_7), (v_3, v_4), (v_3, v_7)\} = \{(v_6, v_1), (v_6, v_5), (v_6, v_7), (v_5, v_4), (v_5, v_7)\}$. Then, the graph map $g\colon G_7 \longrightarrow G_7$ defined by $g\{v_8, v_2, v_9, v_3, v_{10}\} = \{v_{13}, v_6, v_{12}, v_5, v_{11}\}$ is a graph folding, see Figure 2. In this case, g maps the edges (v_1, v_8), (v_8, v_2), (v_2, v_7), (v_2, v_9), (v_9, v_3), (v_3, v_7), (v_3, v_{10}), and (v_{10}, v_4) to the edges (v_1, v_{13}), (v_{13}, v_6), (v_6, v_7), (v_6, v_{12}), (v_{12}, v_5), (v_5, v_7), (v_5, v_{11}), and (v_{11}, v_4), respectively.

Definition 1. For a graph G, if we subdivide each edge once, we get a new graph G_s, and we will call it the subdivision graph.

Theorem 3. *Let G be a graph and G_s the subdivision graph of G. Let $f\colon G \longrightarrow G$ be a graph folding defined by for all $\{v_i, v_j\} \in E(G)$, $f\{v_i, v_j\} = \{v_k, v_l\} \in E(G)$. Then, the graph map $g\colon G_s \longrightarrow G_s$ is defined by*

(i) Mapping the edges vu and uw to themselves if and only if f maps the edge vw to itself

(ii) $g\{v_i, u_r\} = \{v_k, u_s\}$ and $g\{u_r, v_j\} = \{u_s, v_l\}$ where u_r and u_s are the new vertices replaced for the edges $\{v_i, v_j\}$ and $\{v_k, v_l\}$, respectively, is a graph folding

Proof. Suppose that $f\colon G \longrightarrow G$ is a graph folding such that $f\{v_i, v_j\} = \{v_k, v_l\}$ where $\{v_i, v_j\}\{v_k, v_l\} \in E(G)$. Now, replace the edge $\{v_i, v_j\}$ by the new edges

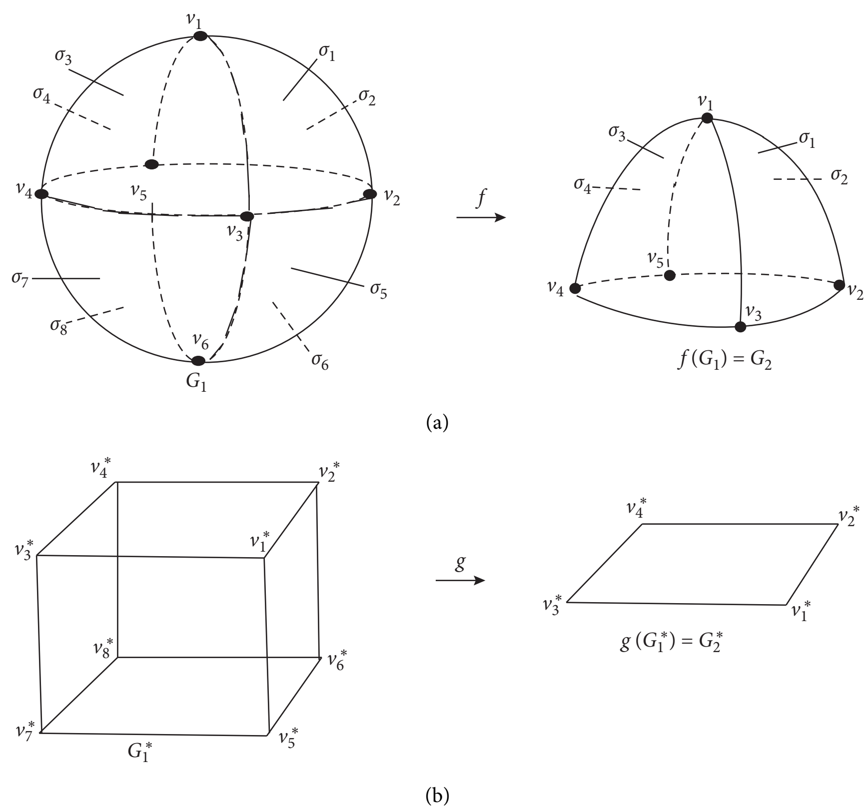

FIGURE 1: Graph folding of a graph and its dual graph.

