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Research Article

Fixed-Point Theorem for Isometric Self-Mappings

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In this paper, we derive a fixed-point theorem for self-mappings. That is, it is shown that every isometric self-mapping on a weakly compact convex subset of a strictly convex Banach space has a fixed point.

1. Introduction

Let $\mathcal{X}$ be a Banach space and $\mathcal{C}$ be a closed convex subset $\mathcal{X}$. Let $T\colon \mathcal{C} \longrightarrow \mathcal{C}$ be a self-mapping of $\mathcal{C}$. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad (1)$$

for all $x, y \in \mathcal{C}$. The fixed-point set of T is $\mathrm{Fix}(T)\colon = \{x \in \mathcal{C}\colon Tx = x\}$. We say that the subset $\mathcal{C}$ of $\mathcal{X}$ is said to have an approximate fixed-point sequence for a nonexpansive self-map T if

$$\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0, \quad (2)$$

for any sequence $\{x_n\}_{n\geq 1} \subset \mathcal{C}$. When the closed convex subset $\mathcal{C}$ is bounded, then such a sequence always exists; indeed, by letting $\varepsilon_n \in (0,1)$, for all $n \geq 1$, be a null sequence and defining the maps $T_n x = \varepsilon_n x_0 + (1-\varepsilon_n)Tx$ where arbitrarily $x_0 \in \mathcal{C}$, one can see that $\|T_n y - T_n x\| \leq (1-\varepsilon_n)\|Ty - Tx\| \leq (1-\varepsilon_n)\|y - x\|$, implying that T_n is a contraction mapping with contraction constant $1-\varepsilon_n$. By the Banach contraction mapping theorem, it follows that there exists a unique $x_n \in \mathcal{C}$ such that $T_n x_n = x_n$, which implies that

$$x_n = \varepsilon_n x_0 + (1-\varepsilon_n)Tx_n, \quad (3)$$

from which we get $\|x_n - Tx_n\| = \varepsilon_n\|x_0 - Tx_n\| \leq \varepsilon_n \sup_{x,y\in\mathcal{C}}\|x - y\|$. Given that $\varepsilon_n \longrightarrow 0$ and $\sup_{x,y\in\mathcal{C}}\|x - y\| < \infty$, it follows that $\{x_n\}_{n\geq 1}$ is an approximate fixed-point sequence.

Another way of constructing an approximate fixed-point sequence is to require or assume that $\mathrm{Fix}(T)$ is nonempty. Now due to the assumption that $\mathrm{Fix}(T) \neq \varnothing$, the sequence $\{x_n\}_{n\geq 1} \subset \mathcal{C}$ is bounded (indeed, $\|x_n - p\| \leq \|x_0 - p\|$ for all $p \in \mathrm{Fix}(T)$ and x_0 taken arbitrarily in $\mathcal{C}$). Hence,

$$\|x_n - Tx_n\| = \varepsilon_n\|x_0 - Tx_n\| \longrightarrow 0, \quad (4)$$

and $\{x_n\}_{n\geq 1}$ is an approximating fixed point for T.

A mapping $T\colon \mathcal{C} \longrightarrow \mathcal{C}$ of a set $\mathcal{C}$ in a Banach space $\mathcal{X}$ is called isometric if

$$\|Tx - Ty\| = \|x - y\|, \quad (5)$$

for all $x, y \in \mathcal{C}$. Note that an isometric mapping is just a nonexpansive mapping in which the inequality is always an equality. A well-known result of Brodskii and Milman [1] asserts that if $\mathcal{C}$ is a weakly compact convex subset of $\mathcal{X}$ and $\mathcal{X}$ has normal structure, then $\mathcal{X}$ has the fixed-point property for isometric mappings. In particular, any compact convex subset of $\mathcal{X}$ has the fixed-point property (see [2]).

References [3–10] can be consulted for fixed-point problems on isometric mappings.