$\{v_i, u_r\}\{u_r, v_j\} \in E(G_s)$ and the edge $\{v_k, v_l\}$ by the new edges $\{v_k, u_s\}\{u_s, v_l\} \in E(G_s)$. Since $g\{v_i, u_r\} = \{v_k, u_s\}$ and $g\{u_r, v_j\} = \{u_s, v_l\}$, then g maps edges to edges of G_s.

Now, let $e = vw$ be an edge of G such that $f(vw) = vw$. The subdivision of e replaces e by a new vertex u and two new edges vu and uw. Since $g(vu) = vu$ and $g(uw) = uw$, then g maps edges to edges of G_s. Thus, g is a graph folding. □

Example 3. Consider the graph G and its subdivision G_s shown in Figure 3. Let $f\colon G \longrightarrow G$ be a graph folding defined by $f(v_1) = v_3$ and $f\{e_1, e_2\} = \{e_4, e_3\}$. Then, the graph map $g\colon G_s \longrightarrow G_s$ defined by $g(v_1, u_1, u_2) = (v_3, u_4, u_3)$ and $g\{e_1', e_2', e_7', e_8'\} = \{e_4', e_3', e_6', e_5'\}$ is a graph folding.

4. Graph Folding of the Web and Crown Graphs

Theorem 4. *Let C_n be a cycle graph, where $V(C_n) = \{v_1, \ldots, v_n\}$, n is even, and $W_{n,r}$ be the web graph where $V(W_{n,r}) = \{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, v_{2n+1}, \ldots, v_{3n}, \ldots, v_{(r-1)n+1}, \ldots, v_{rn}\}$. Let $f\colon C_n \longrightarrow C_n$ be a graph folding defined by for all $v_i \in V(C_n)$, $f(v_i) = v_j \in V(C_n)$. Then, the graph map $g\colon W_{n,r} \longrightarrow W_{n,r}$ is defined by*

(i) For all $v_i \in V(C_n)$, $g(v_i) = f(v_i) = v_j$

(ii) For all $v_{i+(s-1)n} \in V(W_{n,r})$, $s = 2, \ldots, r$, then $g(v_{i+(s-1)n}) = v_{j+(s-1)n}$

Proof. Let C_n be a cycle graph with even vertices and $f\colon C_n \longrightarrow C_n$ a graph folding defined by for all $v_i \in V(C_n)$, $f(v_i) = v_j \in V(C_n)$. Consider the vertices $v_i, v_j, v_k, v_l \in V(C_n)$ such that f maps the edge $\{v_i, v_k\}$ to the edge $\{v_j, v_l\}$. For the vertices $v_i \in V(C_n)$, $g(v_i) = f(v_i) = v_j$, and hence g is a graph folding. If $s = 2$, then $g\{v_i, v_{i+n}\} = \{v_j, v_{j+n}\}$ and $g\{v_{i+n}, v_{k+n}\} = \{v_{j+n}, v_{l+n}\}$, i.e., g maps edges to edges. The same procedure can be done if $s = 3, 4, \ldots, r$. Thus, g is a graph folding. For illustration, see Figure 4. □

Example 4. Consider the cycle graph C_4. Let $f\colon C_4 \longrightarrow C_4$ be a graph folding defined by $f(v_2) = (v_4)$ *and* $f\{(v_2, v_1), (v_2, v_3)\} = \{(v_4, v_1), (v_4, v_3)\}$. The graph map $g\colon W_{4,3} \longrightarrow W_{4,3}$ defined by $g\{v_2, v_6, v_{10}\} = \{v_4, v_8, v_{12}\}$ and $g\{(v_2, v_1), (v_2, v_3), (v_2, v_6), (v_6, v_5), (v_6, v_7), (v_6, v_{10}), (v_{10}, v_9), (v_{10}, v_{11})\} = \{(v_4, v_1), (v_4, v_3), (v_4, v_8), (v_8, v_5), (v_8, v_7), (v_8, v_{12}), (v_{12}, v_9), (v_{12}, v_{11})\}$ is a graph folding, see Figure 5.

Theorem 5. *Any crown graph G of $2n$ vertices can be folded to an edge.*

Proof. Let G be a crown graph on $2n$ vertices, and let $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_n\}$ be the two sets of vertices of G. Now, a graph folding $f\colon G \longrightarrow G$ can be defined by mapping all the vertices of A to one vertex of A, say u_i, and all the vertices of B to one vertex of B, say v_j. Thus, the image $f(G)$ is the edge $\{u_i, v_j\} \in E(G)$. Thus, f is a graph folding. For illustration, see Figure 6. □

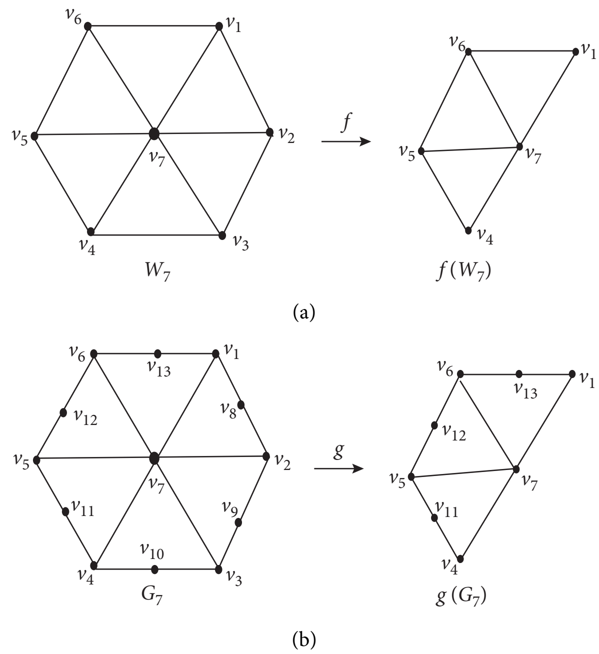

Figure 2: Graph folding of the wheel graph W_7 and the gear graph G_7.

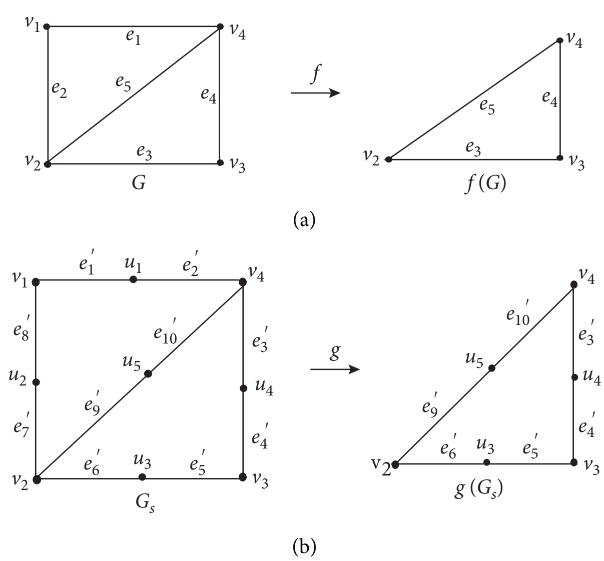

Figure 3: Graph folding of a graph and its subdivision graph.

5. Graph Folding of the Simplex and Crossed Prism Graphs

Theorem 6. *Let G be a graph and $f\colon G \longrightarrow G$ a graph folding. Then, the graph map $g\colon k(G) \longrightarrow k(G)$ is defined by*

(i) *For a zero vertex $v \in V(k(G))$, if $f(v_i) = v_k$, then $g\{v_i, v\} = \{v_k, v\}$ where $v_i, v_k \in V(G)$.*

(ii) *If $\{v_i, v_j\}$ and $\{v_k, v_l\}$ are cliques of G such that $f\{v_i, v_j\} = \{v_k, v_l\}$, then $g\{v_i, v_r\} = \{v_k, v_s\}$ and $g\{v_r, v_j\} = \{v_s, v_l\}$ where v_r and v_s are the new vertices of the cliques $\{v_i, v_j\}$ and $\{v_k, v_l\}$, respectively.*

(iii) *If $\sigma = \{v_i, v_j, v_k\}$ and $\delta = \{v_l, v_m, v_n\}$ are cliques of G such that $f\{v_i, v_j, v_k\} = \{v_l, v_m, v_n\}$, then $g\{u, v_{r_\mu}\} = \{w, v_{s_\mu}\}$, $\mu = 1, 2, 3$, where u and w are the new vertices of the two cliques σ and δ and v_{r_μ} and v_{s_μ} are the new vertices of the edges of σ and δ, respectively.*

And so on.