Definition 1 (strictly convex Banach space). A strictly convex Banach space is a Banach space such that whenever $x \neq 0 \neq y$, then $\|x + y\| = \|x\| + \|y\|$ if and only if $x = \lambda y$ for some $\lambda > 0$.

An example of a strictly convex Banach space is a Hilbert space.

Definition 2 (convex linear). Let $\mathscr{X}$ be a Banach space and $\mathscr{C} \subseteq \mathscr{X}$ be a closed convex subset of $\mathscr{X}$. Then, the map $T: \mathscr{C} \longrightarrow \mathscr{X}$ is said to be a convex linear if

$$T(ax + (1-a)y) = aTx + (1-a)y, \qquad (6)$$

for all $x, y \in \mathscr{C}$ and $a \in (0,1)$.

An example of a convex linear is a linear map.

2. Preliminaries

We introduce the following useful theorems that will be used in the proof of our main result.

Theorem 1 (Mazur's theorem). *Every nonempty convex subset of a Banach space is strongly closed if and only if it is weakly closed.*

Theorem 2 (Cantor's intersection theorem). *Let $\mathscr{X}$ be a topological space. A decreasing nested sequence of nonempty compact closed subset of $\mathscr{X}$ has nonempty intersection. In other words, suppose that $\mathscr{C}_k$ is a sequence of nonempty compact closed subset of $\mathscr{X}$ satisfying $\mathscr{C}_0^{\mathscr{C}_1^{,\ldots,\mathscr{C}_n^{\mathscr{C}_{n+1},\ldots}}}$, and it follows that*

$$\left(\bigcap {}_k \mathscr{C}_k \right) \neq \varnothing. \qquad (7)$$

3. Main Result

We give the proof of the main result of this paper, which is accomplished in Theorem 3 below. The following lemma, corollary, and proposition shall aid us in arriving at the conclusion of the main result.

Lemma 1. *Let $\mathscr{X}$ be a strictly convex Banach space and $\mathscr{C} \subseteq \mathscr{X}$ be a closed convex subset and $T: \mathscr{C} \longrightarrow \mathscr{C}$ be an isometric mapping. Then, T is convex linear on $\mathscr{C}$. That is, $T(ax + (1-a)y) = aTx + (1-a)Ty$ for all $x, y \in \mathscr{C}$ and $a \in (0,1)$.*

Proof. Let $w = ax + (1-a)y$ and $a \in (0,1)$. Without loss of generality, assume $x \neq y$. Then, $w - x = (1-a)(y-x)$ implies that

$$\|w - x\| = (1-a)\|y - x\|. \qquad (8)$$

Similarly, $w - y = a(x-y)$ which also implies that

$$\|w - y\| = a\|x - y\|. \qquad (9)$$

First, we show that $Tx \neq Ty \neq Tw$. To see this, we observe that if $Tx = Tw$, then from (9), we have

$$\begin{aligned} \|w - y\| &= a\|x - y\| \\ &= a\|Tx - Ty\| \\ &= a\|Tw - Ty\| \\ &= a\|w - y\|, \end{aligned} \qquad (10)$$

leading to the contradiction that $a = 1$.

Similarly, if $Ty = Tw$, then from (8), we have

$$\begin{aligned} \|w - x\| &= (1-a)\|y - x\| \\ &= (1-a)\|Ty - Tx\| \\ &= (1-a)\|Tw - Tx\| \\ &= (1-a)\|w - x\|, \end{aligned} \qquad (11)$$

leading to the contradiction that $a = 0$.