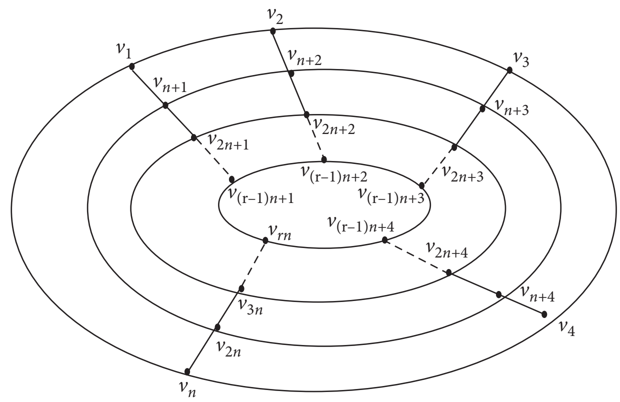

Figure 4: The web graph.

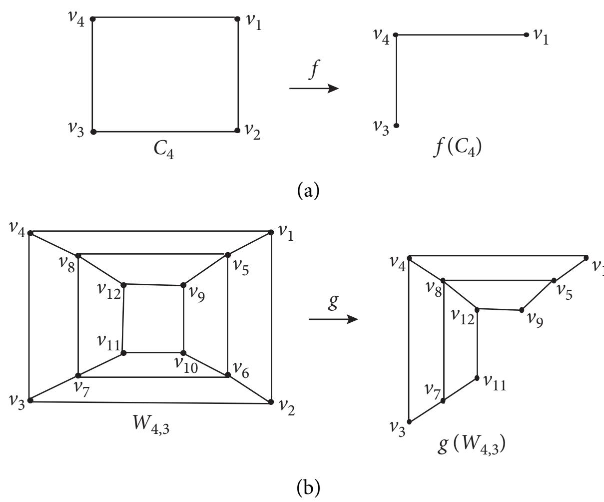

Figure 5: Graph folding of the cycle graph C_4 and the web graph $W_{4,3}$.

Proof. Let G be a graph and $f\colon G \longrightarrow G$ a graph folding

(i) Consider the vertices v_i, $v_k \in V(G)$ such that $f(v_i) = v_k$. Let v be a zero vertex of $V(k(G))$, and then by the given definition of g, it maps the vertex v onto itself. Then, we get new edges $\{v_i, v\},\{v_k, v\} \in E(k(G))$ but $g\{v_i, v\} = \{v_k, v\}$, i.e., g maps edges to edges of $(k(G))$.

(ii) Consider the cliques $\{v_i, v_j\}$ and $\{v_k, v_l\}$ of G such that $f\{v_i, v_j\} = \{v_k, v_l\}$. Let v_r and v_s be the new vertices of the two cliques, respectively; then we have new four edges $\{v_i, v_r\}$, $\{v_r, v_j\}$, $\{v_k, v_s\}$, and $\{v_s, v_l\} \in E(k(G))$ but $g\{v_i, v_r\} = \{v_k, v_s\}$ and $g\{v_r, v_j\} = \{v_s, v_l\}$, i.e., g maps edges to edges.

(iii) Finally, let $\sigma = \{v_i, v_j, v_k\}$ and $\delta = \{v_l, v_m, v_n\}$ be cliques of G such that $f\{v_i, v_j, v_k\} = \{v_l, v_m, v_n\}$. And, considering the new vertices u and w of the two cliques σ and δ, respectively. Then, we have new edges $\{u, v_{r_\mu}\}\{w, v_{s_\mu}\} \in E(k(G))\mu = 1, 2, 3,$ where v_{r_μ} and v_{s_μ} are the new vertices of the edges of the cliques σ and δ, respectively. Then, the map g maps the new edges of the boundary of σ to the new edges of the boundary of δ according to the rule (ii), and $g\{u, v_{r_\mu}\} = \{w, v_{s_\mu}\}$ where $\{u, v_{r_\mu}\}\{w, v_{s_\mu}\} \in E(k(G))$. For illustration, see Figure 7. Hence, g is a graph folding of the simplex graph $k(G)$. □

Example 5. Let G be the graph shown in Figure 8(a) and $f\colon G \longrightarrow G$ the graph folding defined by $f\{v_3, v_6, v_8\} = \{v_1, v_4, v_7\}$ and $f\{(v_5, v_6), (v_2, v_3), (v_3, v_6), (v_6, v_8), (v_3, v_8)\} = \{(v_5, v_4), (v_2, v_1), (v_1, v_4), (v_4, v_7), (v_1, v_7)\}$. Then, the graph map $g\colon k(G) \longrightarrow k(G)$ defined by $g\{v_3, v_6, v_8, u_7, u_8, u_9, u_{10}, u_{11}, w_2\} = \{v_1, v_4, v_7, u_4, u_5, u_3, u_2, u_1, w_1\}$ and $g\{(v_3, V), (v_6, V), (v_8, V), (v_5, u_8), (u_8, v_6), (v_2, u_7), (u_7, v_3), (v_3, u_9), (u_9, v_6), (v_6, u_{10}), (u_{10}, v_8), (v_3, u_{11}), (u_{11}, v_8), (w_2, u_9), (w_2, u_{10}), (w_2, u_{11})\} = \{(v_1, V), (v_4, V), (v_7, V), (v_5, u_5), (u_5, v_4), (v_2, u_4), (u_4, v_1), (v_1, u_3), (u_3, v_4), (v_4, u_2), (u_2, v_7), (v_1, u_1), (u_1, v_7), (w_1, u_3), (w_1, u_2), (w_1, u_1)\}$ is a graph folding, see Figure 8(b).

Theorem 7. *Let C_n be a cycle graph of even vertices and $f\colon C_n \longrightarrow C_n$ a graph folding. Consider the graph map $g\colon CP_n \longrightarrow CP_n$ such that for all $e \in E(C_n)$, $g\{e\} = f\{e\}$:*

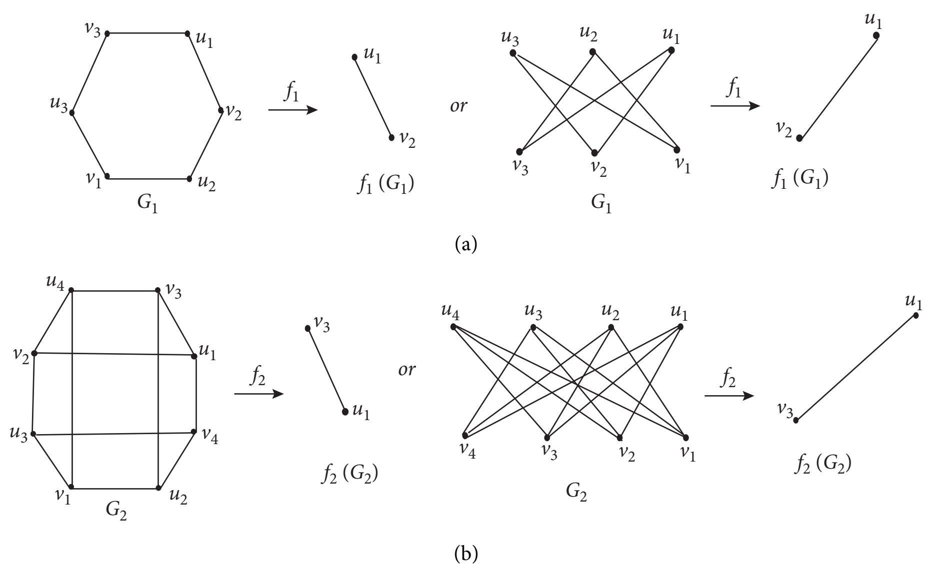

Figure 6: Folding crown graphs with six and eight vertices to an edge. (a) $f_1 = \{u_1, u_2, u_3\} = u_1, f_1 = \{v_1, v_2, v_3\} = v_2$ and (b) $f_2 = \{u_1, u_2, u_3, u_4\} = u_1, f_2 = \{v_1, v_2, v_3, v_4\} = v_3$.

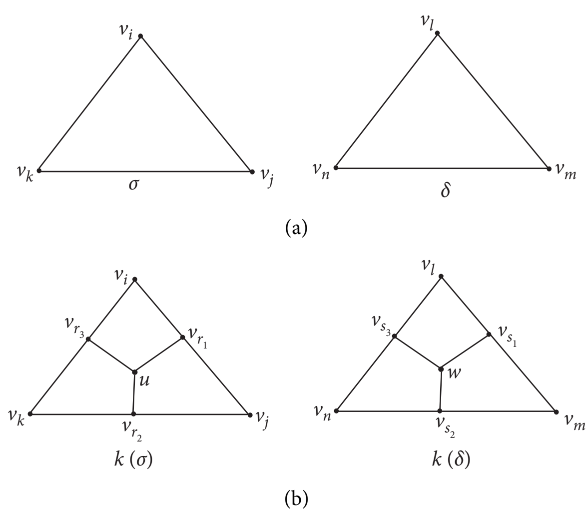

Figure 7: The simplex graph.