Now since T is an isometry, it follows that

$$\begin{aligned} \|Tx - Ty\| &= \|Tx - Tw + Tw - Ty\| \\ &\leq \|Tx - Tw\| + \|Tw - Ty\| \\ &= \|x - w\| + \|w - y\| \\ &= (1-a)\|x - y\| + a\|x - y\| \\ &= \|x - y\| \\ &= \|Tx - Ty\|, \end{aligned} \qquad (12)$$

which implies that $\|Tx - Tw\| + \|Tw - Ty\| = \|Tx - Ty\|$. Since $\mathscr{X}$ is strictly convex and $Tx \neq Tw \neq Ty$ implies that there exist $\lambda > 0$ such that

$$\begin{aligned} Tx - Tw &= \lambda(Tw - Ty), \\ (1+\lambda)Tw &= Tx + \lambda Ty, \\ Tw &= \frac{1}{1+\lambda}Tx + \frac{\lambda}{1+\lambda}Ty, \end{aligned} \qquad (13)$$

we obtain

$$Tw = \beta Tx + (1-\beta)Ty, \qquad (14)$$

where $\beta \in (0,1)$ and $\beta = 1/(1+\lambda)$. We finally show that $\beta = a$.

From (14), we have $Tw - Tx = (1-\beta)(Ty - Tx)$ which implies that

$$\|Tw - Tx\| = (1-\beta)\|Ty - Tx\|. \qquad (15)$$

Also, $Tw - Ty = \beta(Tx - Ty)$ implies that

$$\|Tw - Ty\| = \beta\|Tx - Ty\|. \qquad (16)$$

From (16), $\beta\|Tx - Ty\| = \|Tw - Ty\| = \|w - y\| = a\|x - y\| = a\|Tx - Ty\|$ which implies that $\beta = a$. Hence, from (14), we have shown that $T(ax + (1-a)y) = aTx + (1-a)Ty$ which establishes the convex linearity of T. □

Corollary 1. *Let $\mathscr{X}$ be a strictly convex Banach space, $\mathscr{C} \subseteq \mathscr{X}$ be a closed convex subset, and $T: \mathscr{C} \longrightarrow \mathscr{C}$ be an isometric mapping. Then, the function $f: \mathscr{C} \longrightarrow \mathbb{R}$, $x \longrightarrow \|(I - T)x\|$ is a continuous convex function.*

Proof. From Lemma 1, T is a convex linear. Hence, subtracting the term $ax + (1-a)y$ from both sides of $T(ax + (1-a)y) = aTx + (1-a)Ty$ for all $x, y \in \mathscr{C}$ and $a \in (0,1)$, we have

$$(I-T)(ax+(1-a)y) = a(I-T)x + (1-a)(I-T)y. \quad (17)$$

We have the following evaluation:

$$\begin{aligned} f(ax+(1-a)y) &= \|(1-T)(ax+(1-a)y)\| \\ &= \|a(1-T)x + (1-a)(1-T)y\| \\ &\le a\|(1-T)x\| + (1-a)\|(1-T)y\| \\ &= af(x) + (1-a)f(y). \end{aligned} \quad (18)$$

Thus, f is a convex function and continuous (because T is continuous). □

Proposition 1. *Let $f: C \longrightarrow \mathbb{R}$ be continuous convex function on a weakly compact convex subset $\mathscr{C}$ of any Banach space $\mathscr{X}$. Then, f attains its minimum on $\mathscr{C}$. That is, there exist $\overline{x} \in \mathscr{C}$ such that*

$$f(\overline{x}) = \min\{f(x): x \in \mathscr{C}\}. \quad (19)$$

Proof. Let $m = \inf\{f(x): x \in \mathscr{C}\}$. We show that $m > -\infty$ and that $f(\overline{x}) = m$ for some $\overline{x} \in \mathscr{C}$.

Suppose that $m = -\infty$, and for each $n \in \mathbb{N}$, define $\mathscr{C}_n = \{x \in \mathscr{C}: f(x) \le -n\}$. For each $n \in \mathbb{N}$, the set $\mathscr{C}_n$ is closed (and weakly closed by Theorem 1), convex, and nonempty (since $m = -\infty$). Therefore, $(\mathscr{C}_n)_{n=1}^{\infty}$ forms a nested decreasing sequence of weakly compact nonempty sets. By Theorem 2, $\cap_{n=1}^{\infty}\mathscr{C}_n \ne \varnothing$. But this implies that there is some $\overline{x} \in \mathscr{C}$ such that $f(\overline{x}) \le -n$ for all $n \in \mathbb{N}$, an impossibility. Consequently, $m > -\infty$.