(i) If $g\{v_{2k}, v_{2k+1}\} = \{v_{2k+i}, v_{2k+i+1}\}$ *whenever* $f\{v_k, v_{k+1}\} = \{v_{k+i}, v_{k+i+1}\}$*, then* g *is a graph folding*

(ii) If $g\{v_{2k}, v_{2k+1}\} = \{v_{2k+i}, v_{2k+1}\}$ *whenever* $f\{v_k, v_{k+1}\} = \{v_{k+i}, v_{k+1}\}$*, then* g *is not a graph folding*

Proof (i) Let $f\colon C_n \longrightarrow C_n$ be a graph folding, and consider the edges $\{v_k, v_{k+1}\}\{v_{k+i}, v_{k+i+1}\} \in E(C_n)$ such that $f\{v_k, v_{k+1}\} = \{v_{k+i}, v_{k+i+1}\}$. Now, $g\{e\} = f\{e\}$ for all $e \in E(C_n)$. Also, $g\{v_{2k}, v_{2k+1}\} = \{v_{2k+i}, v_{2k+i+1}\}$, i.e., g maps the edge $\{v_{2k}, v_{2k+1}\} \in E(CP_n)$ to the edge $\{v_{2k+i}, v_{2k+i+1}\} \in E(CP_n)$. Consider one of the new edges of CP_n, e.g., $\{v_{k+1}, v_{2k}\}$. Then, $g\{v_{k+1}, v_{2k}\} = \{v_{k+i+1}, v_{2k+i}\} \in E(CP_n)$, i.e., g maps edges to edges, and hence g is a graph folding.

(ii) Now, let $f\{v_k, v_{k+1}\} = \{v_{k+i}, v_{k+1}\}$, and we know that for all $e \in E(C_n)$, $g\{e\} = f\{e\}$. Also, for the edge $\{v_{2k}, v_{2k+1}\} \in E(CP_n) g\{v_{2k}, v_{2k+1}\} = \{v_{2k+i}, v_{2k+1}\} \in E(CP_n)$, but if we consider one of the new edges of CP_n, e.g., $\{v_k, v_{2k+1}\}$, then $g\{v_k, v_{2k+1}\} = \{v_{k+i}, v_{2k+1}\}$ which is not an edge of CP_n, and hence g is not a graph folding. □

Example 6. Consider the cycle graph C_4, and let $f_1\colon C_4 \longrightarrow C_4$ be a graph folding defined by $f_1(v_1) = (v_3)$

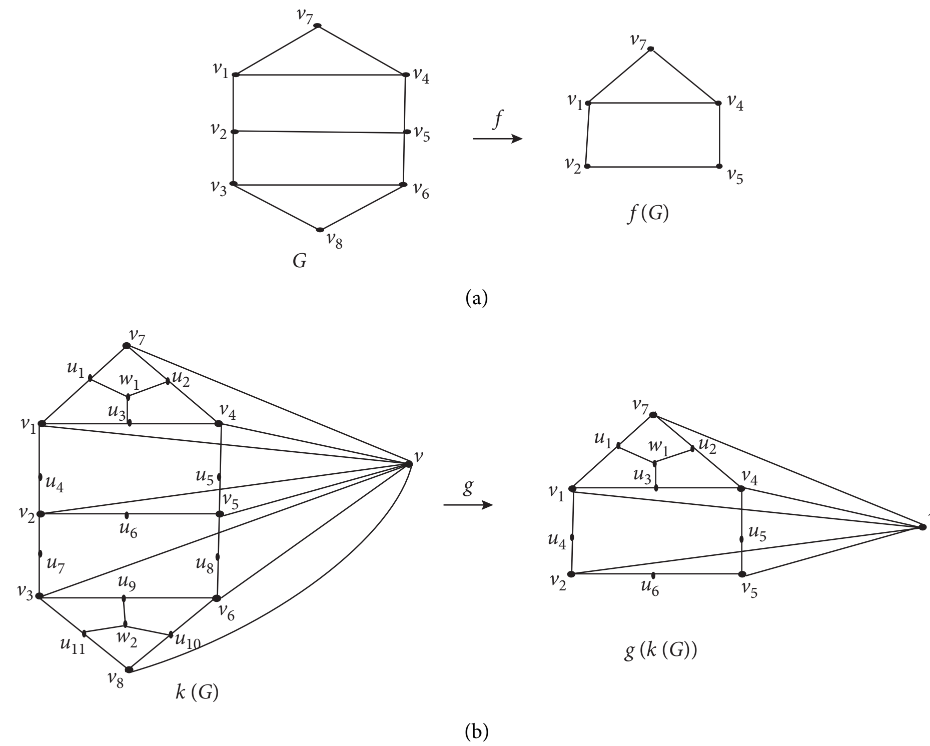

Figure 8: Graph folding of the simplex graph.

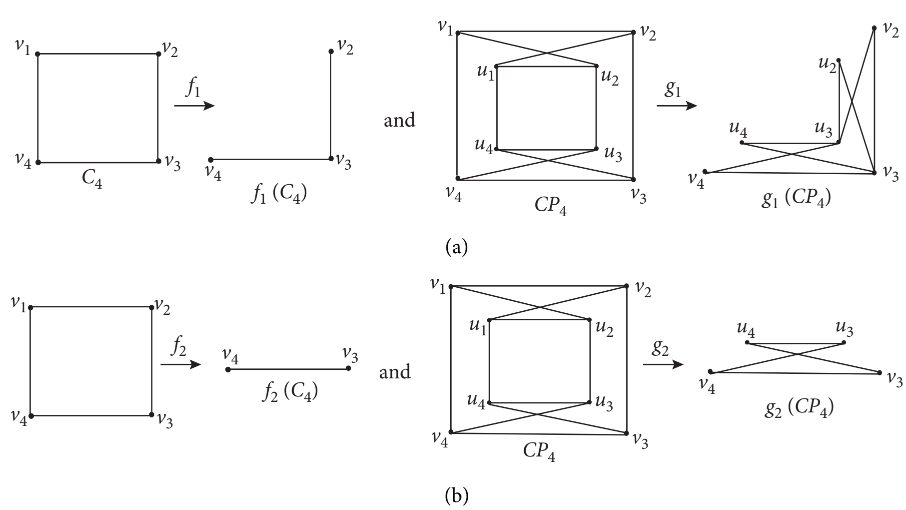

Figure 9: The crossed prism graph may or may not be folded.

and $f_1\{(v_1, v_2), (v_1, v_4)\} = \{(v_3, v_2)(v_3, v_4)\}$. The graph map $g_1\colon CP_4 \longrightarrow CP_4$ defined by $g_1\{v_1, u_1\} = \{v_3, u_3\}$ is not a graph folding since $g_1(v_1, u_2) = (v_3, u_2) \notin E(CP_4)$, see Figure 9(a). While if $f_2\colon C_4 \longrightarrow C_4$ is a graph folding defined by $f_2(v_1, v_2) = (v_3, v_4)$ *and* $f_2\{(v_1, v_2), (v_1, v_4), (v_2, v_3)\} = \{(v_3, v_4), (v_3, v_4), (v_4, v_3)\}$, then the graph map $g_2\colon CP_4 \longrightarrow CP_4$ defined by $g_2\{v_1, v_2, u_1, u_2\} = \{v_3, v_4, u_3, u_4\}$ and $g_2\{(v_1, v_2), (v_1, v_4), (v_2, v_3), (u_1, u_2), (u_1, u_4),$

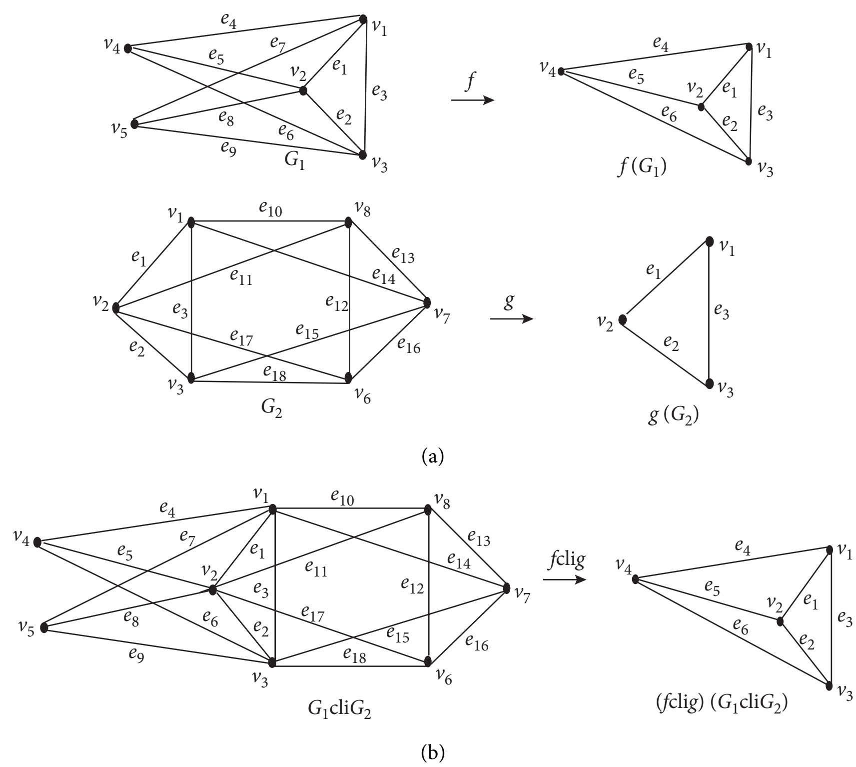

FIGURE 10: Graph folding of the clique-sum graph.