So define $n \in \mathbb{N}$ a sequence of sets $\mathscr{C}'_n = \{x \in \mathscr{C}: f(x) \le m + (1/n)\}$ for all n belonging to N (natural numbers). As before, $(\mathscr{C}'_n)_{n=1}^{\infty}$ is a nested sequence of weakly compact nonempty sets and so $\cap_{n=1}^{\infty}\mathscr{C}'_n \ne \varnothing$. If $\overline{x} \in \cap_{n=1}^{\infty}\mathscr{C}'_n$, then $\overline{x} \in \mathscr{C}$ and $f(\overline{x}) = m$ as required. □

Theorem 3 (main result). *Every isometric self-mapping $T: \mathscr{C} \longrightarrow \mathscr{C}$ on a weakly compact convex subset $\mathscr{C}$ of a strictly convex Banach Space $\mathscr{X}$ has a fixed point.*

Proof. We know from Corollary 1 that $f(x): = \|(I-T)x\|$ is a continuous convex function. So, by Proposition 1, f attains its minimum on C, say $\overline{x}$. By the approximate fixed point of T, it is always possible to choose a sequence $\{x_n\}_{n\ge 1} \subset \mathscr{C}$ such that $f(x_n) = \|x_n - Tx_n\| \le (1/n)$ for all $n \ge 1$. Hence, $\|(I-T)\overline{x}\| = \inf_{x\in\mathscr{C}} \|(I-T)x\| = 0$ and so $T\overline{x} = \overline{x}$. □

Example 1. Let $\mathscr{C} = [0,1]$ and $T: [0,1] \longrightarrow [0,1], Tx = 1 - x$ for all $x \in [0,1]$. Then, T is an isometry, and by Theorem 3, T has a fixed point.

As an application of Theorem 3, it is desired to solve the linear problem

$$x - v = \lambda(Tx - v), \quad \lambda \in (0,1), \quad (20)$$

for some nonzero scalar λ and constant vector v, where $T: \mathscr{C} \longrightarrow \mathscr{C}$ is linear and $\mathscr{C}$ is closed and convex. The following theorem gives a solution.

Theorem 4. *Let $\mathscr{C}$ be a closed convex subset of a Banach space $\mathscr{X}$ and let $T: \mathscr{C} \longrightarrow \mathscr{C}$ be linear and continuous. Then, equation (20) has a solution if the following holds:*

(1) $\|T\|_{B(\mathscr{X},\mathscr{X})} = \lambda^{-1}$
(2) $\mathscr{C}$ is weakly compact
(3) $\mathscr{X}$ is strictly convex

Proof. By defining the auxiliary mapping

$$T': \mathscr{C} \longrightarrow \mathscr{C}, x \longmapsto \lambda Tx + (1-\lambda)v, \quad (21)$$

it is clearly seen that T' is well defined and invariant on $\mathscr{C}$ since $\mathscr{C}$ is convex and $\lambda \in (0,1)$. Since $T'y - T'x = \lambda(Ty - Tx)$, then by taking norms gives us

$$\left\|T'y - T'x\right\|_{\mathscr{X}} = \lambda\|Ty - Tx\|_{\mathscr{X}}. \quad (22)$$

Since T is bounded, then

$$\begin{aligned} \left\|T'y - T'x\right\|_{\mathscr{X}} &\le \lambda\|T\|_{B(\mathscr{X},\mathscr{X})}\|y - x\|_{\mathscr{X}} = \lambda \cdot \lambda^{-1}\|y - x\|_{\mathscr{X}} \\ &= \|y - x\|_{\mathscr{X}}. \end{aligned} \quad (23)$$

Since T' is an isometry, then by Theorem 3, one can find a solution $\overline{x} = T'\overline{x} = \lambda T\overline{x} + (1-\lambda)v$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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