(u_2, u_3), (v_1, u_2), $(v_2, u_1)\} = \{(v_3, v_4)$, (v_3, v_4), (v_4, v_3), (u_3, u_4), (u_3, u_4), (u_4, u_3), (v_3, u_4), $(v_4, u_3)\}$ is a graph folding, see Figure 9(b).

6. Graph Folding of the Clique-Sum Graph

We will denote the clique-sum of the two graphs G and H by G cli H.

Definition 2. Let G_1, G_2, G_3, and G_4 be graphs. Let $f\colon G_1 \longrightarrow G_3$ and $g\colon G_2 \longrightarrow G_4$ be graph maps. Then, we can define a map from the clique-sum of G_1 and G_2 to the clique-sum of G_3 and G_4 denoted by $f\,\text{cli}\,g\colon G_1\,\text{cli}\,G_2 \longrightarrow G_3\,\text{cli}\,G_4$ as follows:

$$(f\,\text{cli}\,g)(e) = \begin{cases} f(e), & e \in G_1, \\ f(e) = g(e) = e, & \text{for all edges of the} \\ g(e), & e \in G_2, \end{cases}$$

shared cliques.

This map is called the clique-sum map of the maps f and g.

Theorem 8. *Let G_1, G_2, G_3, and G_4 be graphs. Let $f\colon G_1 \longrightarrow G_3$ and $g\colon G_2 \longrightarrow G_4$ be graph foldings. Then, the clique-sum map $f\,\text{cli}\,g\colon G_1\text{cli}\,G_2 \longrightarrow G_3\text{cli}G_4$ is a graph folding.*

Proof. Suppose f and g are graph foldings. Now, let $e \in G_1\text{cli}\,G_2$, and then either $e \in G_1$ or $e \in G_2$. In these two cases and since each of f and g is a graph folding, then $(f\text{cli}g)(e) \in (G_3\text{cli}\,G_4)$. Thus, $(f\,\text{cli}\,g)$ maps edges to edges, and hence the clique-sum map is a graph folding. □

Example 7. Consider the two graphs G_1 and G_2 shown in Figure 10(a). Let $f\colon G_1 \longrightarrow G_1$ be a graph folding defined by $f\{v_5\} = \{v_4\}$, $f\{e_7, e_8, e_9\} = \{e_4, e_5, e_6\}$, $g\{v_6, v_7, v_8\} = \{v_1, v_2, v_3\}$ and $g\{e_{17}, e_{18}, e_{12}, e_{16}, e_{10}, e_{11}, e_{13}, e_{14}, e_{15}\} = \{e_1, e_3, e_3, e_1, e_3, e_2, e_2, e_1, e_2\}$. Then, the clique-sum map $f\,\text{cli}\,g\colon G_1\text{cli}\,G_2 \longrightarrow G_1\text{cli}G_2$ defined by $(f\,\text{cli}\,g)\{v_5, v_6, v_7, v_8\} = \{v_4, v_1, v_2, v_3\}$ and $(f\,\text{cli}\,g)\{e_7, e_8, e_9, e_{17}, e_{18}, e_{12}, e_{16}, e_{10}, e_{11}, e_{13}, e_{14}, e_{15}\} = \{e_4, e_5, e_6, e_1, e_3, e_3, e_1, e_3, e_2, e_2, e_1, e_2\}$ is a graph folding, see Figure 10(b).

7. Conclusion

We obtained the necessary and sufficient conditions, if exist, for folding new graphs obtained from a given graph by some techniques like dual, gear, subdivision, web, crown, simplex, crossed prism, and clique-sum graphs. We can examine the relation between folding a given pair of graphs and folding of a new graph generated from these given pair of graphs by some operations like join, Cartesian product, normal product, and tensor product. Also, we can lift the definition of folding from graphs to digraphs which has close connections to important industrial applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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